On a strange resonance noticed by M. Herman

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General introduction

The early history of the theory of the Moon's motion is marked with a famous controversy. Let us explain it briefly. The orbit of the moon is well described as a "moving Keplerian ellipse", in such a way that one can speak about the respective motions of two geometric lines, namely the line of apsides and the line of nodes. The first is the principal axis of the ellipse, the second is the intersection of the plane of the ellipse with the plane of the geocentric motion of the sun (called plane of the ecliptic). The first computations, by Newton, Machin, Euler or Clairaut, based on the Newton's laws, for the motion of these lines, gave a wrong result [Smi].

In these computations the main perturbation, the effect of the Sun, was taken into account at the first order. The predicted result was that both lines of apsides and nodes should undergo uniform rotations, the first being direct with a period of about 18 years, the second being retrograde with the *same* period. The result about the nodes is quite in agreement with the data obtained from the observations. But the period the apsis is of only 9 years. The explanation for this discrepancy was given by Clairaut in 1750, who found an important contribution of the higher order terms.

We are not interested here in the discrepancy, but in the strange phenomenon occurring at the first order, that makes the two periods above equal. Indeed, a nice integrable model for planetary systems appeared in the works of Lagrange and Laplace. Let us call it for short the first secular system. It is essentially a linear system with constant coefficients. It comes just after the "Keplerian model", where the planets all describe fixed keplerian orbits, and it is an interesting approximation if the Keplerian model is good, and if the orbits are nearly circular and nearly in the same plane. This includes the Lunar problem, even if it is quite strange to take the earth as a central body and to consider the moon and the sun as "planets".

Generally this approximation is not considered as sufficiently accurate, but it is used as a step to go further. The strange phenomenon pertains to the first secular system. It is a well-known fact, but the literature is not very talkative about it. Delaunay [Del] speaks about "un résultat singulier". Poincaré [Po1] calls it a "très petit diviseur analytique", and tells that the discrepancy prevents it to be a "très petit diviseur numérique". He then sketches a proof of the phenomenon. Finally, the same facts are known to exist for asteroids [Mil].

In his unpublished attempts to adapt KAM theory to the case of a general planetary system, M. Herman had to study carefully the first secular system. He remarked a strange relation between the frequencies of this linear system: their sum is always zero. In the lunar case above the orbit of the sun is nearly fixed and Herman's relation tells simply that the sum of the two frequencies concerning the moon is zero. This is our "strange phenomenon". The same is true for asteroids, because the Keplerian orbit of Jupiter is nearly fixed. But apparently the general case was not noticed before Herman. However, the proof is a quite standard computation.

The aim of this paper is to give a lemma which in turn gives an infinite sequence of unexpected relations between the coefficients of the classical expansion of the perturbating function. The first of these relations is the source of Herman's resonance.

1. An introduction to complex variable in perturbation theory

The first step in the description of a motion by a "moving keplerian ellipse" is to express the position of a body giving the elements (i.e. the coordinates) of the ellipse where it is supposed to move, and an angle to place it on the ellipse. The past decades were marked by a more and more systematic use of complex numbers for this expression. It is convenient to avoid the traditional e and i for eccentricity and inclination of the orbit, denoting them respectively by ε and δ.

Our starting point is the classical recipe for the "2-body" or "fixed center" problem, in a plane with cartesian coordinates:

$$x = a(\cos u - \varepsilon), \qquad y = a\sqrt{1-\varepsilon^2}\sin u, \qquad l = u - \varepsilon \sin u. \tag{1}$$

The angle l (defined mod. 2π) is called the "mean anomaly" and is such that dl/dt is a constant. The variable u is also an angle called the "eccentric anomaly", and a is a positive number called "semi major axis". The recipe (1) puts the perihelion at the point $x = a(1-\varepsilon)$, $y = 0$. We always suppose that the orbit is elliptic: the eccentricity satisfies $0 \le \varepsilon < 1$.

We must now put our ellipse in a general position in the *plane*. The use of complex numbers is quite natural to make the required rotation. Also, it will be useful to expand $\sqrt{1-\varepsilon^2}$. The complex position of the body now reads

$$R = (x+iy)e^{i\omega} = ae^{i\omega}\big(\cos u - \varepsilon + i(1 - \frac{1}{2}\varepsilon^2 + \cdots)\sin u\big), \tag{2}$$

where ω is an angle indicating the direction of the perihelion. Now we observe that nor this angle nor the eccentric anomaly u is defined when $\varepsilon = 0$. But when we keep a and the *sum* $\omega + u$ constant, making $\varepsilon \to 0$, the position R has a limit. To regularize the case $\varepsilon = 0$ we set $\tilde{u} = \omega + u$ and $L = \varepsilon e^{i\omega}$. The formula (2) becomes

$$\frac{R}{a} = e^{i\tilde{u}} - L - \frac{1}{4}(e^{iu} - e^{-iu})L\bar{L}e^{i\omega} + \cdots = e^{i\tilde{u}} - L - \frac{1}{4}e^{i\tilde{u}}L\bar{L} + \frac{1}{4}e^{-i\tilde{u}}L^2 + \cdots . \tag{3}$$

The other terms in the expansion also appear to be monomials in $e^{i\tilde{u}}$, L and their conjugates.

Rationalizing elements. In the following we will just need the terms in the square of eccentricity, so we could content ourself with the displayed terms in the expansion (3). But it is worth noting that there exists an *exact* rational formula as simple as the truncated expansion (3). The computations giving it are closely related to [Po2], p. 291. Let us put $\varepsilon = \sin\phi$. As $\cos\phi$

appears in (1), we may rationalize putting $\tau = \tan\frac{\phi}{2}$. Then we use instead of L the complex number $k = \tau e^{i\omega}$. Starting again from (1), we find

$$(1+\tau^2)\frac{R}{a} = e^{i\omega}\big((1+\tau^2)\cos u - 2\tau + i(1-\tau^2)\sin u\big) = e^{i\omega}(e^{iu} - 2\tau + \tau^2 e^{-iu})$$
$$= e^{i\omega}e^{iu}(1-\tau e^{-iu})^2 = e^{i\tilde{u}}(1-ke^{-i\tilde{u}})^2.$$

The resulting factorization is quite remarkable, but it will not be useful in this work. We obtained the rational expression for the position R of the body in the plane :

$$R = \frac{ae^{i\tilde{u}}(1-ke^{-i\tilde{u}})^2}{1+k\bar{k}}. \tag{4}$$

The ellipse in 3-space. We choose in the Euclidean space a conventional orthonormal frame. The first two vectors generate the reference plane, called "horizontal". The first vector is the origin of the "longitudes". What should we do with complex numbers in the 3D case? We give us the position of the body by a pair (r_c, r_z), where $r_c \in \mathbb{C}$ is the projection on the horizontal plane, and $r_z \in \mathbb{R}$ is the height. As we shall see, the ellipse will be given by a and two complex numbers, that curiously enough appears quite symmetrically in the formulas.

The intersection of the ellipse with the reference plane is called the line of nodes, and is oriented in the direction of the "ascending node". Its angle with the first reference vector is the "longitude of the ascending node" Ω. The inclination of the plane is called δ and satisfies $0 \le \delta \le \pi$.

Formula (4) gives us the position R of the body in the plane of the ellipse. To obtain its projection on the horizontal plane, we transform R by the affine transformation

$$R \mapsto (\cos^2\frac{\delta}{2})R + (\sin^2\frac{\delta}{2})\bar{R}, \tag{5}$$

and we multiply by $e^{i\Omega}$. One will check easily that (5) fixes the real axis, i.e. the line of nodes, and contracts the imaginary axis by a ratio $\cos\delta$. And this is exactly the effect of the projection on the horizontal plane.

We should now introduce the complex element $s = -i\tan\frac{\delta}{2}e^{i\Omega}$. As k above, this element is helpful both in the question of the indetermination $\delta = 0$ and for the rationalization of the formulas. The factor $-i$ is a convention that makes some formulas nicer. Transformation (5) becomes

$$R \mapsto (1+s\bar{s})^{-1}(R + s\bar{s}\bar{R}). \tag{6}$$

Applying this to (4), we get

$$(1+s\bar{s})(1+k\bar{k})r_c = a\big(e^{i\tilde{u}}(1-ke^{-i\tilde{u}})^2 + e^{-i\tilde{u}}(1-\bar{k}e^{i\tilde{u}})^2 s\bar{s}\big)e^{i\Omega}.$$

We introduce the new variables $\varpi = \omega + \Omega$, called longitude of the perihelion, $\hat{u} = \tilde{u} + \Omega = u + \varpi$, called eccentric longitude, and $g = ke^{i\Omega} = \tan\frac{\phi}{2}e^{i\varpi}$. We obtain

$$(1+s\bar{s})(1+g\bar{g})\frac{r_c}{a} = e^{i\hat{u}}(1-ge^{-i\hat{u}})^2 - e^{-i\hat{u}}(1-\bar{g}e^{i\hat{u}})^2 s^2$$
$$= e^{i\hat{u}}\big(1+\bar{g}s-(g+s)e^{-i\hat{u}}\big)\big(1-\bar{g}s-(g-s)e^{-i\hat{u}}\big).$$

Here again a nice factorization, useless in this paper, appears as a good surprise in the computations. Similarly we compute the vertical component r_z and the "weight" $l_u = dl/du$, useful in the process of averaging:

$$(1+s\bar{s})(1+g\bar{g})\frac{r_z}{a} = -e^{i\hat{u}}(1-ge^{-i\hat{u}})^2\bar{s} - e^{-i\hat{u}}(1-\bar{g}e^{i\hat{u}})^2 s,$$
$$(1+g\bar{g})l_u = (1-\bar{g}e^{i\hat{u}})(1-ge^{-i\hat{u}}).$$

2. Various expressions of the Lemma

Possible variables in the Lemma. The above pair $(g,s) \in \mathbb{C}^2$ is one among various possible choices of coordinates for the set of Keplerian ellipses with semi major axis a. We will state the Lemma using the pair $(w_1, w_2) \in \mathbb{C}^2$, which is one of the following pairs of coordinates: $(2g, 2s)$ or (L_c, S_c) or $(\xi_c/\sqrt{2}, \eta_c/\sqrt{2})$ or (X, Y). We use in these notations the index c as above: a vector $\vec{L}$ in 3-space is written as a pair $(L_c, L_z) \in \mathbb{C} \times \mathbb{R}$, giving its horizontal and vertical components. We now explain the meaning of these variables and the relations between them.

The vectors $\vec{L}$ and $\vec{S}$ are respectively the *eccentricity vector* and the *normalized angular momentum*. They satisfy $\|\vec{L}\| = \varepsilon$, $\|\vec{S}\| = \sqrt{1-\varepsilon^2}$, and $\vec{L}\cdot\vec{S} = 0$. If we call μ the gravitational constant (defined later), the vector $\sqrt{\mu a}\vec{S}$ is the angular momentum. Our convention here is that $\vec{L}$ points the perihelion of the orbit. A computation shows that

$$L_c = \frac{2(g-\bar{g}s^2)}{(1+s\bar{s})(1+g\bar{g})}, \qquad S_c = \frac{2s(1-g\bar{g})}{(1+s\bar{s})(1+g\bar{g})}.$$

The difference $(L_c, S_c) - (2g, 2s)$ is small. More precisely, if we expand it in $(g, s, \bar{g}, \bar{s})$, the series begins at order three (the terms are odd monomials). From this we can conclude that the two set of variables are exactly equivalent for the purpose of the Lemma below.

The vectors $\vec{\xi}$ and $\vec{\eta}$ are the "Souriau vectors" [Sou]. They are defined by the relations $\vec{\xi} = \vec{S}+\vec{L}$ and $\vec{\eta} = \vec{S} - \vec{L}$, and satisfy $\|\vec{\xi}\| = \|\vec{\eta}\| = 1$.

Finally,

$$X = \frac{2g}{\sqrt{1+g\bar{g}}}, \qquad Y = \sqrt{\frac{1-g\bar{g}}{1+g\bar{g}}}\,\frac{2s}{\sqrt{1+s\bar{s}}}.$$

These variables possess the following property. Let $I_l = \sqrt{\mu a}$ be the conjugate variable to the mean anomaly l. We set $\sqrt{I_l}X = x_1 + ix_2$ and $\sqrt{I_l}Y = y_1 + iy_2$. Then the canonical symplectic form is $dI_l \wedge d\hat{l} + dx_2 \wedge dx_1 + dy_2 \wedge dy_1$. The canonical elements $(I_l, \hat{l}, x_2, x_1, y_1, -y_2)$ are Poincaré's variables (cf. [Po2], p. 30). They are introduced in the complex framework in [Las] and [LaR].

We can now give a Lemma due to the first author [Abd].

Lemma. Let a be a non negative real number. Let $\mathcal{E}(w_1, w_2)$ be the Keplerian ellipse in 3-space with focus at the origin O, semi major axis a and complex coordinates (w_1, w_2) as explained above. Let B be a point in the complementary of the "reference circle" $\mathcal{E}(0,0)$. We consider the average

$$D_\lambda(w_1,w_2)=\frac{1}{2\pi}\int_0^{2\pi}\|\vec{\mathrm{AB}}\|^{2\lambda}d\hat{l},$$

where A is the point of $\mathcal{E}(w_1,w_2)$ with mean longitude $\hat{l}$ (the mean longitude is the sum of the already defined mean anomaly l and of the longitude of the perihelion ϖ.) We have

$$\Big(\frac{\partial^2}{\partial w_1\partial\bar{w}_1}+\frac{\partial^2}{\partial w_2\partial\bar{w}_2}\Big)D_\lambda\big|_{w_1=w_2=0}=\frac{1}{2}\lambda(2\lambda+1)a^2D_{\lambda-1}\big|_{w_1=w_2=0}.$$

Proof. We set $\vec{\mathrm{OA}}=(A_c,A_z)\in\mathbb{C}\times\mathbb{R}$, $\vec{\mathrm{OB}}=(B_c,B_z)\in\mathbb{C}\times\mathbb{R}$. We work in the variables (g,s), and make the substitution $d\hat{l}=l_u d\hat{u}$. We want to expand D_λ at the neighborhood of $(g,s)=(0,0)$ and look at the coefficients of the terms $g\bar{g}$ and $s\bar{s}$. We can truncate at each step of the computation, excluding any term of order greater than 2, and any term of order 2 which is not $g\bar{g}$ or $s\bar{s}$. Thus

$$A_c=ae^{i\hat{u}}+\varphi\qquad\text{with}\qquad\varphi=-a(s\bar{s}+g\bar{g})e^{i\hat{u}}-2ag+\cdots,$$
$$A_z=-a(\bar{s}e^{i\hat{u}}+se^{-i\hat{u}}+\cdots)\qquad\text{and}\qquad l_u=1-\bar{g}e^{i\hat{u}}-ge^{-i\hat{u}}+\cdots.$$

Now

$$\|\vec{\mathrm{AB}}\|^2=(A_c-B_c)(\bar{A}_c-\bar{B}_c)+(A_z-B_z)^2=(J+\varphi)(\bar{J}+\bar{\varphi})+(A_z-B_z)^2=K+\epsilon,$$

with $J=ae^{i\hat{u}}-B_c$, $K=J\bar{J}+B_z^2$ and $\epsilon=J\bar{\varphi}+\bar{J}\varphi+\varphi\bar{\varphi}-2B_zA_z+A_z^2$. The expression to integrate in eccentric longitude $\hat{u}$ is

$$\|\vec{\mathrm{AB}}\|^{2\lambda}l_u=(K^\lambda+\lambda K^{\lambda-1}\epsilon+\frac{1}{2}\lambda(\lambda-1)K^{\lambda-2}\epsilon^2+\cdots)l_u.$$

We substitute ϵ and ϵ^2 using:

$$\begin{aligned}\epsilon=&-a(s\bar{s}+g\bar{g})(Je^{-i\hat{u}}+\bar{J}e^{i\hat{u}})-2a(\bar{g}J+g\bar{J})\\&+4a^2g\bar{g}+2B_z(se^{-i\hat{u}}+\bar{s}e^{i\hat{u}})+2a^2s\bar{s}+\cdots\\\epsilon^2=&\,8a^2J\bar{J}g\bar{g}+8a^2B_z^2s\bar{s}+\cdots\end{aligned}$$

We must now make the sum of the respective coefficients of $g\bar{g}$ and $s\bar{s}$. The contribution of the ϵ^2 term is

$$\lambda(\lambda-1)K^{\lambda-2}(4a^2J\bar{J}+4a^2B_z^2)=4a^2\lambda(\lambda-1)K^{\lambda-1}.$$

The contribution of the ϵ term

$$2a\lambda K^{\lambda-1}(-Je^{-i\hat{u}}-\bar{J}e^{i\hat{u}}+3a+Je^{-i\hat{u}}+\bar{J}e^{i\hat{u}})=6a^2\lambda K^{\lambda-1}.$$

The last two terms in the parenthesis come from the expansion of l_u. The sum is finally $2a^2\lambda(2\lambda+1)K^{\lambda-1}$. This is the formula in the lemma, before averaging on $\hat{u}$, and before the required division by 4 due to $(w_1,w_2)=(2g,2s)$.

An equivalent formulation. Let a be a non negative real number. Let $\mathcal{E}(\vec{\xi},\vec{\eta})$ be the Keplerian ellipse in 3-space with focus at the origin O, semi major axis a and Souriau variables $\vec{\xi}$ and $\vec{\eta}$. Let B be a point such that $\|\vec{\mathrm{OB}}\| \neq a$. We consider the time average

$$\begin{array}{ccc} \mathcal{S}\times\mathcal{S} & \longrightarrow & \mathbb{R}, \\ (\vec{\xi},\vec{\eta}) & \longmapsto & D_\lambda = \dfrac{1}{T}\displaystyle\int_0^T \|\vec{\mathrm{AB}}\|^{2\lambda} dt, \end{array}$$

where A is the position at the time t of a point moving on the orbit $\mathcal{E}(\vec{\xi},\vec{\eta})$, T is the period, and $\mathcal{S}$ is the unit sphere with center O (the function D_λ is the same as in the Lemma, but we now avoid the use of a reference frame.) Let Δ be the Laplace-Beltrami operator on $\mathcal{S}\times\mathcal{S}$. We have on the diagonal $\vec{\xi}=\vec{\eta}$ (corresponding to circular orbits) the identity

$$\Delta D_\lambda = \lambda(2\lambda+1)a^2 D_{\lambda-1}.$$

Use of the Lemma. The most remarkable fact is the cancellation of the second member when $\lambda=-1/2$, i.e. when the averaged function is the newtonian potential corresponding to the interaction between A and B. This result is significant in the dynamics of the *first secular system*, that we wish to construct briefly now.

3. Secular dynamics

The *first secular system* is obtained from the Newtonian equations of motion after an *averaging process* and a *linearization*. The averaging process starts with the choice of an *uncoupled system* and then defines an *averaged system*.

Uncoupled system. It is a differential system sufficiently close to the system of Newtonian equations and whose solutions can be completely described as sums of *uncoupled Keplerian motions*. The freedom in the choice is mainly to decide which are the vectors whose motions are Keplerian motions. We make here the "barycentric" choice, equally good in the case of an "Earth, Moon, Sun" type system and in a "Sun+two planets" system, but which leads to difficulties in the case "Sun+n planets". It is related to what is called "Jacobi reduction of the center of mass".

Let $\vec{r}_0$, $\vec{r}_1$, $\vec{r}_2$ be the position vectors of the three bodies with masses m_0, m_1, m_2. We always suppose that

$$m_0\vec{r}_0 + m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{7}$$

for all time. Thus the data of two of the position vectors $\vec{r}_i$ and their first derivative $\dot{\vec{r}}_i = \vec{v}_i$ with respect to time is sufficient to describe the state of the system. The system of Newtonian equations is

$$\ddot{\vec{r}}_i = \sum_{k\neq i} m_k \frac{\vec{r}_k - \vec{r}_i}{\|\vec{r}_k - \vec{r}_i\|^3}. \tag{N}$$

The uncoupled system is defined introducing the center of mass J of the Earth-Moon system. We denote by $\vec{r}_\mathrm{J}$ the vector satisfying $(m_0+m_1)\vec{r}_\mathrm{J} = m_0\vec{r}_0 + m_1\vec{r}_1$. The two vectors $\vec{r}_{01} =$

$\vec{r}_1 - \vec{r}_0$, $\vec{r}_{J2} = \vec{r}_2 - \vec{r}_J$, together with the constraint (7), are sufficient to recover $\vec{r}_0$, $\vec{r}_1$ and $\vec{r}_2$. The uncoupled system is chosen to be

$$\ddot{\vec{r}}_{01} = -(m_0 + m_1)\frac{\vec{r}_{01}}{\|\vec{r}_{01}\|^3}, \qquad \ddot{\vec{r}}_{J2} = -(m_0 + m_1 + m_2)\frac{\vec{r}_{J2}}{\|\vec{r}_{J2}\|^3}. \tag{U}$$

This system appears as a limit when one writes $\ddot{\vec{r}}_{01}$ and $\ddot{\vec{r}}_{J2}$ in the Newtonian system, and neglect small terms using the assumption that $\|\vec{r}_{01}\|$ is much smaller than $\|\vec{r}_{J2}\|$.

Averaged system. We can give the state of the system giving $\vec{r}_{01}$, $\vec{r}_{J2}$, $\vec{v}_{01} = \dot{\vec{r}}_{01}$ and $\vec{v}_{J2} = \dot{\vec{r}}_{J2}$, but also, after some restriction in the domain, giving the elements $(a_1, w_1, w_2, \hat{l}_1)$ of the elliptic orbit described by the vector $\vec{r}_{01}$ in the uncoupled system, $(a_2, w_3, w_4, \hat{l}_2)$ of the elliptic orbit described by the vector $\vec{r}_{J2}$. This is just a change of variables, and the Newtonian system in this set of variables reads:

$$\dot{a}_i = A_i(a_1, a_2, w_1, \dots, w_4, \hat{l}_1, \hat{l}_2), \quad \dot{w}_i = W_i(a_1, \dots), \quad \dot{\hat{l}}_i = L_i(a_1, \dots).$$

The *averaged system* is a differential system defined by

$$\dot{a}_i = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} A_i d\hat{l}_1 d\hat{l}_2, \quad \dot{w}_i = \frac{1}{(2\pi)^2}\int W_i d\hat{l}_1 d\hat{l}_2, \quad \dot{\hat{l}}_i = \frac{1}{(2\pi)^2}\int L_i d\hat{l}_1 d\hat{l}_2.$$

The following characterization makes clear that the definition does not depend on the choice of the elliptic elements. It also makes clear that our definition of averaging coincides with Moser's definition in [Mos].

Proposition 1. Let X_1 and X_2 be the vector fields respectively associated to the differential systems

$$\dot{a}_1 = \cdots = \dot{w}_4 = \dot{\hat{l}}_2 = 0, \quad \dot{\hat{l}}_1 = 1; \quad \dot{a}_1 = \cdots = \dot{w}_4 = \dot{\hat{l}}_1 = 0, \quad \dot{\hat{l}}_2 = 1.$$

Let b be any function of the elements such that $\partial_{X_1} b$ and $\partial_{X_2} b$ are constant functions. Then if in the system (N)

$$\dot{b} = B(a_1, a_2, w_1, \dots, w_4, \hat{l}_1, \hat{l}_2), \tag{N_1}$$

in the averaged system

$$\dot{b} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} B d\hat{l}_1 d\hat{l}_2. \tag{A_1}$$

Proof. In the averaged system $(2\pi)^2\dot{b} = (\partial b/\partial a_1)\int A_1 + \cdots + (\partial b/\partial \hat{l}_2)\int L_2$. But the coefficients does not depend on the $\hat{l}_i$ by the hypothesis, so they can pass under the symbol of integration. But $B = (\partial b/\partial a_1)A_1 + \cdots + (\partial b/\partial \hat{l}_2)L_2$ by its definition.

Symplectic properties. System (N) is associated to the Lagrangian

$$\mathcal{L}_N = \frac{1}{2}K + U = \frac{1}{2}(m_0\|\vec{v}_0\|^2 + m_1\|\vec{v}_1\|^2 + m_2\|\vec{v}_2\|^2) + \frac{m_0 m_1}{\|\vec{r}_{01}\|} + \frac{m_0 m_2}{\|\vec{r}_{02}\|} + \frac{m_1 m_2}{\|\vec{r}_{12}\|},$$

Using (7) the vis viva K may be written

$$K = \frac{m_0 m_1}{m_0 + m_1} \|\vec{v}_{01}\|^2 + \frac{(m_0 + m_1) m_2}{m_0 + m_1 + m_2} \|\vec{v}_{\mathrm{J}2}\|^2.$$

System (U) is associated to any Lagrangian of the family

$$\alpha(\|\vec{v}_{01}\|^2 + \frac{m_0 + m_1}{\|\vec{r}_{01}\|}) + \beta(\|\vec{v}_{\mathrm{J}2}\|^2 + \frac{m_0 + m_1 + m_2}{\|\vec{r}_{\mathrm{J}2}\|}),$$

with α and β two non-zero numbers. It is an important property of the uncoupled system (U) that one can choose α and β such that the vis viva for (U) is K:

$$\mathcal{L}_U = \frac{1}{2} K + \frac{m_0 m_1}{\|\vec{r}_{01}\|} + \frac{(m_0 + m_1) m_2}{\|\vec{r}_{\mathrm{J}2}\|}.$$

We can make the classical identification of the tangent space to the cotangent space using the Legendre transform associated to K. There is now just one space, where are defined the energy functions or Hamiltonians

$$H_N = \frac{1}{2} K - \frac{m_0 m_1}{\|\vec{r}_{01}\|} - \frac{m_0 m_2}{\|\vec{r}_{02}\|} - \frac{m_1 m_2}{\|\vec{r}_{12}\|}, \qquad H_U = \frac{1}{2} K - \frac{m_0 m_1}{\|\vec{r}_{01}\|} - \frac{(m_0 + m_1) m_2}{\|\vec{r}_{\mathrm{J}2}\|}$$

and the symplectic form

$$\omega = \frac{m_0 m_1}{m_0 + m_1} d\vec{v}_{01} \wedge d\vec{r}_{01} + \frac{(m_0 + m_1) m_2}{m_0 + m_1 + m_2} d\vec{v}_{\mathrm{J}2} \wedge d\vec{r}_{\mathrm{J}2}, \tag{8}$$

using the abusive notation $d\vec{v} \wedge d\vec{r} = dv_x \wedge dr_x + dv_y \wedge dr_y + dv_z \wedge dr_z$, where (v_x, v_y, v_z) and (r_x, r_y, r_z) are coordinates for $\vec{v}$ and $\vec{r}$ in an orthonormal frame.

Proposition 2. Let $H_N(a_1, \ldots, w_4, \hat{l}_1, \hat{l}_2)$ be the Hamiltonian of (N) expressed in the Keplerian elements. The averaged system is a Hamiltonian system with Hamiltonian function

$$\bar{H} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} H_N(a_1, \ldots, w_4, \hat{l}_1, \hat{l}_2) d\hat{l}_1 d\hat{l}_2.$$

Proof. Using the Poisson bracket notations, equation (N_1) in Proposition 1 reads $\dot{b} = \{H_N, b\}$. We must prove that equation (A_1) takes the form $\dot{b} = \{\bar{H}, b\}$, i.e. that

$$\{\bar{H}, b\} = \frac{1}{(2\pi)^2} \int \{H_N, b\} d\hat{l}_1 d\hat{l}_2.$$

Let us denote by γ_i, $1 \le i \le 8$, the elements $a_1, \ldots, \hat{l}_2$. We have

$$\{H_N, b\} = \sum_{i,j} c_{ij} \frac{\partial H_N}{\partial \gamma_i} \frac{\partial b}{\partial \gamma_j},$$

where the c_{ij} are the coefficients of the Poisson form. We claim that these coefficients does not depend on the $\hat{l}_i$. This is the expression in coordinates of the fact that the Lie derivative of the Poisson form with respect to the vector fields X_1 and X_2 is zero (i.e. these vector fields are symplectic fields). We also know, from proposition 1, that the $\partial b/\partial\gamma_j$ does not depend on the $\hat{l}_i$. Thus we can put these quantities and the c_{ij} out of the symbol of integration. The end of the computation is easy.

Reduction of the averaged system. By its very (non hamiltonian) definition, the averaged system possesses "ignorable" variables, namely, the $\hat{l}_i$. In the Hamiltonian framework, the a_i appear as the associated integrals : we have $\dot{a}_i = \{\bar{H}, a_i\} = 0$ (Lagrange theorem). For each choice of the a_i, we can thus define a reduced space, the "secular space", endowed with a symplectic form. This form is very nice in Souriau spherical elements, and very simple in Poincaré's variables, but here we will content ourselves of its expression at the circular-non-inclined point. This is sufficient in for the present work.

Deduction of the equations for the first secular system. The first secular system is obtained from the reduced averaged system by linearization at the equilibrium point $w_1 = \cdots = w_4 = 0$, corresponding circular motions in the plane of reference. The quickest way to obtain its expression is to compute the symplectic form of the secular space at this point and the quadratic part of the Hamiltonian function.

The linearized symplectic form. Let us consider first the system

$$\ddot{\vec{r}} = -\mu\frac{\vec{r}}{\|\vec{r}\|^3}.$$

We call μ the *gravitational constant.* We want the expression of the symplectic form $\sigma = d\vec{v}\wedge d\vec{r}$ in the elements of the Keplerian motion. In complex notation, we write $\vec{r} = (r_x, r_y, r_z) = (r_c, r_z)$, with $r_c = r_x + ir_y$, and $\vec{v} = (v_c, v_z)$, and we get

$$2\sigma = dv_c \wedge d\bar{r}_c + d\bar{v}_c \wedge dr_c + 2dv_z \wedge dr_z.$$

The recipe giving $\vec{r}$ is at the beginning of the paper. To obtain $\vec{v}$, the additional formula $\nu^2a^3 = \mu$, where $\nu = dl/dt$ is the frequency, is required. We get $\vec{v} = \nu l_u^{-1} d\vec{r}/du$. Thus

$$(1+s\bar{s})(1+g\bar{g})\frac{l_u v_c}{i\nu a} = e^{i\hat{u}}(1-g^2e^{-2i\hat{u}}) + e^{-i\hat{u}}(1-\bar{g}^2e^{2i\hat{u}})s^2,$$
$$(1+s\bar{s})(1+g\bar{g})\frac{l_u v_z}{i\nu a} = -e^{i\hat{u}}(1-g^2e^{-2i\hat{u}})\bar{s} + e^{-i\hat{u}}(1-\bar{g}^2e^{2i\hat{u}})s.$$

We just need the symplectic form at the origin $s = g = 0$, so we can truncate all these formulas at the first order in $(g, s, \bar{g}, \bar{s})$. This gives (fixing $\hat{u}$)

$$dr_c = -2adg + \cdots, \quad dr_z = a(-e^{i\hat{u}}d\bar{s} - e^{-i\hat{u}}ds) + \cdots,$$

$$dv_c = i\nu a(dg + e^{2i\hat{u}}d\bar{g}) + \cdots, \quad dv_z = i\nu a(-e^{i\hat{u}}d\bar{s} + e^{-i\hat{u}}ds) + \cdots,$$
$$\sigma = i\sqrt{\mu a}(-2dg\wedge d\bar{g} - 2ds\wedge d\bar{s}) + \cdots.$$

We write now the expression (8) for the symplectic form ω. There are two uncoupled Keplerian motions in (U), corresponding to different values of μ. Thus

$$\omega = -2i\Lambda_1(dg_1 \wedge d\bar{g}_1 + ds_1 \wedge d\bar{s}_1) - 2i\Lambda_2(dg_2 \wedge d\bar{g}_2 + ds_2 \wedge d\bar{s}_2) + \cdots,$$

with

$$\Lambda_1 = \frac{m_0 m_1 \sqrt{a_1}}{\sqrt{m_0 + m_1}}, \qquad \Lambda_2 = \frac{(m_0 + m_1) m_2 \sqrt{a_2}}{\sqrt{m_0 + m_1 + m_2}}.$$

The perturbating function and the quadratic part of its average. We must take the average $\bar{H}$ of the hamiltonian H_N. An Hamiltonian function equivalent to $\bar{H}$ is obtained subtracting any function of a_1 and a_2, which are constant function in the reduced averaged system. The Hamiltonian H_U is such a function, so we can consider as well the average $\bar{H}_P$ of

$$H_P = H_N - H_U = -\frac{m_0 m_2}{\|\vec{r}_{02}\|} - \frac{m_1 m_2}{\|\vec{r}_{12}\|} + \frac{(m_0 + m_1) m_2}{\|\vec{r}_{J2}\|},$$

called the perturbating function. The last term gives a constant when we take its average in $\hat{l}_2$, so we can also omit it, and consider

$$V = -m_0 m_2 V_0 - m_1 m_2 V_1, \quad \text{with} \quad V_j = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\hat{l}_1 d\hat{l}_2}{\|\vec{r}_{j2}\|}.$$

It appears that we have to take the average of the inverse of the distance between two points moving on Keplerian ellipses. But our Lemma gives a property for such averages, more precisely a property of the quadratic part of the expansion at the neighborhood of the circular coplanar motions.

Equations for the first secular system. Here we consider $w = (w_1, w_2, w_3, w_4)$ as another notation for the elements (g_1, s_1, g_2, s_2) and $l = (l_1, l_2, l_3, l_4)$ as another notation for $(\Lambda_1, \Lambda_1, \Lambda_2, \Lambda_2)$. Taking into account some obvious symmetries we deduce that V_j (and consequently V) possesses an expansion of the form

$$V_j(w) = V_j(0) + \sum \frac{\partial^2 V_j}{\partial w_\mu \partial \bar{w}_\nu} w_\mu \bar{w}_\nu + \cdots$$

Hamilton's equations for the linearized symplectic form are

$$\dot{w}_k = -\frac{i}{2l_k} \frac{\partial V}{\partial \bar{w}_k}, \qquad \dot{\bar{w}}_k = \frac{i}{2l_k} \frac{\partial V}{\partial w_k}.$$

Let us introduce the diagonal matrix Q with diagonal l. We obtain the first secular system in the form

$$\dot{w} = \frac{1}{2i} Q^{-1} \cdot \bar{\partial}\partial V \cdot w, \quad \text{with} \quad \bar{\partial}\partial V = \left(\frac{\partial^2 V}{\partial \bar{w}_\mu \partial w_\nu}\Big|_{w=0} \right)_{\mu\nu}. \tag{9}$$

Proposition 3. The trace of the matrix $Q^{-1} \cdot \bar{\partial}\partial V$ is zero.

Proof. We have

$$\operatorname{tr}(Q^{-1}\cdot\bar\partial\partial V) = -\sum_{j=0}^{1} m_j m_2 \operatorname{tr}(Q^{-1}\cdot\bar\partial\partial V_j)$$

and

$$\operatorname{tr}(Q^{-1}\cdot\bar\partial\partial V_j) = \frac{1}{\Lambda_1}\Big(\frac{\partial^2 V_j}{\partial w_1\partial\bar w_1} + \frac{\partial^2 V_j}{\partial w_2\partial\bar w_2}\Big) + \frac{1}{\Lambda_2}\Big(\frac{\partial^2 V_j}{\partial w_3\partial\bar w_3} + \frac{\partial^2 V_j}{\partial w_4\partial\bar w_4}\Big).$$

Each of the parenthesis is zero. To see this, note that V_j is obtained by a double integration, in $\hat l_1$ and $\hat l_2$, of the function $\|\vec r_{j2}\|^{-1}$. The Lemma shows that fixing $\hat l_2$, the Laplacian of the average in $\hat l_1$ of the same function is zero. Integrating this in $\hat l_2$, we obtain the vanishing of the first parenthesis. For the second, we simply exchange the roles of $\hat l_1$ and $\hat l_2$.

Proposition 4. System (9) can be diagonalized with spectrum $i\lambda_1,\dots,i\lambda_4$. The λ_j are real numbers, satisfying Herman's relation $\lambda_1+\cdots+\lambda_4=0$. One of them is zero.

Remark. M. Herman claimed that no other such relation exists between the λ_j.

Proof. What remains to be proven is quite standard. To prove the first claim, one uses the fact that there exists a unitary transformation U of $\mathbb{C}^4$, endowed with the Hermitian form $\sum l_j w_j\bar w_j$, putting the auto-adjoint matrix $\bar\partial\partial V$ in a diagonal form, with real entries. In equations, this reads ${}^t\bar U Q U = Q$, $\bar\partial\partial V = {}^t\bar U q U$, where q is a diagonal matrix with real entries. If we set $x = Uw$, system (9) takes the diagonal form $\dot x = \frac{1}{2i}Q^{-1}q\cdot x$. The trace is preserved by these transformations. This is the second claim. The claim that one of the eigenvalues is zero is classical. It is due to the first integral of the angular momentum.

Identities at higher order. Let us consider the full expansion of V at the neighborhood of $w=(0,0,0,0)$:

$$V(w) = \sum_{m=0}^{\infty}\sum_{\substack{|j|=m\\|k|=m}} \frac{1}{j!k!}\frac{\partial^{2m}V}{\partial w^j\partial\bar w^k}\Big|_{w=0} w^j\bar w^k,$$

with the usual multi-index notation $j=(j_1,\dots,j_4)$, $|j|=j_1+\cdots+j_4$, $j! = j_1!j_2!j_3!j_4!$, $w^j = w_1^{j_1}\cdots w_4^{j_4}$. The Lemma gives

$$\frac{\partial^2 V}{\partial w_1\partial\bar w_1} + \frac{\partial^2 V}{\partial w_2\partial\bar w_2}\Big|_{w_1=w_2=0} = 0$$

for any value of w_3 and w_4. The proof of Herman's resonance used only this identity at the point $w_3=w_4=0$. It would have been sufficient to prove the Lemma for a point B in the horizontal plane, which is much easier. The full application of the identity gives an infinite number of relations between the coefficients of the expansion of V, such as

$$\frac{\partial^4 V}{\partial w_3\partial\bar w_4\partial w_1\partial\bar w_1} + \frac{\partial^4 V}{\partial w_3\partial\bar w_4\partial w_2\partial\bar w_2}\Big|_{w=0} = 0.$$

It is not clear for us what are the implications of these relations on the dynamics of the averaged system.

Remark on the case of n planets. As this work consists in giving properties of the averaged system, we felt necessary to give a clear status to this system. This is the aim of Propositions 1 and 2, and this is why we chose to restrict our exposition to the case of three bodies and a barycentric uncoupled system. We must insist that Herman's resonance also exists in the case of a system of n planets, and the proof does not present any new difficulty. What is a little bit annoying is that the averaged system may loose part of its intrinsic character in this case.

A reasonable choice in the case of n planets is to use an heliocentric uncoupled system, where all the planets describe Keplerian ellipses around the sun. But it appears that the kinetic energy for the uncoupled system and for the Newtonian system are not identical. Thus the tangent space to the configuration space is identified in two different ways to the cotangent space. The two spaces cannot be considered as the same space, and we have to choose where live the systems we want to study and compare. If our choice is to work in the tangent space, we have to pull back the canonical symplectic form from the cotangent by the Legendre transform, and as we have two Legendre transforms, we have two symplectic forms. This complicates the symplectic properties of the averaged system.

If our choice is to work on the cotangent, everything works perfectly well, but we have a bad surprise, already noticed by [Po3] or [Las], p. 70. To an initial condition of the Newtonian system, we want to associate the so-called "osculating elements", which are the elements of the Keplerian orbits in the uncoupled system with *same initial condition*. Here the initial condition is a covector, not a tangent vector: the three bodies will not start with the same velocities in the Newtonian system and in the uncoupled system. In brief, we gave us a new freedom in the definition of the averaged system: the use of a non-standard relation of osculation.

In a system of n planets, if we choose to define the averaged system using an heliocentric uncoupled system, and to work on the cotangent of the configuration space, one can check that Herman's resonance is still there (apparently Herman made precisely this computation). The sum of the $2n$ eigenvalues of the linearized averaged system is zero.

Remarks on Herman's resonance. Maybe Herman's relation should be called a degeneracy rather than a resonance, as it is true for any value of the parameters. One can ask if its existence makes troubles when one constructs normal forms for the averaged system. The answer is no, according to [MRL] where the case of n planets is studied. Resonant terms do not appear.

Indeed there is a simple way to show that Herman's resonance has no effect on the "main" dynamics of a (averaged) planetary problem, i.e. the *reduced* dynamics, the dynamics after reduction by the symmetry group $SO(3)$. Let us form the Hamiltonian $H_\lambda = \bar{H} + \rho C_z$, where ρ is a real number and C_z is the vertical component of the total angular momentum. The reduced dynamics is the same for any value of ρ. But for different values of ρ the dynamics before reduction "rotates" differently in the absolute space. The spectrum $(\lambda_1, \ldots, \lambda_{2n})$ of the first secular system is changed in $(\lambda_1 + \rho, \ldots, \lambda_{2n} + \rho)$ and Herman relation disappears.

The degeneracy of the Kepler problem, among the family of fixed center problems with the potential $\|\vec{r}\|^\alpha$, is of the same nature. The reduced dynamics of this family of problems is not particular for the case $\alpha = -1$ (Kepler). But the dynamics before reduction is resonant only for this value and for $\alpha = 2$.

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