Generalized Euler's Configurations and Kushnirenko Problem

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Abstract. According to Euler, there is exactly one collinear configuration of three particles of given masses in a given order, that allows a motion of relative equilibrium in the 3-body problem. In Proposition 3 we extend this result to more general homogeneous laws of force. In Propositions 1 and 2 the homogeneous law of force is arbitrary and the masses are arbitrary real numbers; the maximum number of configurations is 3.

Introduction. Consider a system of n equations in n real unknowns, with real parameters. The description of the solution may be complicated. Even the number of solutions may vary in a complicated way with the parameters. Sometimes, one can give the minimum number or the maximum number of solutions. Very few tools are available if we look for the maximum number, or for an upper bound. It is even difficult to determine the conditions for the finiteness. We consider here the equations for central configurations in Celestial Mechanics. Some basic questions about the set of central configurations have remained without answer for several decades (see [Alb]). An interesting attempt is to put our equations in the form of a system of "fewnomials". It is then natural the study together all the force laws in r^b, where r is the distance and b a real parameter. We study in this paper the simplest particular cases according to this point of view.

1. A result on generalized Euler's configurations

1.1. Definitions and Equations. We consider the "collinear configuration" $(x_1, \dots, x_n) \in \mathbb{R}^n$. The "particle" i has the abscissa x_i and the "mass" m_i. The "attraction" γ_i exerted on the particle i by the remaining particles is

$$\gamma_i = \sum_{k \neq i} m_k \rho(x_{ki}), \qquad x_{ki} = x_i - x_k, \qquad \rho(x) = x|x|^{b-1}. \tag{1}$$

We take $(m_1, \dots, m_n, b) \in \mathbb{R}^{n+1}$. In the case $b = -2$, $x_{ij} \neq 0$ for any $i \neq j$, and $m_i > 0$ for any i, the number $-\gamma_i$ is the Newtonian acceleration $\ddot{x}_i$ of the particle i due to the gravitational interaction between the particles. In the case $b = -1$, $x_{ij} \neq 0$ for any $i \neq j$, there is another physical interpretation: we consider the n collinear particles as Helmholtz' vortices in the Euclidean plane, with vorticities $m_i \in \mathbb{R}$. Formula (1) defines now the oriented measure of the velocity of the particle i, which is a vector orthogonal to the line.

Central configurations. Moulton configurations. Euler configurations. The collinear central configurations are, by definition, the collinear configurations $(x_1, \dots, x_n)$ such that there exists a $\lambda \in \mathbb{R}$ with $\gamma_{ij} = \lambda x_{ij}$ for any $i \neq j$ (denoting $\gamma_{ij} = \gamma_j - \gamma_i$). They are also called *Moulton configurations*, and in the case $n = 3$, *Euler configurations*. This terminology comes from Celestial Mechanics. In the n-body problem, with Newtonian attraction, if a motion is homothetic or of relative equilibrium, with a collinear configuration, this configuration is central. In the Helmholtz problem, where $b = -1$, a collinear central configuration of vortices has a motion of relative equilibrium: during the motion, the distances between particles remain constant.

The conditions for central configuration express that the n-uple $(x_1,\dots,x_n)$ and the n-uple $(\gamma_1,\dots\gamma_n)$ are equal up to a translation and a change of scale. This may be written

$$\begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ \gamma_i & \gamma_j & \gamma_k \end{vmatrix} = 0, \qquad \text{for any} \quad 1 \le i < j < k \le n. \tag{2}$$

We expand this formula in the masses. The term in m_i is

$$m_i \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ 0 & \rho(x_{ij}) & \rho(x_{ik}) \end{vmatrix} = m_i\bigl(x_{ij}\rho(x_{ik}) - x_{ik}\rho(x_{ij})\bigr). \tag{3}$$

This suggests an interesting way to write (2):

$$\begin{vmatrix} m_i & m_j & m_k \\ x_{jk} & x_{ki} & x_{ij} \\ \rho(x_{jk}) & \rho(x_{ki}) & \rho(x_{ij}) \end{vmatrix} + \sum_{l\neq i,j,k} m_l \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ \rho(x_{li}) & \rho(x_{lj}) & \rho(x_{lk}) \end{vmatrix} = 0. \tag{4}$$

1.2. Euler configurations. It is the case $n = 3$. System (4) reduces to an equation for Euler configurations similar to that in [Win], §358:

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ x_{23} & x_{31} & x_{12} \\ \rho(x_{23}) & \rho(x_{31}) & \rho(x_{12}) \end{vmatrix} = 0. \tag{5}$$

We want to describe the set of Euler configurations. Our domain will be the cell in the space of configurations defined by the inequalities $x_1 < x_2 < x_3$. If we normalize the configuration putting $x_1 = 0$ and $x_2 = 1$, this cell is the interval $]0,+\infty[$ with variable $s = x_{23}$. We have to find the zeros of the function

$$g(s) = \begin{vmatrix} m_1 & m_2 & m_3 \\ s & -(1+s) & 1 \\ s^b & -(1+s)^b & 1 \end{vmatrix}. \tag{6}$$

Definition 1. We denote by $\mathcal{E}$ the number of normalized Euler configurations in the cell $x_1 < x_2 < x_3$, i.e. the number of roots of the function g with $s \in]0,+\infty[$.
Proposition 1. Let m_1, m_2, m_3 be arbitrary real masses. The number $\mathcal{E}$ of Euler configurations is infinite if and only if g vanishes identically on $]0,+\infty[$. This happens only in the following cases: i) $m_1 = m_2 = m_3 = 0$, ii) $b = 0$ and $m_1 = -m_2 = m_3$, iii) $b = 1$, iv) $b = 2$, $m_2 = 0$ and $m_1 = m_3$, v) $b = 3$ and $m_1 = m_2 = m_3$.
Proposition 2. For any $(m_1, m_2, m_3, b) \in \mathrm{I\!R}^4$, except those listed in Proposition 1, $\mathcal{E} \le 3$.
Proof. We put g in a convenient form and compute its second derivative

$$g(s) = (m_2+m_3)s^b + (m_1+m_3)(1+s)^b + m_3\bigl(s^{b+1} - (1+s)^{b+1}\bigr) - m_1(1+s) - m_2 s, \tag{a}$$

$$\frac{g''(s)}{b(b-1)} = (m_2+m_3)s^{b-2} + (m_1+m_3)(1+s)^{b-2} + m_3\frac{b+1}{b-1}\bigl(s^{b-1} - (1+s)^{b-1}\bigr). \tag{b}$$

We make a substitution in g'' putting $s = -1 + 1/x$, $x \in]0, 1[$. The resulting function is $k(x) = b^{-1}(b-1)^{-1}g''(-1+1/x)$. We write below this expression multiplied by x^{b-1}, and its second derivative in x.

$$x^{b-1}k(x) = (m_2 + m_3)x(1-x)^{b-2} + (m_1 + m_3)x + m_3\frac{b+1}{b-1}\big((1-x)^{b-1} - 1\big). \qquad (c)$$

$$\frac{\big(x^{b-1}k(x)\big)''}{b-2} = (1-x)^{b-4}\Big((b-3)(m_2+m_3)x - \big(2m_2 + (1-b)m_3\big)(1-x)\Big). \qquad (d)$$

The main observation is that the expression (d) possesses at most one root in the interval $x \in]0, 1[$, except if it is identically zero. By successive applications of Rolle's theorem, this implies that expression (c) possesses at most three roots in the closed interval $x \in [0, 1]$, except if it is identically zero. We observe that (c) vanishes at $x = 0$. Thus (c) vanishes at most twice in the open interval $u \in]0, 1[$, except if it is identically zero.

We deduce that expression (b) vanishes at most twice in the interval $s \in]0, +\infty[$, except if it vanishes identically. Thus expression (a) vanishes *at most four times* in the closed interval $[0, +\infty]$, except if it vanishes identically.

Let us look at this more carefully, first giving all the cases where (a) vanishes identically. We must consider separately the cases $b = 0$, $b = 1$ and $b = 2$, because of the denominators in our formulas. If $b = 0$, $g(s) = m_2 + m_3 - s(m_1 + m_2)$, which is identically zero iff $m_1 = -m_2 = m_3$. If $b = 1$, $g(s) = 0$. If $b = 2$, $g(s) = (m_1 + m_2 - m_3)s^2 + (m_1 - m_2 - m_3)s$, identically zero iff $m_1 = m_3$ and $m_2 = 0$. The other cases where g vanishes identically must be such that the affine factor in (d) vanishes at $x = 0$ and $x = 1$. A possibility is $b = 3$, and $2m_2 + (1-b)m_3 = 0$, i.e. $m_2 = m_3$.

At this point we use a useful trick: we exchange the "exterior" masses m_1 and m_3. The set of roots is simply reflected. We continue our argument knowing now that $(b-3)(m_1+m_2) = 0$ and $2m_2 + (1-b)m_1 = 0$. In the case $b = 3$ this gives $m_1 = m_2$. We check that these conditions (v) give $g = 0$.

If $b \neq 3$, then $m_2 + m_3 = 0$. We have $(b+1)m_3 = 0$ by (d), and $(b+1)m_1 = 0$ exchanging m_1 and m_3. If $b \neq -1$ we are in the trivial case (i). If $b = -1$ we also have $m_1 + m_2 = 0$. We check that g is *not* identically zero in this case.

The cases where g is identically zero are now listed and excluded. The function g has at most four roots with $s \in [0, +\infty]$ by the conclusion above. In the case $b > 0$, $g(0) = 0$ and we have at most three roots in the interval $s \in]0, +\infty[$, which is the conclusion required.

The last step is to exclude the possibility of four roots when $b < 0$. By the main observation the expression (d) should vanish exactly once. Let $p = m_2 + m_3$ and $q = 2m_2 + (1-b)m_3$. We have $pq < 0$. In particular $b = -1$ or $m_2 = 0$ are excluded. Let $\hat{p} = m_2 + m_1$ and $\hat{q} = 2m_2 + (1-b)m_1$, obtained exchanging m_1 and m_3. We have $\hat{p}\hat{q} < 0$. But $(1-b)p - q = (1-b)\hat{p} - \hat{q} = -(b+1)m_2$. This shows that p and $\hat{p}$ have the same sign.

Thus $m_2 + m_3$ and $m_1 + m_2$ are non zero and have the same sign, let us say positive. We look now at expression (a). In our case $b < 0$, it tends to $+\infty$ when $s \to 0$, and to $-\infty$ when $s \to +\infty$. But this is impossible, because g has exactly four roots, so it changes of sign four times (the four roots should be non-degenerate because degenerate roots would give other roots for g' and contradict the main argument). QED.

Remark. Cases with $\mathcal{E} = 3$ are easy to find, even with the symmetry condition $m_1 = m_3$.

2. Other results in the case of positive masses

2.1. Euler's result and some extensions. Euler gave the following result: if $m_i > 0$,

$i = 1, 2, 3$, and $b = -2$, then $\mathcal{E} = 1$. At §8 of [Eul], he computes the following polynomial

$$\begin{aligned}(1+s)^2 s^2 g(s) &= m_1 s^2\Big(1-(1+s)^3\Big)+m_2(1+s)^2(1-s^3)+m_3\Big((1+s)^3-s^3\Big)\\ &= -(m_1+m_2)s^5-(3m_1+2m_2)s^4-(3m_1+m_2)s^3+(m_2+3m_3)s^2+(2m_2+3m_3)s+m_2+m_3\end{aligned} \tag{7}$$

and writes "eumque unicum elici, cum unica signorum variatio occurat": the sequence of coefficients changes of sign exactly once; thus by Descartes' rule of signs there is exactly one positive root. The success of this argument is surprising: it is not often that Descartes' rule of signs gives an answer for all the required values of the parameters. Moreover, we can check that the argument works as well for the other negative integer values of b.

We will obtain below the same conclusion, $\mathcal{E} = 1$, for any $b < 1$ and any choice of positive masses. Elementary proofs of this are well known for $b < 0$. For $b > 0$, a main change occurs, that was explained to me by Carles Simó. We have $g(0) = 0$, and the collision configuration $x_1 < x_2 = x_3$ is a central configuration.

Proposition 3. Let m_1, m_2, m_3 be any positive masses. If $b < 1$ then $\mathcal{E} = 1$. If $1 < b \leq 2$ then $\mathcal{E} = 1$ if $\big(m_1 + m_2 - (b-1)m_3\big)\big(-(b-1)m_1 + m_2 + m_3\big) > 0$, and $\mathcal{E} = 0$ elsewhere.

Proof. We normalize differently the configuration putting $x_1 = 0$ and $x_3 = 1$. Our cell is the interval $]0, 1[$ with variable $x = x_{12}$. We compute

$$f(x) = \begin{vmatrix} m_1 & m_2 & m_3 \\ 1-x & -1 & x \\ (1-x)^b & -1 & x^b \end{vmatrix}, \quad f'(x) = \begin{vmatrix} m_1 & m_2 & m_3 \\ -1 & 0 & 1 \\ (1-x)^b & -1 & x^b \end{vmatrix} + b \begin{vmatrix} m_1 & m_2 & m_3 \\ 1-x & -1 & x \\ -(1-x)^{b-1} & 0 & x^{b-1} \end{vmatrix}.$$

We have to find the zeros of f. If $b \leq 0$, a surprising feature of the formula for f' is that the 8 terms of its brutal expansion are non negative. We have $f' > 0$; so f is strictly increasing in the interval $]0, 1[$, from $-\infty$ to $+\infty$. This gives the required result. Now $f^{(3)}(x) = b(b-1)C_3(x)$, with

$$C_3(x) = 3\begin{vmatrix} m_1 & m_2 & m_3 \\ -1 & 0 & 1 \\ (1-x)^{b-2} & 0 & x^{b-2} \end{vmatrix} + (b-2)\begin{vmatrix} m_1 & m_2 & m_3 \\ 1-x & -1 & x \\ -(1-x)^{b-3} & 0 & x^{b-3} \end{vmatrix}.$$

Again the complete expansion gives only positive terms, if $b < 2$. Thus $C_3(x) > 0$, if $b \leq 2$. We have at most three roots of f in the interval $[0, 1]$, under the hypotheses of Proposition 3. But if $b > 0$, $f(0) = f(1) = 0$. There is at most one root in the open interval. The discussion of the existence is easily done looking at the following table

$$\begin{aligned} 0 < b < 1: \quad & f'(x) + b(m_1+m_2)x^{b-1} \text{ is bounded as } x \to 0, \\ & f'(x) + b(m_2+m_3)(1-x)^{b-1} \text{ is bounded as } x \to 1, \\ b > 1: \quad & f'(0) = m_1 + m_2 - (b-1)m_3, \quad f'(1) = -(b-1)m_1 + m_2 + m_3. \text{ QED} \end{aligned}$$

2.2. Moulton's result. Moulton's result is an extension of Euler's result in the case of $n > 3$ particles on the line. The proof of Moulton [Mou] has been clarified by Smale and Shub in [Sma]. Saari [Saa] made a subsequent step, indicating that Moulton's uniqueness result was a mere application of the result: a convex function possesses at most one critical point. A technical simplification was recommended in [Yoc]: rather than the usual normalization by the moment of inertia I, one should fix the multiplier λ.

Definition 2. We denote by $\mathcal{M}$ the number of normalized Moulton configurations with $x_1 < x_2 < \ldots < x_n$ (an example of normalization is $x_1 = 0$, $x_n = 1$).

Proposition 4 (Moulton). If $m_i > 0$, $i = 1, \dots, n$, and if $b < 0$, then $\mathcal{M} = 1$.
Proof. Let $x > 0$ and $P_b(x) = x^{b+1}/(b+1)$, if $b \neq -1$, and $P_{-1}(x) = \ln x$. We have $P_b'(x) = x^b$. Let x_G is the center of mass of the n particles, well defined because $M = m_1 + \cdots + m_n \neq 0$. Let

$$V(x_1, \dots, x_n) = \sum_{1 \le i < j \le n} m_i m_j P_b(|x_{ij}|), \quad I(x_1, \dots, x_n) = \sum_{i=1}^{n} m_i (x_i - x_G)^2.$$

We have

$$m_i \gamma_i = \frac{\partial V}{\partial x_i}, \qquad 2m_i(x_i - x_G) = \frac{\partial I}{\partial x_i}.$$

Comparing these relations with the definition of a central configuration, we check that a critical point $(x_1, \dots, x_n)$ of the function $-2V + I$ is a central configuration with multiplier $\lambda = 1$. Reciprocally, if $(x_1, \dots, x_n)$ is a central configuration, the definition gives a λ such that $\gamma_{ij} = \lambda x_{ij}$. We use the identity $\sum m_i \gamma_i = 0$ to deduce $\gamma_i = \lambda(x_i - x_G)$. We use the identity $\sum m_i x_i \gamma_i = \sum_{i<j} m_i m_j |x_{ij}|^{b+1}$ to deduce that $\lambda > 0$. The normalized configuration

$$(\hat{x}_1, \dots, \hat{x}_n) = \lambda^{1/(1-b)}(x_1, \dots, x_n)$$

satisfies $\hat{\gamma}_i = \hat{x}_i - \hat{x}_G$, and thus is a critical point of the function $-2V + I$. We look for the normalized central configurations looking for the critical points of the function $-2V + I$. Using the so-called Leibniz identity, we find

$$-2V + I = \sum_{1 \le i < j \le n} m_i m_j \big(-2P_b(|x_{ij}|) + \frac{x_{ij}^2}{M}\big).$$

Each term of the sum is a convex function of $(x_1, \dots, x_n)$ if $b \le 0$. The sum is then a convex function, but we need a strictly convex function to conclude. The sum is constant if we translate the x_i's together; we must pass to the quotient space $\mathbb{R}^n/[(1, 1, \dots, 1)]$. In this space a critical point must be non-degenerate as shown by the following expression of the second derivative of $-2V + I$ on an arbitrary vector $(q_1, \dots, q_n) \in \mathbb{R}^n$:

$$\sum_{1 \le i < j \le n} 2m_i m_j \big(-b|x_{ij}|^{b-1} + \frac{1}{M}\big) q_{ij}^2.$$

This quadratic form is strictly positive if $(q_1, \dots, q_n)$ is not proportional to $(1, \dots, 1)$. This gives the uniqueness. The existence comes from the fact that $-2V + I$ is increasing when going to the border of the domain $0 = x_1 < x_2 < \cdots < x_n = 1$. QED

3. Discussion. How could we improve these results?

The techniques used in this paper to find upper bounds on the number of solutions of our system are very poor. In Propositions 1 to 3, the system is reduced to one equation in one unknown, and Rolle's theorem is applied several times. In Proposition 4, the convexity is used, and the upper bound is one. To go further, we would need techniques giving an upper bound greater than one in cases that cannot be simply reduced to the dimension one. It would help to prove the following fact, that is clear from the numerical experiments of [Sim].
Conjecture. There are at most four planar central configurations (defined up to isometry and rescaling) of four particles with given positive masses, such that the first particle is inside the triangle formed by the three remaining particles.

This arrangement of the particles is called the "non-convex" case. In the "convex" case, where no particle is inside the triangle of the others, the corresponding conjectured upper bound is one. In each of these questions, we should first prove the finiteness of the number of solutions, a famous open problem risen by Chazy, Wintner and Smale.

I consider Propositions 1 and 2 as a model in trying to solve these conjectures. The finiteness is easy to obtain when we have an efficient way to give upper bounds. Also the idea to work with a free exponent b instead of its value -2 seems to be efficient. If we fix $b=-2$ and reduce the system to a polynomial system, we find too complicated equations, as (7), which is not the best tool to prove Proposition 2. Another example is my work on the symmetric planar central configurations of equal masses [Al2]. I found the following polynomial system in two variables (z,u) and could prove, using standard techniques of computer algebra, it has only one root in the domain $u>0$, $z>0$. I guess that keeping free the exponent of the attraction, one could obtain this result in a simpler way and generalize it.

$$A(z)=0, \qquad (z^3-1)zC(z)+uB(z)=0,$$

$$A(z)=z^{37}-61\,z^{34}+336\,z^{33}-240\,z^{32}+2052\,z^{31}-12120\,z^{30}+8400\,z^{29}-30456\,z^{28}+175113\,z^{27}-88548\,z^{26}+241040\,z^{25}$$
$$-1364385\,z^{24}+338994\,z^{23}-1081984\,z^{22}+6241506\,z^{21}+642162\,z^{20}+2319507\,z^{19}-15790278\,z^{18}-12287376\,z^{17}$$
$$+1386909\,z^{16}+11212992\,z^{15}+55894536\,z^{14}-19889496\,z^{13}+53738964\,z^{12}-128353329\,z^{11}+44215308\,z^{10}-172452240$$
$$z^{9}+160917273\,z^{8}-42764598\,z^{7}+217615248\,z^{6}-115440795\,z^{5}+17124210\,z^{4}-139060395\,z^{3}+39858075\,(z^2+1)$$

$$B(z)=3\,z^{32}-2\,z^{30}-108\,z^{29}-60\,z^{28}+128\,z^{27}+1875\,z^{26}+2424\,z^{25}-3121\,z^{24}-22362\,z^{23}-35088\,z^{22}+44802\,z^{21}$$
$$+186900\,z^{20}+262764\,z^{19}-392367\,z^{18}-1066896\,z^{17}-1140663\,z^{16}+2171932\,z^{15}+4108782\,z^{14}+2897544\,z^{13}-7895660$$
$$z^{12}-10281168\,z^{11}-4046067\,z^{10}+18998496\,z^{9}+15345693\,z^{8}+2381886\,z^{7}-28348380\,z^{6}-11534238\,z^{5}+131220\,z^{4}$$
$$+22491108\,z^{3}+3247695\,z^{2}-472392\,z-7085880$$

$$C(z)=3\,z^{30}-21\,z^{28}-105\,z^{27}+146\,z^{26}+639\,z^{25}+1464\,z^{24}-3830\,z^{23}-8517\,z^{22}-9486\,z^{21}+40393\,z^{20}+67767\,z^{19}+30408$$
$$z^{18}-247009\,z^{17}-366192\,z^{16}-72252\,z^{15}+1084985\,z^{14}+1363620\,z^{13}+364053\,z^{12}-3596987\,z^{11}-3510702\,z^{10}-1332243$$
$$z^{9}+7705152\,z^{8}+6206706\,z^{7}+1637577\,z^{6}-8200278\,z^{5}-6567561\,z^{4}+85293\,z^{3}+2650644\,z^{2}+2657205\,z-472392$$

Kushnirenko problem. Consider a system of n equations in n unknowns $f_1(x_1,\dots,x_n)=0,\dots,f_n(x_1,\dots,x_n)=0$. The unknowns x_i are real and positive. The f_i are finite sums of "generalized monomials" of the form $ax_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, where in each monomial the α_i's are real numbers. Kushnirenko asked for an upper bound to the number of solutions of such a system. In contrast to Bézout's result, the upper bound should not depend on the magnitudes of the exponents α_i, but rather on the number of monomials used to write the system. The f_i are called "fewnomials".

The set of equations (4), with $1\le i<j<k\le n$, is of such form, except for the fact that it is overdetermined: there are $n(n-1)/2$ unknowns x_{ij}, $1\le i<j\le n$, and much more equations.

The Kushnirenko problem has a perfect solution in the case $n=1$. It is Laguerre's extension to Descartes' rule of signs. The upper bound is the number of monomials minus one (or better, the number of changes of signs in the ordered list of the coefficients of the monomials.)

One can obtain a nearly as good answer to the Kushnirenko problem in the case $n=2$, when one of the equations is a trinomial, i.e. consists in three generalized monomials. The trinomial may be put into the form $x_1+x_2=1$ by multiplications and ordinary changes of variables. Successively "killing monomials" by derivations after convenient multiplications, we can check that if N is the number of monomials of the second equation, 2^N-2 is an upper bound for the number of roots. This bound is probably not optimal, but is not bad in the case $N = 3$, where an example with 5 roots has been recently published by Haas.

Our Euler system is in this class. However, to obtain our optimal bound $\mathcal{E} = 3$, we had to apply the technique of "killing monomials" associated with other ideas. In this example it was not difficult to control the finiteness, and probably the same is true in examples of the same class, i.e. examples that can be explicitly reduced to the dimension one.

Khovanskii obtained remarkable results in the general case (and indeed for equations of a much more general type). He obtained upper bounds for the number of nondegenerate roots in the Kushnirenko problem, that could be applied to our system (4). His technique has in common with the elementary technique we just explained the successive "killing" of monomials. But we were not able to decide if his technique allows a control on the finiteness, similar to our Proposition 1. Also it is well-known that the upper bounds are far from being optimal. In the case above of a trinomial and a N-nomial, the theorem at p. 12 of [Kho] gives, counting as does Haas, the upper bound $3^{N+2}2^{(N+2)(N+1)/2}$. Note that in a similar theory with non-zero complex variables, due to Bernstein, the finiteness can be controlled, and the upper bounds are optimal (see [Moe]).

As a conclusion, we wish to recall that the problem of counting the real roots of an equation or of a system is very old. The introduction of [Lag] emphasizes some questions in the case of one polynomial equation. The theorem published by Sturm in 1835 answers most of them. For other cases general ideas may be found in the complete works of Hermite, vol. 1 and 3, Laguerre, vol. 1 or Picard, vol. 4, and in the references given by Khovanskii and Haas. The question of lower bounds made a considerable progress with the development of topology, but the question of upper bounds seems to be of another nature.

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Addendum (30/3/2006). The main result of this preprint (Proposition 2) is reproduced in the article: Alain Albouy, Yanning Fu, *Euler configurations and quasi-polynomial systems* (2006) http://arxiv.org/abs/math-ph/0603075
Abstract. In the Newtonian 3-body problem, for any choice of the three masses, there are exactly three Euler configurations (also known as the three Euler points). In Helmholtz' problem of 3 point vortices in the plane, there are at most three collinear relative equilibria. The “at most three” part is common to both statements, but the respective arguments for it are usually so different that one could think of a casual coincidence. By proving a statement on a quasi-polynomial system, we show that the “at most three” holds in a general context which includes both cases. We indicate some hard conjectures about the configurations of relative equilibrium and suggest they could be attacked within the quasi-polynomial framework.

Proposition 2 leaves the possibility of a total of 9 Euler configurations, 3 for each ordering of the particles. With Yanning Fu we arrived at the conclusion that 3 is the maximal total number if $b < 0$, which includes the Newtonian case $b = -2$ (with possibly negative masses) and the vortex case $b = -1$ (for which this result is trivial). For $0 < b < 1$, the maximal total number of Euler configurations is 5. We leave open the case $1 < b$.

Erratum (30/3/2006). The part of Proposition 3 which is not reproduced in the above article had a sign mistake in the previous version (13/8/2003), which is corrected here, thanks to Zheng Dong Li.