On a paper of Moeckel on Central Configurations

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17/10/2002

Abstract. This paper is devoted to *general* properties of the central configurations: we do not make any restriction on the number $n \geq 3$ of particles nor on the dimension $d \geq 1$ of the configuration. Part 6 considers however the particular case $n = d + 2$, of interest because the case $n = 4$, $d = 2$ is the first for which we cannot solve the equations for central configurations. Our main result is Proposition 6, which gives some estimates implying an important estimate due to [Moe]. Our main tool is Equation (10).

Introduction

In the Newtonian n-body problem, the simplest possible motions are such that the configuration is constant up to rotation and scaling, and that each body describes a Keplerian orbit. Only some special configurations of point particles are allowed in such motions. A. Wintner called them "Central Configurations".

A central configuration appears for the first time in a work of Euler, at the occasion of a strange claim: if the moon was four times farther from the earth, its motion could be such that it would appear to us as an eternal full moon. Euler argued that this motion would be stable[1], and on this point was wrong. Liouville [Lio] proved much later the instability, and doing this put an end to some irritating philosophical speculations about the perfection of the universe [Lüt].

Several aspects of the n-body problem motivate the study of central configurations. The introduction of [ALl] gives a partial review on this matter[2]. Many questions were raised about the set of central configurations. The main *general* open problem is: n positive masses $m_1, \ldots, m_n$ being given, is there a finite number of central configurations? We mean of course "relative configurations", i.e. configurations defined up to isometry and rescaling. J. Chazy confessed his inability to solve this problem[3], as did A. Wintner[4] later.

[1] [Eul] §10: "Sin autem motus impressus tantillum ab hac lege discrepet, non quidem perpetuo Soli, vel coniunctum, vel oppositum, maneret, sed exiguas excursiones hinc inde quasi oscillando conficeret"

[2] Please note that we attributed there erroneously to P. Pedersen the result of [Lin], p. 239

[3] Stated in [Cha] p. 327: "*à tous les chocs possibles des n corps correspondent un nombre fini de figures-limites de formes distinctes*"

[4] [Win] §365: "In §360 it appeared to be a reasonable conjecture that such is never the case, i.e. that the integer $q(n; m_1, \cdots, m_n)$ defined at the beginning of §360 always exists. But no proof is known for the truth of this hypothesis."

It is interesting to note that both Chazy and Wintner were considering the question of the "total collapse", and thus did not restrict the conjecture to the planar case. Both guessed that the conjecture is true, but both also believed in a stronger statement, respectively that any central configuration is non-degenerate, and that the number of planar central configuration with n given masses possesses a bound independent of n, statements which are now known to be false. Palmore [Pal] gave a simple example of a degenerate central configuration and lower bounds on the number of planar central configurations were obtained, on the impulsion of S. Smale, using Morse theory (or Ljusternik-Schnirelmann theory, see [McC].) These topological approaches, as well as the method of [Xia], show that the number of central configurations increases dramatically with n.

S. Smale frequently insists on the Chazy-Wintner conjecture[5], which is open already for $n = 4$, except in the case $m_1 = m_2 = m_3 = m_4$ (see [Al2].) Recent advances on it were published in [Rob] and in [Mo3]. Numerical studies [Sim] also suggest some very simple *particular* statements about the case $n = 4$, but proofs are still missing, mainly for what concerns upper bounds (see however [Ham], [Lea], [LoS].)

The present work was inspired by the famous "On Central Configurations" by R. Moeckel. It is not devoted to the enumeration of central configurations, but rather to their general properties, in particular to the restrictions on their shape. Some previous work convinced us that this topic may help in the question of the enumeration. We insist on the strength of very elementary methods, and present in Prop. 6 simple inequalities, that take a still nicer aspect in the "Dziobek case" $d = n-2$. We expose the theory of Dziobek configurations more simply and more completely than in any published paper.

Acknowledgements. I wish to thank Alan Almeida Santos, Dominique Bang, Alain Chenciner, Rick Moeckel and David Sauzin for helpful remarks.

1. Some remarks on the equations of the n-body problem

Let $q_1, \dots, q_n$ represent the positions at time t of n particles with respective masses $m_1, \dots, m_n$. Let

$$\gamma_i = \sum_{k \neq i} m_k S_{ik}(q_i - q_k), \qquad S_{ik} = S_{ki} = \|q_i - q_k\|^{2a}, \qquad a = -3/2. \tag{1}$$

The above value of a plays a fundamental role in questions of dynamics, but not in the questions of statics we wish to study in this paper. We will consider other possible values of a. We consider $a < 0$ as the "standard hypothesis" and always assume that the q_i's occupy n distinct positions. The hypothesis $a > 0$ barely changes some results, but turns less natural the exclusion of the configurations such that $q_i = q_j$ for some (i, j).

[5] See [Sm3]. This short account also discusses the question of the integral manifolds. For this matter, I suggest that the reader should also consider the statement in the introduction of [ALl]. Smale claims that Birkhoff's problem for n bodies is *solved* in the planar case, and *open* in the 3D case. One can also consider that both problems are *unsolved* because in general we don't know the set of central configurations with given masses. I claim that I "solved" the 3D case in [Alb] "as much" as Smale solved the planar case in [Sma].

The motion of the system of particles is described by the so-called Newton's equations

$$\frac{d^2 q_i}{dt^2} + \gamma_i = 0. \tag{2}$$

Let q_O be the position (at time t) of an arbitrary point. The fundamental properties of the γ_i's are

$$\text{i)} \quad \sum_i m_i \gamma_i = 0, \qquad \text{ii)} \quad \sum_i m_i (q_i - q_O) \wedge \gamma_i = 0. \tag{3}$$

The introduction of an arbitrary point q_O to write the last formula is a question of style. What we denote by q_i is a point, while $q_i - q_O$ is a vector. Rather than hiding arbitrary choices in identifications between points and vectors, we make them explicit. The wedge product is between two elements of a vector space E. The natural E is Euclidean and of dimension 3, but a surprising rule has been successful in simplifying the proofs of our results: do not insist in taking into account these specifications. Thus, the two following well-known relations

$$\text{iii)} \quad W = \sum_i m_i (q_i - q_O) \cdot \gamma_i, \qquad \text{iv)} \quad m_i \gamma_i = \frac{\partial V}{\partial q_i},$$

with

$$W = \sum_{i<j} m_i m_j \|q_i - q_j\|^{2a+2},$$

$$V = \frac{1}{2a+2} W \quad \text{if} \quad a \neq -1, \qquad V = \sum_{i<j} m_i m_j \log \|q_i - q_j\| \quad \text{if} \quad a = -1,$$

which involve strongly the Euclidean structure, are of lesser importance for us and will not be used in this work.

To finish this section we recall an important fact that has often been neglected (see however [Win], §356 and [AlC].) A matrix $Z = (Z_{ij})_{ij}$, that we will study in the last section, is involved in the expression of Newton's equations. Putting

$$Z_{ki} = -m_k S_{ki} \quad \text{if} \quad i \neq k, \qquad Z_{ii} = \sum_{k \neq i} m_k S_{ki} \tag{4}$$

and introducing again an arbitrary point q_O, Equation (1) takes the form

$$\gamma_i = \sum_{k=1}^{n} Z_{ki} (q_k - q_O). \tag{5}$$

Indeed, one can go further (compare [AlM]) and observe that a 3-tensor is involved in the equations. The linearity in the masses of the Z_{ij}'s may be expressed introducing the tensor ζ_{hij} such that $Z_{ij} = \sum_h m_h \zeta_{hij}$. Its components are the following: $\zeta_{hij} = 0$ if $h \neq i$ and $j \neq i$; $\zeta_{hhj} = -S_{hj}$ if $h \neq j$; $\zeta_{hjj} = S_{hj}$ if $h \neq j$; $\zeta_{iii} = 0$. The tensor appears to be antisymmetric in the first and third indices. This remark is linked with nice properties of some "inverse problems", when the question is to determine the masses compatible with some particular solution of the n-body problem (see for example [AlM].) It seems to be useless in problems where the masses are fixed.

2. Central configurations

Following [AlM], it is easy to extend the usual definition of a central configuration as suggested by the consideration of a "brother problem", the problem of the motion of point vortices (or Helmholtz vortices) in the plane. To write the differential equations for this problem, consider the q_i and the γ_i as complex numbers, and make $a = -1$ in Equation (1). The system is $\dot{q}_i = \sqrt{-1}\,\gamma_i$. A new feature is that negative m_i's are now allowed. Thus a good definition for a central configuration should not suppose that the m_i are positive, and should allow the singular case $M = m_1 + \cdots + m_n = 0$.

Definition 1. A configuration is central if there exists a vector γ_O, a point q_O, and a $\lambda \in \mathbb{R}$ such that for all i, $\gamma_i - \gamma_O = \lambda(q_i - q_O)$.

When the total mass $M = m_1 + \cdots + m_m$ is non-zero, we can define the center of mass q_G, characterized by the relation $\sum m_i(q_i - q_G) = 0$.

Proposition 1. Suppose $M = m_1 + \cdots + m_n \neq 0$. A configuration is central if and only if there exists a $\lambda \in \mathbb{R}$ such that $\gamma_i = \lambda(q_i - q_G)$ for all i.

Proof. Suppose $\lambda = 0$ in the definition. Using (3i) we get $M\gamma_O$=0. So $\gamma_i = \gamma_O = 0$ and $\gamma_i = \lambda(q_i - q_G)$ is true. Suppose now $\lambda \neq 0$. Making $q = q_O - \gamma_O/\lambda$, the equation of definition becomes $\gamma_i = \lambda(q_i - q)$. Finally, (3i) gives $q = q_G$.

Important remark concerning the case of zero masses. The naive way to treat the case of $n-p$ positive masses and p infinitesimal masses is to make $m_1 = \cdots = m_p = 0$ in Equation (2). This system will define the motion of the infinitesimal particles as well (but it is not hamiltonian is this case.) Definition 1 is compatible with the study of (2) in this particular case. So we call it the "naive definition" of a central configuration.

Another way (see for example [Mo2]) to define the central configurations with p infinitesimal particles is to introduce positive numbers $\mu_1, \ldots, \mu_p$, a small parameter ε, and to make $m_1 = \varepsilon\mu_1, \ldots, m_p = \varepsilon\mu_p$. The limiting segments of orbit, when $\varepsilon \to 0$, are not always solution of system (2) with p zero masses. The neighborhood of a binary collision between infinitesimal particles provides obvious counterexamples.

The distinction between both ways is important in the problem of central configurations. Consider the case of a big particle and two or three infinitesimal particles on a plane. With the naive definition, any configuration such that the infinitesimal particles lie on the same circle centered at the big one is a central configurations. But with the "other way", taking limits of central configurations with positive masses, this constraint is necessary, but not sufficient (see for example [ALl].) If there are two infinitesimal particles, they should either coincide at the limit, or arrive at opposite points on the circle, or finally form an equilateral triangle together with the big particle. In the case of three infinitesimal particles, the resulting central configurations depend on the values of the μ_i's. Hall [Hal] solved the case $\mu_1 = \mu_2 = \mu_3$.

3. The first central configurations

Proposition 2. If in (1) we set $S_{12} = S_{13} = \cdots = S_{n-1,n} = 1$, then the γ_i's are independent of i in the case $M = m_1 + \cdots + m_n = 0$, whereas $\gamma_i = M(q_i - q_G)$ in the case $M \neq 0$.

This elementary fact has two striking corollaries.

Corollary 1. If in (1) we set $a = 0$, any configuration is central, and system (2) is immediately integrated.

Corollary 2. For any value of a in Equation (1), the regular simplex is a central configuration.

Remark. The integration of the case $a = 0$ (harmonic oscillator) goes back to Proposition 64 of Newton's *Principia*. The case $n = 3$ in Corollary 2 is the equilateral triangle discovered by Lagrange [Lag]. The first known mention of a higher dimensional analogue of Lagrange configuration is a statement by Lehmann-Filhés [Leh] in the case $n = 4$.

Proposition 3. Let us take any value $a \neq 0$ in Equation (1). A configuration of n non-zero masses, of dimension exactly $n - 1$, is central if and only if it is the regular simplex.

Proof. In the case $M \neq 0$, Prop. 1 and 2 imply that the equation for central configurations is

$$0 = \sum_{k \neq i} m_k \check{S}_{ik}(q_k - q_i), \qquad \text{with} \qquad \check{S}_{ik} = S_{ik} - \frac{\lambda}{M}. \tag{6}$$

For any i the $n-1$ vectors $q_k - q_i$ are independent (it is the hypothesis on the dimension of the configuration.) As $m_k \neq 0$, we have $\check{S}_{ik} = 0$. Therefore, for all (i,k), $i < k$, $S_{ik} = \lambda/M$, and, as $S_{ik} = \|q_i - q_k\|^{2a}$, all the mutual distances are equal. In the case $M = 0$, the vector $Q = \sum_i m_i q_i$ cannot be zero, because $Q = 0$ implies that the configuration is of dimension at most $n - 2$. We can imagine that the center of mass Q/M is at infinity, in the direction of Q. Now (3i) implies $\lambda = 0$ in Definition 1, and (3ii) implies that $\gamma_O = \sigma Q$. As $Q = -\sum m_k(q_i - q_k)$, we can write the equation for central configurations in the form $0 = \sum_{k \neq i} m_k(S_{ik} - \sigma)(q_i - q_k)$, and conclude as we did in the general case that $M \neq 0$.

Remark. In the case where some of the masses are zero, the above Proposition is false when adopting what we called the "naive definition" of a central configuration. The argument just implies that the distances between big particles, and the distances between a big and an infinitesimal particle, are equal, but says nothing about the distance between the infinitesimal particles. A corollary of Prop. 3 is that Prop. 3 is still true in the case of zero masses if we adopt any reasonable "limit" definition.

Another proof of Proposition 3. We use the following computational trick (compare [Pa2] prop. 2 and [Pac] eq. 3.13.) We have

$$\gamma_i - \gamma_j = (m_i + m_j) S_{ij}(q_i - q_j) + \sum_{k \neq i,j} m_k \big(S_{ik}(q_i - q_k) - S_{jk}(q_j - q_k) \big). \tag{7}$$

Putting

$$\Sigma_{ij} = (m_i + m_j)S_{ij} + \frac{1}{2}\sum_{k\neq i,j} m_k(S_{ik} + S_{jk}), \tag{8}$$

we get

$$\gamma_i - \gamma_j = \Sigma_{ij}(q_i - q_j) - \sum_{k\neq i,j} m_k(S_{ik} - S_{jk})\big(q_k - \frac{1}{2}(q_i + q_j)\big). \tag{9}$$

Consider now a central configuration. We set $\check{\Sigma}_{ij} = \Sigma_{ij} - \lambda$ and we have

$$0 = \gamma_i - \gamma_j - \lambda(q_i - q_j) = \check{\Sigma}_{ij}(q_i - q_j) - \sum_{k\neq i,j} m_k(S_{ik} - S_{jk})\big(q_k - \frac{1}{2}(q_i + q_j)\big). \tag{10}$$

Suppose moreover that the dimension of the configuration is exactly $n-1$. The $n-1$ vectors $q_i - q_j$ and $q_k - (q_i + q_j)/2$, $k = 1, \dots, n$, $k \neq i$, $k \neq j$ are independent. All the coefficients in (10) must cancel out. This implies the equality of the S_{hk}, which gives Proposition 3 again.

Comment. The second proof is not much simpler, but it has several good features. First, the differences $S_{ik} - S_{jk}$ appear explicitly in (10). Also, we do not need to distinguish the case $M = 0$. Indeed, we use the definition of a central configuration rather than Proposition 1, as shown by the following obvious statement.

Proposition 4. A configuration is central with multiplier λ if and only if Eq. (10) is satisfied for all (i, j), $1 \leq i < j \leq n$.

The next statement illustrates the use Equation (10).

Proposition 5. Let us take any $a < 0$ in Eq. (1). A central configuration of n non-zero masses, of dimension exactly d, such that $n - b$ particles $q_{b+1}, \dots, q_n$ lie in an affine subspace π of dimension $d - b$, is such that all the particles in π are on a hypersphere $\mathcal{S} \subset \pi$. This is true for any value of the integers n, d, b, with the obvious restrictions $1 \leq d \leq n-1$, $1 \leq b \leq d$. Furthermore the center $q_{\mathcal{S}}$ of the sphere $\mathcal{S}$ forms with the particles $q_1, \dots, q_b$ a (non-degenerate) simplex, such that

$$\|q_1 - q_{\mathcal{S}}\| = \cdots = \|q_b - q_{\mathcal{S}}\| \leq \|q_1 - q_2\| = \|q_1 - q_3\| = \cdots = \|q_{b-1} - q_b\|.$$

A configuration with the same geometry but different values for the masses $m_1, \dots, m_b$ is still central. The sub-configuration formed by the particles in π is central.

Proof. Consider the projection in the direction of the subspace π. The projection of π is a point we call Q_0, while we denote by $(Q_1, \dots, Q_b)$ the image of $(q_1, \dots, q_b)$. The $b+1$-uple $(Q_0, \dots, Q_b)$ is a non-degenerate simplex by the hypothesis, so we can use the projection of Eq. (10) as we used Eq. (10) in Prop. 3. It is sufficient to consider the terms in the sum such that $1 \leq k \leq b$. Choose first $q_i \in \pi$ and $q_j \in \pi$. As $m_k \neq 0$, we get $S_{ik} = S_{jk}$ for $k = 1, \dots, b$. We call $S_{0k} = S_{ik} = S_{jk}$. Looking at the same terms with now $q_i \in \pi$ and $q_j \notin \pi$, we obtain $S_{0k} = S_{jk}$. The conclusion is that all the $S_{ij} = \|q_i - q_j\|^{2a}$, with $1 \leq i < j \leq n$ and $i \leq b$, are equal. Thus $q_{b+1}, \dots, q_n$ live on a sphere, and the b-subspace orthogonal to π and passing by the center q_S of the sphere contains $q_1, \dots, q_b$. The displayed equalities and inequalities follow easily. Note also that $\check{\Sigma}_{ij} = 0$ if $1 \le i < j \le b$, thus, calling $S_0 = S_{ij}$, we get $\lambda = MS_0$. We look again at system (10), and write $\check{\Sigma}_{ij} = (m_i + m_j)(S_{ij} - S_0) + \sum m_k(S_{ik} + S_{jk} - 2S_0)/2$. The masses m_k, $1 \le k \le b$, always appear in (10) multiplied by zero, so their values are indifferent. This gives the last assertions.

Remark. If its dimension $d - b$ is 0, the subspace π contains one particle. If $d - b = 1$, π contains two particles, because a hypersphere is made of two points. This gives a well-known result: a non-collinear central configuration with four particles, three of them being on a line, is impossible. The simplest central configurations of Proposition 5 which are not regular simplices are such that $d = 3$ and $\dim \pi = d - b = 2$. They were, according to G. Meyer, studied by E. Brehm in his thesis (Berlin, 1908; see also [Fay].) Four particles form a central configuration on a circle, for example a square of equal masses, or an isosceles trapezoid with $m_1 = m_2$ and $m_3 = m_4$. The fifth particle is on the orthogonal axis passing through the center.

4. The perpendicular bisector theorem and Eq. (10)

Laura-Andoyer equations. The Laura-Andoyer equations for central configurations are Equations (11) below, in the particular case of a planar configuration. They have been well known since their exposition in [Hag]. Hagihara followed essentially the work of G. Meyer, a disciple of Andoyer, who also presented a generalization to higher dimensions. Eq. (11) may be seen as a corollary of Eq. (10): let us take the wedge product of (7), (9) or (10) with $q_i - q_j$. We get

$$0 = \sum_{k \ne i,j} m_k (S_{ik} - S_{jk}) \Lambda_{ijk}, \qquad \text{where} \qquad \Lambda_{ijk} = (q_i - q_j) \wedge (q_i - q_k). \tag{11}$$

In the case of a planar configuration, the bivector Λ_{ijk} is simply the oriented area of the triangle (q_i, q_j, q_k). In the usual case where the m_i are positive, we can immediately deduce from these equations a qualitative consequence on the configuration.

Perpendicular bisector theorem. Let q_i and q_j be two points of a planar central configuration with positive masses. The line (q_i, q_j) —by convention the "horizontal" line— and the perpendicular bisector —the "vertical" line— divide the plane into four regions. *The other points cannot all pertain to the union of the upper left and the lower right regions. Similarly, they cannot all pertain to the union of the upper right and the lower left regions.*

This theorem is attributed to Conley. [Moe] developed it, associating to each configuration a certain angle θ, with the following properties: if $\theta = 0$, the configuration is collinear, and if $0 < \theta < 45°$, the configuration does not satisfy the perpendicular bisector theorem and consequently is not central. It is known (see [Ham]) that one can deduce from the theorem above that the Lagrange configuration is the unique non-collinear configuration of three bodies. However, our second proof of Prop. 3 suggests that Eq. (10) itself is a simpler and more powerful tool than its corollaries. The following section will confirm this.

5. Estimates implying Moeckel's one

Another consequence of (10) is obtained from the observation of the terms of the sum in k.

Let us adopt the "standard hypothesis" in Equation (1): all the m_i's are positive and $a < 0$. The space is divided into two regions by the perpendicular bisector hyperplane of q_i and q_j. We call them the region of q_i and the region of q_j. As S_{ik} is a decreasing function of $\|q_i - q_k\|$, the factor $S_{ik} - S_{jk}$ is positive if and only if q_k is in the region of q_i. Therefore we add vectors that are all in a half-space, and obtain a vectorial sum in the same half-space. The following statement follows.

Proposition 6. Consider any central configuration of n particles with masses $m_i > 0$. Consider the quantities Σ_{ij} defined by Eq. (8) and appearing in Eq. (10). For any (i, j), $1 \le i < j \le n$, we have $\lambda \le \Sigma_{ij}$. The equality holds if and only if all the points except q_i and q_j lie on the perpendicular bisector hyperplane of q_i and q_j.

This Proposition can also be easily deduced from the lemma in §4 of [Moe]. It is not stated explicitly by Moeckel, who rather emphasizes an important corollary: $\lambda \le \bar{\Sigma}$, where we denote by

$$\bar{\Sigma} = \frac{2}{n(n-1)} \sum_{i<j} \Sigma_{ij} = \frac{1}{n-1} \sum_{i<j} (m_i + m_j) S_{ij} = \frac{1}{n-1} \sum_{i=1}^{n} Z_{ii} \tag{12}$$

the arithmetic mean of the Σ_{ij}, $1 \le i < j \le n$. Proposition 6 itself appeared to be useful in [ALl], where in a particular case it helps to prove the symmetry of some central configurations. We conclude this section giving a simple formula for the mysterious quantities Σ_{ij}. Let $e^1 = (1, 0, \dots, 0), \dots, e^n = (0, \dots, 0, 1)$ be the standard base of $\mathbb{R}^n$. Let $e^{ij} = e^i - e^j$. A straightforward computation gives

$$\Sigma_{ij} = \frac{1}{2} \sum_{kl} e_k^{ij} Z_{kl} e_l^{ij} = \frac{1}{2}(Z_{ii} + Z_{jj} - Z_{ij} - Z_{ji}). \tag{13}$$

6. The particular case of Dziobek configurations

Proposition 7. Consider a configuration $q_1, \dots, q_n$ with non-zero masses $m_1, \dots, m_n$, and suppose the configuration is at most of dimension $n-2$, which can be expressed saying that there exists a non-zero n-uple $\Delta = (\Delta_1, \dots, \Delta_n) \in \mathbb{R}^n$ such that

$$\Delta_1 + \cdots + \Delta_n = 0, \qquad \Delta_1 q_1 + \cdots + \Delta_n q_n = 0. \tag{14}$$

Let us set $d_k = \Delta_k / m_k$ for each index k and substitute in Eq. (1) the definition of S_{ij} by $S_{ij} = d_i d_j$. Then $\gamma_i = 0$ for all index i.

The proof is a straightforward computation. We can now give one of the possible definitions of a Dziobek configuration.

Definition 2. A *Dziobek configuration* is a configuration of n particles, with non-zero masses, such that there exists a non-zero $\Delta \in \mathbb{R}^n$ satisfying (14) and such that for some $(\xi, \eta) \in \mathbb{R}^2$

$$S_{ij} = \|q_i - q_j\|^{2a} = \xi + \eta d_i d_j, \qquad \text{with} \qquad d_k = \Delta_k / m_k. \tag{15}$$

Such a configuration is of dimension at most $n-2$ and is central with multiplier $\lambda = M\xi$ by Propositions 2 and 7.

We remark that if the configuration is of dimension exactly $n-2$, the linear dependence relation (14) is unique and the n-uple Δ is well defined up to a factor. To interpret geometrically this n-uple, in the case $\Delta_1 \neq 0$, one can normalize it using the constraint $\Delta_1 = -1$. The point q_1 is then the barycenter of $q_2, \dots, q_n$ with respective weights $\Delta_2, \dots, \Delta_n$. Thus we can call the Δ_i the *barycentric weights* of the configuration. There is another geometrical interpretation of Δ_i due to Möbius: it is the conveniently oriented volume of the simplex defined by all the q_k, $k \neq i$. The free factor corresponds to the free choice of the unit of volume. In our problem, the Euclidean structure indeed defines a unit of volume, but in all the particular problems we have met up to now we found better to choose another unit.

Proposition 8. Suppose the n masses satisfy $m_i \neq 0$ for all i and $M \neq 0$. Then any central configuration of dimension exactly $n-2$ is a Dziobek configuration.

Proof. For any i the unique relation, up to a factor, between the $q_k - q_i$, $1 \leq k \leq n$, is

$$\sum_k \Delta_k (q_k - q_i) = 0.$$

But (6) is also such a relation. Thus $m_k \check{S}_{ik} = \mu_i \Delta_k$ for some $\mu_i \in \mathbb{R}$. By symmetry $m_i \check{S}_{ik} = \mu_k \Delta_i$ also holds. Thus $\mu_i d_k = \mu_k d_i$, so there exists a $\mu \in \mathbb{R}$ with $\mu_i = \mu d_i$ for all i. This gives $m_k \check{S}_{ik} = \mu d_i \Delta_k$, which is the result.

Remark. We do not know any Dziobek configuration of dimension $\leq n-3$ so we could as well define a Dziobek configuration as a central configuration of dimension $n-2$. But the case $M = 0$ would introduce a discrepancy, as shown by the example of a planar configuration formed by three particles of unit mass in an equilateral triangle with a fourth particle of mass -3 at the barycenter. It is a central configuration with multiplier $\lambda \neq 0$ and $M = 0$ so it does not satisfy $\lambda = M\xi$ and it is not a Dziobek configuration according to Definition 2.

Proposition 9. If in Eq. (1) we make the standard choices $a < 0$, $m_i > 0$, then $\eta < 0$ for any Dziobek configuration.

This important estimate can be established in several ways but it follows immediately from Moeckel's estimate $\lambda \leq \bar{\Sigma}$ and expression (16) below for the quantity $\bar{\Sigma}$, defined by Eq. (12).

Proposition 10. For any Dziobek configuration (Definition 2), we have

$$\Sigma_{ij} = \lambda - \frac{\eta}{2}(\Delta_i - \Delta_j)(d_i - d_j), \qquad \bar{\Sigma} = \lambda - \frac{\eta}{n-1} \sum_{i=1}^{n} \frac{\Delta_i^2}{m_i}. \tag{16}$$

Proof. By the identity $S_{ik} = \xi + \eta d_i d_k$ and (4) we get $Z_{ki} = -m_k \xi - \eta \Delta_k d_i$ and $Z_{ii} = (M - m_i)\xi - \eta \Delta_i d_i$. We know that $\lambda = \xi M$. Eq. (13) gives the first expression and (12) gives the second.

Proposition 11. The estimates of Proposition 6, for a Dziobek configuration with positive masses, are $(\Delta_i - \Delta_j)(d_i - d_j) \geq 0$.

This follows from Propositions 9 and 10. This estimates are interesting in the case of configurations with unequal masses. They constrain the configuration up to an affine transformation. They are strikingly simple and strong in the $1+n$-body problem [ALl]. Also, we should note that the Dziobek configurations with positive m_i's and $(\Delta_1-\Delta_2)(d_1-d_2)=0$ satisfy the case of equality in Proposition 6. Thus we have $\Delta_1=\Delta_2$ by symmetry and (15) gives $d_1=d_2$ and $m_1=m_2$. If we start with masses $m_1\neq m_2$, the conditions $\Delta_1=\Delta_2$ or $d_1=d_2$ are impossible.

Convex and non-convex configurations. A configuration $(q_1,\dots,q_n)$ is called convex if none of the points q_i is strictly inside the convex hull of the other points. If the configuration is of dimension exactly $n-2$ and satisfies (14) then it is convex if and only if at least two of the Δ_i's are positive or zero and at least two of the Δ_i's are negative or zero. In the other cases, by the "barycentric" interpretation of the Δ_i's, one of the q_i's is barycenter of the others with positive weights, and the configuration is non-convex.

Proposition 12. Take $a<0$ in Eq. (1). There does not exist a Dziobek configuration at the frontier between convex and non-convex configurations.

This result is a consequence of the fact that a cospherical configuration is convex and of the following statement.

Proposition 13. Take $a<0$ in Eq. (1). A Dziobek configuration $q_1,\dots,q_n$ with $\Delta_1=0$ is such that the sub-configuration $q_2,\dots,q_n$ is a cospherical Dziobek configuration.

Assuming that the configuration is of dimension exactly $n-2$, this is the particular case $b=1$, $d=n-2$ of Proposition 5. Without this restriction on the dimension, the statement is an obvious consequence of Definition 2. We finish this section mentioning an interesting fact of geometry.

Proposition 14. Consider a configuration $q_1,\dots,q_n$ of dimension exactly $n-2$, with barycentric weight vector Δ. The quantity $\Delta_1\|q_1-q\|^2+\cdots+\Delta_n\|q_n-q\|^2$ is independent of the choice of the point q. It is zero if and only if the q_i's are on a the same sphere.

To prove the second assertion, it is enough to choose q as the center of the hypersphere passing through $n-1$ of the points. If instead we choose $q=q_i$, $1\leq i\leq n$, the Proposition gives the relations between the Δ_i's and the squares of mutual distances used in [Al2].

7. The spectrum of Z

We obtained the previous results without even noticing that the multiplier λ in Definition 1 is an eigenvalue of the matrix Z. However, the consideration of Z as a symmetric operator is useful to get some intuition on the subject. We will thus conclude this paper reviewing some known properties of Z.

The first step is to get rid of the trivial vector $L=(L_i)_{1\leq i\leq n}=(1,\dots,1)$ in the kernel. We have $\sum_i L_iZ_{ij}=0$. This corresponds to the free choice of q_O in (5).

Definition 3. We call the *disposition space* the $n-1$-dimensional vector space $\mathcal{D}=\mathbb{R}^n/[L]$, quotient of $\mathbb{R}^n$ by the line generated by L. If a non-zero n-uple $(m_1,\dots,m_n)$ is given, we call

$\mathcal{D}_m$ the hyperplane of $\mathbb{R}^n$ of equation $m_1x_1 + \cdots + m_nx_n = 0$. Finally, we denote by π be the canonical projection $\pi : \mathcal{D}_m \to \mathcal{D}$.

The matrix Z induces a linear application

$$\begin{aligned} \mathcal{Z}_m : \quad \mathcal{D} \quad &\longrightarrow \quad \mathcal{D}_m \\ (x_i)_{1\le i\le n} &\longmapsto (\sum_k x_k Z_{ki})_{1\le i\le n} \end{aligned}$$

To obtain from it an endomorphism, we can consider either $\pi\circ\mathcal{Z}_m$ or $\mathcal{Z}_m\circ\pi$. In the general case $M \neq 0$ the projection π is invertible and the two possible endomorphisms are conjugate. So both choices may appear as equivalent. However, we believe there is a good choice, which is not the one made by [Pac] and [Moe]. We choose to study the endomorphism $\mathcal{Z} = \pi\circ\mathcal{Z}_m : \mathcal{D} \to \mathcal{D}$. Many arguments may be given for this, but basically, the advantage is to work on $\mathcal{D}$ rather than $\mathcal{D}_m$, a less structured space defined independently of the m_i's. At the technical level, this choice is better because it leads to simpler formulas. For example, it suggests that only the differences between the γ_i should be considered, and our remark at the end of Section 4 is a good illustration of this trick. It also leads to a conceptually accurate "reduction by the translations", constructed in [AlC], where the singular choice $M = 0$ does not cause any trouble. If E is the vector space where the differences $q_i - q_j$ live, the configuration up to translation is a $q \in E \otimes \mathcal{D}$. The reduced equation of motion reads $\ddot{q} + q\mathcal{Z} = 0$, as discussed in [AlC] where $\mathcal{Z}$ is denoted by $-2A$.

Proposition 15. Let $q_1, \dots, q_n$ be a configuration of dimension exactly d. Define $q^1, \dots, q^d$ as the elements of $\mathcal{D}$ obtained from an arbitrary frame $(O, e_1, \dots, e_d)$, projecting the configuration on each line (O, e_i), which gives d elements of $\mathbb{R}^n$, and then applying the canonical projection $\mathbb{R}^n \to \mathcal{D}$. The configuration is central of multiplier λ if and only if the q^i, $1 \le i \le d$, are eigenvectors of $\mathcal{Z} = \pi\circ\mathcal{Z}_m$ with eigenvalue λ.

This is just another way to write Definition 1. Another important observation about $\mathcal{Z}$ follows.

Proposition 16. For any configuration with masses $m_i > 0$, the operator $\mathcal{Z}$ is diagonalizable with real, positive eigenvalues.

Proof. The space $\mathcal{D}$ is endowed with the quadratic form

$$(c_1, \dots, c_n) \longmapsto \frac{1}{M}\sum_{i<j} m_i m_j (c_i - c_j)^2, \tag{17}$$

which is positive definite in the case of positive masses. If we compose with $\mathcal{Z}$ the linear map $\mu : \mathcal{D} \to \mathcal{D}^*$ associated to the quadratic form above, we obtain a symmetric form, associated to

$$(c_1, \dots, c_n) \longmapsto \sum_{i<j} m_i m_j S_{ij} (c_i - c_j)^2. \tag{18}$$

The operator $\mathcal{Z}$ is thus diagonal in a base in which (17) and (18) are both diagonal. The diagonal terms are positive because the second form is positive definite.

In the case of a central configuration, this Proposition gives some information on $\mathcal{Z}$ out of the space generated by the q^i. This is interesting for several applications, in particular the determination of the index of a d-dimensional central configuration ($d = 1$ or $d = 2$) seen as a critical point of the potential function defined on the manifold of $d + 1$-dimensional configurations. It happens that this index is directly related with the spectrum of $\mathcal{Z}$. In the case $d = 1$, the complete determination of this index may be deduced from the following beautiful result due to Conley.

Proposition 17. In the standard hypothesis $a < 0$, $m_i > 0$, all the eigenvalues of the operator $\mathcal{Z}$ associated to a collinear central configuration are greater than the eigenvalue λ.

We do not repeat the proof published in [Pac]. The idea is to consider the form (18) as a function defined on the unit sphere of $\mathcal{D}$ endowed with (17), and to study its gradient vector field. *For any collinear configuration* $(q_1, \dots, q_n) \in \mathbb{R}^n$, *with* $q_1 < \cdots < q_n$, *this vector field points strictly inside the spherical simplex defined by the inequalities* $c_1 < c_2 < \cdots < c_n$. So the unique local minimum, i.e. the global minimum, of the function is inside. It is the disposition represented by $(q_1, \dots, q_n)$ if the configuration is central.

Before Conley, Proposition 17 has been the object of conjectures and attempts of proof, in 1933 by Meyer, later by Smale and by his former student Palmore. Wintner[5] and Saari even thought that the same result is still true if we remove the hypothesis that the central configuration is collinear. Moeckel's inequality indeed gives an indication in this direction: for any central configuration, the arithmetic mean (12) of the $n - 1$ eigenvalues of $\mathcal{Z}$ is greater than λ. This is enough to conclude in the case of a Dziobek configuration, because in this case there is only one eigenvalue distinct from λ. This is in essence our Proposition 9.

This last question was solved by Moeckel [Moe] who gave a counter-example, a planar central configuration of 474 particles for which $\mathcal{Z}$ possesses a simple eigenvalue smaller than λ. The works [Moe] and [MoS] also describe a central configuration of 946 particles for which Wintner's assertion is false, i.e. the multiplicity of the eigenvalue λ is 3 while the dimension of the configuration is 2.

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