Some classical integrable problems

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ABSTRACT: This is a first lecture in classical mechanics, with a basic study of some classical mechanical systems, as tops or bodies rolling on a table, and with emphasis on the integrable cases. We deduce the equations of motion from the principle of virtual velocities and d'Alembert principle, which is the standard way to treat together systems with holonomic or non-holonomic constraints. Energy and angular momentum first integrals are treated in the same way in the holonomic and the non-holonomic cases.

A Chaplygin ball is a spherical ball without spherically symmetric repartition of mass, but with center of mass at the geometric center of the ball. This ball is rolling on a table. This system admits the first integrals of energy and angular momentum. On this and other examples we explain how integrability may be predicted by simply counting the first integrals and the symmetries.

1. Introduction. Statics is the science of equilibrium, Dynamics is the science of motion. Mechanics is Statics and Dynamics together. D'Alembert is said to have reduced Dynamics to Statics. This means that if we know how to write the algebraic equations for equilibrium, we know how to write the differential equations for motion. And the laws of equilibrium can be deduced from the principle of virtual velocities. We will explain this material explaining the laws of Statics first, and only after, the laws of Dynamics. Doing so we follow the presentation by Lagrange in his *Mécanique analytique*, assuming the reader has some knowledge in linear algebra that was not available at the time of Lagrange. We treat holonomic and non-holonomic systems together, deducing the equations of motion and integrating them using the same method. At the time of Euler or Lagrange the holonomic and non-holonomic systems were not distinguished. At the end of the 19th century, Hertz noticed that the Lagrangian and Hamiltonian dynamics fails to describe non-holonomic systems. So d'Alembert principle confirmed its position as more fundamental than the least action principle.

Our presentation is very elementary. We do not need the concept of Lagrangian or Hamiltonian. These concepts should be presented in a more

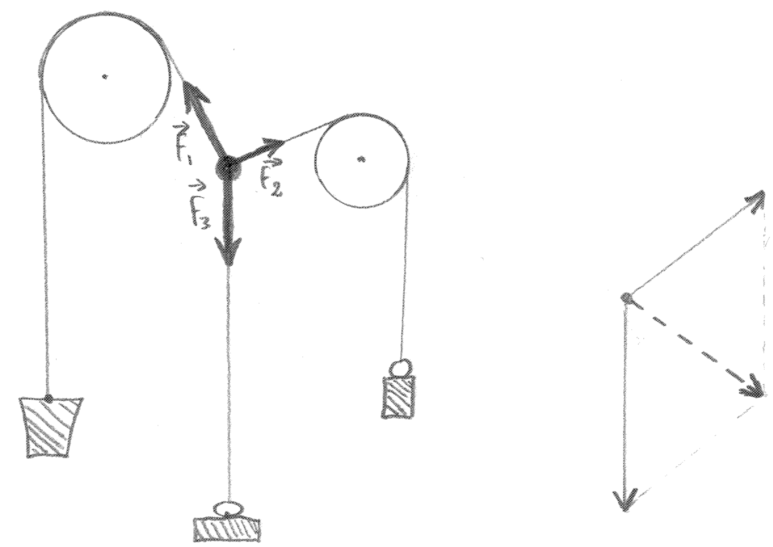

Figure 1: Forces and parallelogram of forces

advanced lecture and related to d'Alembert principle. They were discovered and used after the works of Lagrange, while classical mechanics was already an old and successful science.

2. The laws of equilibrium in 5 examples

EXAMPLE 1 (figure 1). A particle subjected to forces f_1, f_2, f_3 is moving in a plane.

The law of equilibrium is $f_1 + f_2 + f_3 = 0$. This law is expressed using elementary *vector calculus*. Vector calculus was developed at the end of 19th century. Before this time, it was not possible to put a + sign between anything else than numbers. The law of equilibrium above was known much earlier[1].

[1]From antiquity, the addition of vectors has been known through the geometrical construction called parallelogram of velocities: "The idea of a parallelogram of velocities may be found in various ancient Greek authors, and the concept of a parallelogram of forces was not uncommon in the sixteenth and seventeenth centuries" (Crowe, *History of vector analysis*, p. 2). Another keyword is "composition". The motions and then the forces were "composed". It is striking to see how these old words disappeared when the + sign was introduced. Much before vectors were added, Cauchy broke a taboo in 1812 by denoting a substitution with a symbol and by "multiplying" substitutions. In 1843 Hamilton invented the quaternions and one of the reasons he was proud of them is that he could use them as we now use vectors, putting a + or a $\times$ sign between two of these "numbers".

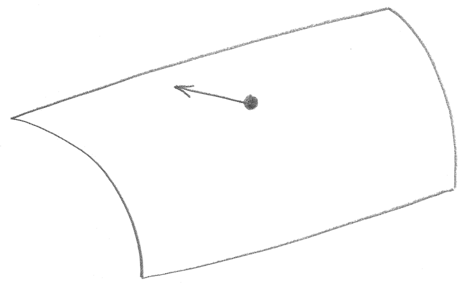

Figure 2: Particle on a surface

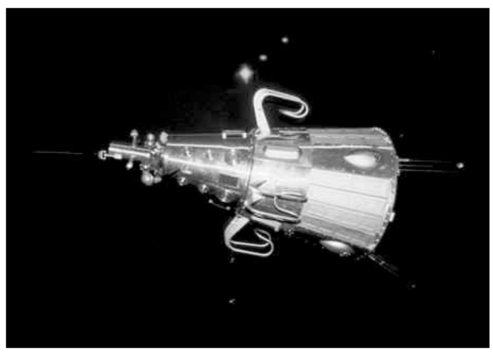

Figure 3: Rigid body in space (source: Space news international)

This law is enough to treat free particles subjected to forces, but not to treat problems with constraints. We present 4 problems with constraints.

EXAMPLE 2 (figure 2). A particle subjected to a force is moving on a fixed surface.

The law of equilibrium is: the force is orthogonal to the surface.

EXAMPLE 3 (figure 3). A free rigid body moving in an n-dimensional Euclidean space is subjected to a field of forces f.

The laws of equilibrium are (a) $\int f|_q \,\mathrm{d}^n q = 0$, (b) $\int (q - q_0) \wedge f|_q \,\mathrm{d}^n q = 0$.

Remark 1. The integral is extended to the domain where the forces are applied. The form $\mathrm{d}^n q$ is the standard volume element on the Euclidean space E. The vector f is a force by unit of volume. The standard example is the force of gravity in a laboratory. To express it, we introduce the density of the solid, which is a real function m. The gravity by unit of volume is $f = mg$, where g is a constant vector directed downwards. The integral is

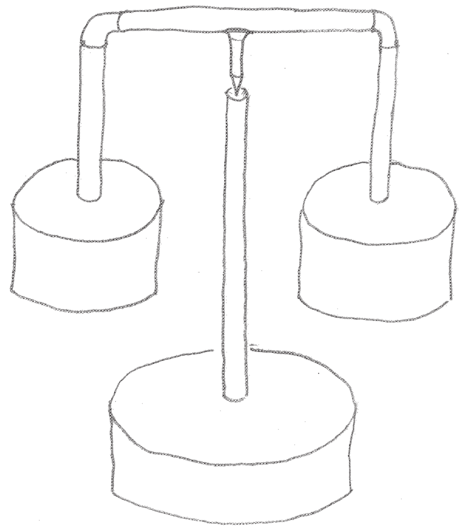

Figure 4: Rigid body with a fixed point

extended to the solid body seen as a subset of E.

Remark 2. In the formulas q and q_0 are points, and $q - q_0$ is the vector from q_0 to q. The symbol $\wedge$ is the wedge product (or exterior product). It is not the vector product. The quantity $(q - q_0) \wedge f$ is a bivector. The laws of dynamics are consistent in dimension $\neq 3$. To apply the usual theories it is sufficient to replace the vector product by the wedge product. Higher dimensional mechanics has been studied since the 19th century. The motivation for this abstract study is almost similar to the motivation for studying geometry in dimension $\neq 3$.

Remark 3. In (a) and (b), q is the variable point in the integration process. The point q_0 in (b) is fixed in the integration process. According to (a), the choice of q_0 is indifferent. In dynamics it can be a particle, a fixed point or a point moving arbitrarily. Of course there exists a more satisfying way to write (a)+(b). One considers that the body moves on an affine hyperplane of an $n+1$-dimensional vector space (a hyperplane not passing through zero). Then the system is simply $\int q \wedge f|_q \, \mathrm{d}^n q = 0$. Now q is a vector with $n+1$ coordinates.

EXAMPLE 4 (figure 4). A rigid body with a fixed particle q_F is subjected to a field of forces.

Figure 5: Rolling ball (source: Sportibel)

The law of equilibrium is $\int (q - q_F) \wedge f|_q \, \mathrm{d}^n q = 0$.

EXAMPLE 5 (figure 5). A smooth rigid body subjected to forces is rolling without sliding on a fixed smooth surface. There is no friction except what is required to avoid sliding. There is a unique contact particle q_K. We assume that the geometry allows rolling: at the contact point, the surface is "less convex" than the body.

The law of equilibrium is $\int (q - q_K) \wedge f|_q \, \mathrm{d}^n q = 0$.

3. How to deduce these laws of equilibrium from a single principle?

The principle is called "principle of virtual velocities" and is due, in the form we use here, to Johann Bernoulli. We assume the system is made of N particles in the Euclidean space E, where N is possibly big. The particles are possibly the "atoms" of a rigid body. Not all the motions of the particles are allowed: there are constraints. A "virtual velocity" is a "first order motion" which is allowed by the constraints. For the type of constraints we consider the space of virtual velocities of a given configuration is a linear subspace of $(\vec{E})^N$.

Principle of virtual velocities. *A mechanical system with constraints, made of N particles subjected to force vectors $(f_1, \ldots, f_N)$ is at equilibrium if and only if $\sum_{i=1}^N \langle f_i, q_i' \rangle = 0$ for any virtual velocity $(q_1', \ldots, q_N')$.*

It is more convenient to think the rigid bodies as continuous repartitions of mass, subjected to a field of force acting on small units of volume. In this case the velocity is a vector field, and we replace the $\sum$ symbol by an integration. The principle is $\int \langle f, q' \rangle \, \mathrm{d}^n q = 0$.

EXAMPLE 1. There is no constraint. The virtual velocity is any vector q' from the only particle q. The law of equilibrium is $\langle f, q' \rangle = 0$ for any q'. This is $f = 0$, i.e. the force applied to q is zero. It is the law we announced.

EXAMPLE 2. The virtual velocity q' is tangent to the surface at the only particle q. The law $\langle f, q' \rangle = 0$ means that f is orthogonal to the surface, as announced.

Important remark. One may also say: the particle is subjected to a *reaction force* R which is orthogonal to the surface. The law of equilibrium is: there is a vector R orthogonal to the surface (called the reaction) such that $R + f = 0$. So we treat the constraint of staying on the surface introducing a force. The equilibrium law looks the same as in Example 1. When we explain Mechanics using the principle of virtual velocities, we do not need to introduce a reaction force and to assume it is orthogonal to the surface.

EXAMPLE 3. The position of the rigid body is an isometric affine map $\mathcal{R} : \vec{F} \to E$. The space $\vec{F}$ is "the space where the particles are fixed" while in E the particles are moving. A point in E can be called a "place". So E is "the space where the places are fixed", while in $\vec{F}$ the places are moving. What is moving is indeed the linear map $\mathcal{R}$, which establishes a one-to-one correspondence between particles and places.

The space $\vec{F}$ is a Euclidean vector space. The space E is a Euclidean space. We mean: in the second space no origin is specified, while in $\vec{F}$ the point are called vectors and there is a special origin, the zero vector. These choices are just conventions. Sometimes we decide that this special origin is the center of mass of the rigid body. Other times it is a geometric center.

We call $\vec{E}$ the Euclidean vector space intrinsically associated to E (it may be seen as the space of translations of E, or as the tangent space at any point of E). Then at any time t an $R : \vec{F} \to \vec{E}$ is associated to $\mathcal{R}$. For any $X \in \vec{F}$ we have $q = \mathcal{R}X = q_0 + RX$, where $q_0 = \mathcal{R}(0)$. Then $\dot{q} = \dot{q}_0 + \dot{R}X$. But R

is an isometry, so $\dot{R} = \omega R$, where $\omega : \vec{E} \to \vec{E}$ satisfies $\omega = -{}^t\omega$ (we make $\vec{E} = (\vec{E})^*$ using the Euclidean structure). It is the classical "instantaneous rotation". Now $\dot{q} = \dot{q}_0 + \omega(q - q_0)$. A field of virtual velocities is given by any pair (q_0', ω), through the formula $q' = q_0' + \omega(q - q_0)$. The principle of virtual velocities is $\int \langle q', f\rangle \, \mathrm{d}^n q = 0$ for any (q_0', ω). It splits immediately into two conditions: $\int \langle q_0', f\rangle \, \mathrm{d}^n q = 0$ for any q_0' and $\int \langle \omega(q - q_0), f\rangle \, \mathrm{d}^n q = 0$ for any antisymmetric ω. The first condition gives $\int f|_q \, \mathrm{d}^n q = 0$. To treat the second condition, we write $\langle \omega(q - q_0), f\rangle = \langle \omega, f \otimes (q - q_0)\rangle = \langle \omega - {}^t\omega, f \otimes (q - q_0)\rangle/2 = \langle \omega, f \otimes (q - q_0) - (q - q_0) \otimes f\rangle/2 = \langle \omega, f \wedge (q - q_0)\rangle/2$. So the second condition is $\langle \omega, \int (q - q_0) \wedge f|_q \, \mathrm{d}^n q\rangle = 0$ for any ω. It is $\int (q - q_0) \wedge f|_q \, \mathrm{d}^n q = 0$. So we get the equations we announced.

EXAMPLE 4. The position of the rigid body is an isometric linear map $R : \vec{F} \to \vec{E}$. The image space $\vec{E}$ is now a Euclidean vector space, because it is convenient to decide that q_F is the origin of $\vec{E}$. Thus we set $q_F = 0$ and write q instead of $q - q_F$. The deduction of the law of equilibrium is obtained by simplifying the process in Example 3. We have $q = RX$, $\dot{q} = \omega q$. We obtain immediately the condition $\int q \wedge f|_q \, \mathrm{d}^n q = 0$.

EXAMPLE 5. The virtual velocities are among the virtual velocities of a rigid body. They are given by a pair $(q_0', \omega) \in \vec{E} \times \bigwedge^2 \vec{E}$ through the formula $q' = q_0' + \omega(q - q_0)$. The rolling without sliding condition is simply $\dot{q}_K = 0$, where q_K is the particle at the contact point. So $0 = q_0' + \omega(q_K - q_0)$. Subtracting from the previous equation gives $q' = \omega(q - q_K)$. A virtual velocity is thus given by an $\omega \in \bigwedge^2 \vec{E}$. For any such ω we have $\int \langle f, \omega(q - q_K)'\rangle \, \mathrm{d}^n q = 0$. By the same computation as in Example 3, it gives $\int (q - q_K) \wedge f|_q \, \mathrm{d}^n q = 0$, as announced.

4. The laws of Dynamics from d'Alembert principle

Le *Traité de Dynamique* de d'Alembert, qui parut en 1743, mit fin à ces espèces de défis, en offrant une méthode directe et générale pour résoudre, ou du moins pour mettre en équations tous les problèmes de Dynamique que l'on peut imaginer. Cette méthode réduit toutes les lois du mouvement des corps à celles de leur équilibre et ramène ainsi la Dynamique à la Statique. (Lagrange, Mécanique analytique, œuvres 11, p. 255)

D'Alembert principle. *Consider the motion of a mechanical system. Suppose the system is subjected at time t to a field of forces f. The field of accelerations $\ddot{q}$ at time t is such that the system would be at equilibrium if its velocity was zero and if it was subjected to the field of force $f - m\ddot{q}$.*

If we consider an isolated particle, what we called "field of force" or "field of acceleration" is simply a vector. In the case of a continuum of particles, it can be a field along a curve, on a surface or in a volume. The mass ratio m is simply the mass of the isolated particle, or it is the density of mass per standard unit of length, area or volume in the case of a continuum of particles.

Now we have the principle of virtual velocities and d'Alembert principle. In front of a statement called principle the reader should raise questions.

Question 1. To which class of mechanical systems do the principles apply? They apply to the systems we give as examples, i.e. systems of rigid bodies with holonomic or non-holonomic constraints, subjected to forces, and where the only friction is the one required to have contact points that do not slip. But they appear to apply to other systems. Lagrange discovered that they give Euler's equations in fluid dynamics. The reader can see P. Duhem, *Des principes fondamentaux de l'hydrostatique*, Annales de la faculté des sciences de Toulouse (1890) for a classical discussion. These principles were challenged in many situations, and we were not able to find in the huge literature a discussion of a case where they fail to give a correct description. But most authors exclude systems where the constraint works, e.g. where there is a friction related to the constraint. If we put a coin on a horizontal table, and then incline slightly the table, the coin does not slide because of the friction. A force which models the friction should be introduced after some experimentation, and it appears that this force is related to the reaction force. We spoke about the reaction force in the "important remark" above. We present the dynamics in a way which avoids the introduction of this reaction force. But dealing with friction we would have to introduce a friction force and soon after discuss its relation with the reaction force. So maybe d'Alembert principle is not an elegant way to present Dynamics as soon as we discuss friction.

Question 2. Are they really "principles", i.e. basic laws of physics, or are they deduced from more basic laws, such as Newton's law? Again, this is a difficult question which would depend on a precise definition and a model of the type of constraints we accept. We can introduce an interaction between the particles of a rigid body which allows the rigid body motion. If it consists in central forces satisfying the action=reaction axiom, it will give a correct description. But concerning the laws of motion for a rolling object this seems less easy. Many "proofs" of d'Alembert principle may be found in the classical books, but sometimes the more basic principles from which d'Alembert is deduced are not clearly stated.

Question 3. Does d'Alembert principle determine uniquely the field of acceleration $\ddot{q}$? To answer this question and the previous one we should answer Question 1 first. What we will do is less ambitious. We will deduce from d'Alembert principle the value of the field $\ddot{q}$ for some systems in our class.

Let us repeat Question 3 more precisely. At any time t, is there a unique field $\ddot{q}$ satisfying d'Alembert principle and compatible with: the constraints, the position of the system at time t and the field of velocities at time t?

To answer the question for some class of systems we should be able to write the equations of equilibrium for this class of system. For us it is natural to use the principle of virtual velocity to write the equations of equilibrium. Here both principle are clearly associated. We understand why, from the beginning of the 20th century, some authors used the new terminology "principle of virtual works" or "principle of virtual powers" to associate them in a single principle. However we prefer not to use this terminology and rather use Lagrange's words.

A second difficulty raised by Question 3 is that in each particular system we have to deduce what are the $\ddot{q}$ compatible with the position at time t and the velocity at time t. Any new system requires a new deduction. In the next section will make the deduction in some examples.

But anyway let us try to answer Question 3 with some generality. The system has N particles, where N is possibly big. The particles are possibly the "atoms" of a rigid body. A "velocity" is an element $v = (v_1, \ldots, v_N) \in (\vec{E})^N$. To each particle is associated a velocity vector. We suppose we can give the holonomic or non-holonomic constraints by an equation $Lv = 0$, where $L : (\vec{E})^N \to \mathcal{W}$ is a linear operator depending on time and $\mathcal{W}$ is some vector space. Then v is a virtual velocity if and only if $Lv = 0$.

To obtain the constraint on $\ddot{q}$ we remark that $L\dot{q} = 0$ at any time. Differentiating we obtain $\dot{L}\dot{q} + L\ddot{q} = 0$. So if some acceleration $\ddot{q} = \gamma_0$ is possible, all the possible accelerations have the form $\ddot{q} = \gamma_0 + w$, where $w \in \ker L$.

Now we write the condition given by d'Alembert principle associated to the principle of virtual velocities: for any $v \in \ker L$, $\sum_{i=1}^N \langle f_i - m_i\ddot{q}_i, v_i\rangle = 0$. Substituting $\ddot{q} = \gamma_0 + w$, and setting $(\gamma_1)_i = f_i/m_i - (\gamma_0)_i$ this condition is $\langle\langle \gamma_1 - w, v\rangle\rangle = 0$ for all $v \in \ker L$, where $\langle\langle v, w\rangle\rangle$ is by definition $\sum_{i=1}^N m_i\langle v_i, w_i\rangle$. Let γ_2 be the orthogonal projection of γ_1 on $\ker L$. We obtain the solution $w = \gamma_2$ and $\ddot{q} = \gamma_0 + \gamma_2$ solves the problem uniquely.

We will not use this uniqueness argument, but rather write down the equations of motion in each particular case. We will check that d'Alembert principle gives uniquely the value of the acceleration at time t as a function of the position and the velocity at time t.

5. The laws of Dynamics in 5 examples

EXAMPLE 1. The law of equilibrium is $f = 0$. The law of dynamics is $f - m\ddot{q} = 0$. This is Newton's law[2].

EXAMPLE 2. The law of equilibrium is: f is orthogonal to the surface. Thus the law of dynamics is: $f - m\ddot{q}$ is orthogonal to the surface. If the equation of the surface is $h(q) = 1$, the velocity satisfies $\langle dh, \dot{q}\rangle = 0$ and the acceleration $\langle \dot{dh}, \dot{q}\rangle + \langle dh, \ddot{q}\rangle = 0$. This constraint decides which is the orthogonal component of $\ddot{q}$, while d'Alembert principle gives the tangential component. Thus $\ddot{q}$ is uniquely determined.

FREE MOTION IN EXAMPLE 3. Applying d'Alembert principle, where we assume there is no force, the equations of motion are (a) $\int m\ddot{q}\,\mathrm{d}^n q = 0$, (b) $\int m(q - q_0) \wedge \ddot{q}\,\mathrm{d}^n q = 0$. Now we shall find the possibilities for $\ddot{q}$ and use (a) and (b) to choose among them. We know that $\dot{q} = \dot{q}_0 + \omega(q - q_0)$. Differentiating we get $\ddot{q} = \ddot{q}_0 + \dot{\omega}(q - q_0) + \omega(\dot{q} - \dot{q}_0) = \ddot{q}_0 + (\dot{\omega} + \omega^2)(q - q_0)$. The free parameters for the acceleration are $\ddot{q}_0$ and $\dot{\omega}$. To determine them we first express Condition (a). Let $M = \int m\,\mathrm{d}^n q$ be the total mass of the system. Condition (a) is $0 = M\ddot{q}_0 + (\dot{\omega} + \omega^2)\int m(q - q_0)\,\mathrm{d}^n q$. But $\int m(q - q_0)\,\mathrm{d}^n q = M(q_G - q_0)$ where q_G is the center of mass.

A good convention is to take the center of mass as the origin of $\vec{F}$, "the space where the particles are fixed". Then $q_0 = q_G$, and the first condition is simply $\ddot{q}_G = 0$. The center of mass has a rectilinear uniform motion. It is a well-known trick in such cases to choose a Galilean frame in which q_G is fixed. Then the rigid body is moving around its center of mass, which is fixed. We reduced the free motion in Example 3 to the free motion in Example 4.

FREE MOTION IN EXAMPLE 4. INERTIA TENSOR. D'Alembert principle gives the equation of motion $\int mq \wedge \ddot{q}\,\mathrm{d}^n q = 0$. We have $\dot{q} = \omega q$ and thus $\ddot{q} = (\dot{\omega} + \omega^2)q$. We use the equation of motion to choose the correct $\dot{\omega}$.

[2]Newton's law appeared in 1687 but at that time people were not accustomed to vector relations. Newton did not explain what is a vector. Of course he did not write the equation in vector form. He did not present the relation decomposed by projecting onto three coordinate axes: it was Maclaurin who first explained this decomposition in 1742 in his *Treatise of Fluxions*. Before him Newton's law was not clearly understood and mechanics could not progress. The first theoretical step in classical mechanics after Newton's law, d'Alembert principle, appeared in 1743. Maupertuis and Euler published their principles of least action in 1744.

We write $\alpha = \dot\omega + \omega^2$ and put this operator out of the $\int$ symbol as follows.

$$0 = \int mq \wedge \alpha q \,\mathrm{d}^n q = \int m(q \otimes \alpha q - \alpha q \otimes q)\,\mathrm{d}^n q =$$

$$= (\int mq \otimes q \,\mathrm{d}^n q) \circ {}^t\alpha - \alpha \circ (\int mq \otimes q\,\mathrm{d}^n q).$$

The $\circ$ is, as one prefers, the composition of the linear maps, the product of matrices, or the contracted product of tensors. We will omit this symbol. We set $b = \int_{\vec{E}} mq \otimes q\,\mathrm{d}^n q$. This is the inertia tensor, related to the so-called moments of inertia. It is a kind of universal integration of quadratic expressions in q according to the mass density. It is the integral of $q \otimes q$ which is, if we write $q = (x, y, z)$, the matrix

$$q \otimes q = \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}.$$

There is another inertia tensor, which appears implicitly in what follows through the formula $b\omega + \omega b$. It is the integral of the matrix

$$\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}.$$

To define b we use the first matrix and not the second one. In dimension 4 the first matrix is a 4×4 matrix, while the second is a 6×6 matrix.

The equation of motion becomes $b(-\dot\omega + \omega^2) = (\dot\omega + \omega^2)b$. This equation gives $\dot\omega$ as a function of ω and the position.

We can check this equation in the particular case of a body with spherical symmetry around the fixed point. In this case $b = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{R}$. The equation of motion is $\dot\omega = 0$. The motion is a uniform rotation of the spherical object.

In the general case, all the variables in the equation $b(-\dot\omega+\omega^2) = (\dot\omega+\omega^2)b$ are moving with t. The inertia tensor b moves with the configuration. It is better to write the equation in the space $\vec{F}$, where the particle are fixed. There the tensor of inertia is constant.

We set $B = \int_{\vec{F}} mX \otimes X\,\mathrm{d}^n X$. Then B is the pull-back of b by the isometry R. We set $\dot R = \omega R = R\Omega$. Then $\Omega \in \bigwedge^2 \vec{F}$ is the pull-back by R of ω. Finally $\ddot R = (\dot\omega + \omega^2)R = R(\dot\Omega + \Omega^2)$. Strangely $\dot\Omega$ is the pull-back by R of $\dot\omega$. We obtained the so-called Euler-Poisson equations $B(-\dot\Omega + \Omega^2) = (\dot\Omega + \Omega^2)B$.

The angular momentum is $a = -R(B\Omega + \Omega B)^t R$. Its first derivative is $\dot{a} = -R(\Omega(B\Omega + \Omega B) + B\dot{\Omega} + \dot{\Omega}B - (B\Omega + \Omega B)\Omega)^t R$. So the conservation of angular momentum is exactly the equation of motion! Also the pull-back of a is $A = -B\Omega - \Omega B$. We get $\dot{A} = -[B, \Omega^2] = [A, \Omega]$, a famous form of the Euler-Poisson equations.

EXAMPLE 5. If we want the possible accelerations we take the time derivative of the relation $\dot{q} = \omega(q - q_K)$. Here we must be careful. At time t there is a contact particle q_K on the rolling object and a contact particle on the table. They have the same velocity because they are in contact. The table is fixed: this is why we wrote previously $\dot{q}_K = 0$. But these particles are not in contact at other times. We will call K the *contact point* which moves on the table. The relation above is indeed $\dot{q} = \omega(q - K)$. Differentiating we get $\ddot{q} = \dot{\omega}(q - K) + \omega(\dot{q} - \dot{K}) = (\dot{\omega} + \omega^2)(q - K) - \omega\dot{K}$.

Assuming there is a constant gravitational attraction vector g, the equation of dynamics is $\int m(q - K) \wedge (g - (\dot{\omega} + \omega^2)(q - K) + \omega\dot{K})\, \mathrm{d}^n q = 0$. The first and third terms are linear in q. They give $M(q_G - K) \wedge (g + \omega\dot{K})$, where M is the total mass, and q_G the center of mass. The second term is computed as above, introducing the inertia tensor b_K with respect to the contact point K. Finally, the equation of motion is:

$$M(q_G - K) \wedge (g + \omega\dot{K}) + b_K(\dot{\omega} - \omega^2) + (\dot{\omega} + \omega^2)b_K = 0. \qquad (\star)$$

It seems clear that $\dot{K}$ is known as soon as ω is known, but the relation depends on the geometry of the surface of the rolling object and the surface of the table. We restrict us to the simplest possible geometry, a ball rolling on a plane.

Rolling ball on the plane. The rigid ball has a perfectly spherical aspect but the repartition of mass inside is arbitrary. We call q_C the particle at the center of the sphere. It is clear that $\dot{q}_C = \dot{K}$. We also know that $\dot{q}_C = \omega(q_C - K)$. The equation is

$$M(q_G - K) \wedge (g + \omega^2(q_C - K)) + b_K(\dot{\omega} - \omega^2) + (\dot{\omega} + \omega^2)b_K = 0. \qquad (*)$$

Usual ball. In order to check this equation we try the most common case of a ball with spherical symmetry. We have $q_C = q_G$. The inertia tensor b_C with respect to q_C is $\lambda\mathrm{Id}$. We have $b_K = \int m(q - K) \otimes (q - K)\, \mathrm{d}^n q = \int m(q - q_C) \otimes (q - q_C)\, \mathrm{d}^n q + M(q_C - K) \otimes (q_C - K)$. The first term is $b_C = \lambda\mathrm{Id}$. Setting $k = q_C - K$ the equation is $Mk \wedge g + Mk \wedge \omega^2 k + b_K\dot{\omega} + \dot{\omega}b_K - Mk \otimes \omega^2 k + M\omega^2 k \otimes k = 0$, which is $Mk \wedge g + b_K\dot{\omega} + \dot{\omega}b_K = 0$. If the plane is horizontal, $k \wedge g = 0$ and the equation possesses the unique solution $\dot{\omega} = 0$. The ball is rolling in a uniform way, with a constant ω. This is the solution of the problem in a space E of any dimension.

The equation $(*)$ for a general rolling ball on a horizontal plane deserves further studies. Note that we can pull-back it to $\vec{F}$ as we did for Example 4. The pull-back B_K of the inertia tensor b_K with respect to the contact point K is still not constant, because K, or more correctly $\mathcal{R}^{-1}(K)$, is moving in $\vec{F}$. We should use B_G or B_C, which are fixed. However, it is better to consider the angular momentum integral before considering the pull-back.

6. The first integrals of angular momentum and linear momentum

First integrals are also known as constants of motion. To study any mechanical system it is extremely useful to know its first integrals.

We begin with the linear momentum and the angular momentum. These are the names for the main first integrals which are linear forms in the velocity (for any given configuration). The linear momentum is also called impulsion or quantity of motion. The angular momentum is also called constant of areas. We will see in each example the precise formula.

EXAMPLE 1. A particle is subjected to a force f directed toward a fixed point C (one also says a fixed "center". The force is said "central").

Newton equation of motion is $m\ddot{q} = f$. The bivector $a = (q - C) \wedge \dot{q}$ is constant, because $\dot{a} = (q - C) \wedge \ddot{q} + \dot{q} \wedge \dot{q} = (q - C) \wedge f/m = 0$.

Instead of the central force we may consider a force f which is always proportional to given vector e_1. Then $e_1 \wedge \dot{q}$ is a first integral which can be called linear momentum. The force with constant direction may be seen as a central force with center at infinity. Together with Remark 2.3, the unification of central force and force with constant direction, of angular momentum and linear momentum, calls attention to the projective aspects in dynamics.

EXAMPLE 2. A particle is moving on a surface which admits the vertical axis (C, e_3) as an axis of symmetry, and is subjected to a vertical force vector.

The angular momentum trivector $a = (q - C) \wedge \dot{q} \wedge e_3$ is constant (we can replace C in this formula by any point on the axis of symmetry.) The first integral a is usually called Clairaut first integral, but it was given before Clairaut in Proposition 55 of Newton's *Principia*.

Proof. The equation of motion is $m\ddot{q} = \lambda\nu + f$, where ν is the normal vector to the surface, and λ is real. By the symmetry around the e_3 axis, $\nu \wedge (q-C) \wedge e_3 = 0$. Then $\dot{a} = 0$.

Exercise. We used the ambiguous terminology "vertical force". It could mean parallel to e_3 or tangent to the surface along a meridian. Is there any trouble with this ambiguity?

Remark. Here the surface is symmetric but the force is not. We will discuss the relation between the symmetry and a later. The central force explanation of Example 1 may be extended into a "central axis" explanation and then it applies to this case. Indeed if the normal ν remains in the $(q-C, e_3)$ plane the surface has the symmetry of revolution.

EXAMPLE 3 (again with zero force). The first equation of motion is $\ddot{q}_G = 0$. The first integral of linear momentum is $\dot{q}_G$. The second equation of motion and the first integral of angular momentum reduce to the next example.

EXAMPLE 4 (again the free motion, i.e. zero force). We already gave without justification a formula for the angular momentum: $a = -R(B\Omega + \Omega B)^t R$. A more predictable formula is $a = \int mq \wedge \dot{q}\, \mathrm{d}^n q$. We use $\dot{q} = \omega q$ and the same computation as in §5.4. This gives $a = -b\omega - \omega b$. To check again that this quantity is conserved, we can also compute in the physical $\vec{E}$ space. We have $\dot{b} = \int m(\dot{q} \otimes q + q \otimes \dot{q})\, \mathrm{d}^n q = \omega b - b\omega$. So $\dot{a} = -\omega b\omega - b\omega^2 - b\dot{\omega} - \dot{\omega} b - \omega^2 b + \omega b\omega$. The equation of motion $b(-\dot{\omega} + \omega^2) = (\dot{\omega} + \omega^2)b$ is exactly $\dot{a} = 0$.

EXAMPLE 5. We set $a = \int m(q-K) \wedge \dot{q}\, \mathrm{d}^n q$. We have $\dot{a} = \int m(q-K) \wedge \ddot{q}\, \mathrm{d}^n q - \int m\dot{K} \wedge \dot{q}\, \mathrm{d}^n q$. The first term reminds clearly the equation of motion as obtained directly from d'Alembert principle. The second is $-M\dot{K} \wedge \dot{q}_G$. If the force is $f = mg$, the equation of motion is

$$\int m(q-K) \wedge g\, \mathrm{d}^n q = \int m(q-K) \wedge \ddot{q}\, \mathrm{d}^n q = \dot{a} + \int m\dot{K} \wedge \dot{q}\, \mathrm{d}^n q.$$

With a constant g it simplifies in $M(q_G - K) \wedge g = \dot{a} + M\dot{K} \wedge \dot{q}_G$. This is equivalent to $(\star)$ but of somewhat simpler aspect.

Exercise. Deducing $\dot{b}_K = \omega b_K - b_K\omega - M\dot{K} \otimes (q_G - K) - M(q_G - K) \otimes \dot{K}$ by using the symmetric part of $\int m(\dot{q} - \dot{K}) \otimes (q-K)\, \mathrm{d}^n q$, check that $\dot{a} = b_K\omega^2 - \omega^2 b_K - M\dot{K} \wedge \dot{q}_G - M(q_G - K) \wedge \omega\dot{K} - b_K\dot{\omega} - \dot{\omega} b_K$ and use $(\star)$ to obtain again the above formula.

EXAMPLE 5, CHAPLYGIN CASE. If the table is a horizontal plane, if the rolling object is a ball, if the center of mass is the center of the sphere, we have $\dot{K} = \dot{q}_G$ and the equation of motion reduces to $\dot{a} = 0$. Chaplygin integrated these equations. The ball may have a strange non-uniform motion but still the system is integrable, i.e. the motion is given by nice formulas. We will see how the integrability may be predicted.

7. Why are the angular and linear momentum constant?

Given a mechanical system, how can we predict that first integrals exist? There are two main answers, which both extend examples given by Newton in his *Principia*. The angular and linear momentum are constant when the system is made of particles which mutually attract each other by central forces satisfying the action=reaction law. The second answer is: the linear momentum is related to a symmetry of translation, while the angular momentum is related to a symmetry of rotation. Concerning the symmetries, we need to discuss the energy integral first. So we postpone the discussion.

We discuss Example 3, where there is the constraint of rigidity but neither the constraint of a fixed particle nor a particle in contact. We don't need a sophisticated model of constraint. We think the rigid body as a big number N of atoms. Each atom is subjected to forces but the distance between two atoms remains the same. We need to introduce a force in order to maintain the rigidity. This force is a "reaction". We already discussed reaction forces in an important remark. We claim that there exist reaction forces that maintain the rigidity and which are central forces satisfying the action=reaction axiom. Here the terminology is confusing, the word reaction being employed twice.

The *law of action and reaction* was stated by Newton. The tradition to write laws in words rather than in formulas is not satisfying. In words, the law is: if a particle acts on a second particle by a force, the second particle acts on the first by the opposite force. But on a particle is applied a unique force vector. We do not know which is the "part" of this force vector which is "due" to a given particle rather than another. We can only give a meaning to this law in situations where we have more information. In formulas, when N particles with masses $m_1, \dots, m_N$ interact, a system of force vectors $f_1, \dots, f_N$ of the form

$$f_i = \sum_{j \neq i} \sigma_{ij}(q_j - q_i)$$

where the σ_{ij}'s are real numbers, satisfies the action=reaction law if and only if $\sigma_{ij} = \sigma_{ji}$. If the system of particle is subjected to such a system of force vectors, i.e. if the law of motion is

$$m_i \ddot{q}_i = f_i,$$

it is obvious that, for any reference point q_0,

$$G = \frac{\sum m_i q_i}{\sum m_i} \quad \text{and} \quad a = \sum m_i (q_i - q_0) \wedge \dot{q}_i$$

are such that $\ddot{G} = 0$ and $\dot{a} = 0$. These are the conservation laws we are discussing in this paragraph.

We want to prove that the rigidity of the system of particles may be the effect of a system of "reaction forces" $R_i = \sum_{j \neq i} \sigma_{ij}(q_j - q_i)$, with $\sigma_{ij} = \sigma_{ji}$, which compensate the arbitrary external forces. We just want a mathematical result which is not supposed to be physically realistic. At each time t the σ_{ij}'s will be functions of the external forces and the velocities.

We claim that the rigidity is still possible if we furthermore set $\sigma_{ij} = 0$ for any (i,j), $4 \leq i < j \leq N$. Here we assume $\dim E = 3$ to facilitate intuition. The first three particles q_1, q_2 and q_3 play a special role. They form a kind of system of reference. We assume that they are not on the same line and that none of the remaining particles is in the plane they define.

We call $\mathcal{S}$ the complementary set of (i,j)'s, $1 \leq i < j \leq N$, those which satisfy $1 \leq i \leq 3$. We claim the linear map $(\sigma_{ij})_{(i,j)\in\mathcal{S}} \mapsto (\alpha_{ij})_{(i,j)\in\mathcal{S}}$, where $\alpha_{ij} = \langle r_i - r_j, q_i - q_j \rangle$ and $m_i r_i = R_i = \sum_{k \neq i} \sigma_{ik}(q_k - q_i)$, is invertible. Here is a "mechanical" proof of this statement. We interpret the r_i as the velocities of the particles instead of accelerations. Suppose that $\alpha_{ij} = 0$ for all $(i,j) \in \mathcal{S}$. This means that the distances $\|q_i - q_j\|$, $(i,j) \in \mathcal{S}$, are not modified by the velocities r_i. With our hypothesis on the first three particles, this means that the motion is rigid at the first order. But the identities $\sum m_i r_i = 0$ and $\sum m_i (q_i - q_0) \wedge r_i = 0$ show that there is neither translation nor rotation. All the points are fixed, i.e. $r_i = 0$ for all i. But R_j, $j \geq 4$, is sum of 3 independent vectors, one from each particle q_1, q_2 and q_3. As $r_j = 0$, they are all zero. Thus $\sigma_{1j} = \sigma_{2j} = \sigma_{3j} = 0$. Then we easily see that $\sigma_{23} = \sigma_{13} = \sigma_{12} = 0$. We supposed that the α_{ij}'s are zero and proved that the σ_{ij}'s are zero, so we proved the claimed invertibility.

Suppose the particle q_i is subjected to the external force f_i and the reaction force $R_i = m_i r_i$. Is it always possible to choose the σ_{kl}'s, $(k,l) \in \mathcal{S}$, such that the $\|q_i - q_j\|$ are constant? It is sufficient to consider the $\|q_i - q_j\|$, $(i,j) \in \mathcal{S}$, since if these mutual distances are constant then the others are also constant.

First of all, we take the velocities such that $\langle q_i - q_j, \dot q_i - \dot q_j\rangle = 0$. Then, the accelerations should be such that $\langle q_i - q_j, \ddot q_i - \ddot q_j\rangle + \|\dot q_i - \dot q_j\|^2 = 0$. This equation is

$$\langle q_i - q_j, r_i - r_j\rangle = -\|\dot q_i - \dot q_j\|^2 - \langle q_i - q_j, f_i/m_i - f_j/m_j\rangle.$$

We call the right hand side α_{ij} and use the invertibility of the linear map above. We find that there exist σ_{kl}'s solving this equation at any time t. These σ_{kl}'s maintain the rigidity of the system of particles.

We see that if there is no external forces, the angular momentum of the rigid system of particles is constant. If several rigid bodies are subjected to classical gravitational forces (which are described as a system of central forces between atoms), the total angular momentum is constant. So remembering this simple fact, that the rigidity of a system of particles is obtained by a system of central forces, we can predict the conservation of the linear and the angular momentum in most natural situations.

8. The first integral of energy

Suppose the system is made of N particles, where N is possibly big. The particles are possibly the "atoms" of a rigid body. By the two principles the equation of motion is: for any virtual velocity $(q'_1, \dots, q'_N)$, $\sum_{i=1}^N \langle f_i - m_i \ddot q_i, q'_i\rangle = 0$. In particular, the true velocity is a virtual velocity. So

$$\sum_{i=1}^N \langle f_i - m_i \ddot q_i, \dot q_i\rangle = 0. \qquad (o)$$

Force function. We introduce the concept of force function. It is a function of the configuration $U : E^N \to \mathbb{R}$ such that $dU = (f_1, \dots, f_N)$. Note that dU is naturally an element of $(\vec E^*)^N$ but when E is Euclidean we make the conventional identification $E = E^*$.

We introduce the kinetic energy $T = \sum_{i=1}^N m_i \langle \dot q_i, \dot q_i\rangle/2$. Then Equation (o) is $\dot U - \dot T = 0$, and we have the first integral of energy $H = T - U$. So the new thing we have to assume is the existence of a force function. When the force is the gravity in the laboratory, the hypothesis is fulfilled. We set $U = g \sum_{i=1}^N m_i z_i$ where z_i is the height of particle i. Of course for the treatment of a continuous repartition of mass we change the $\sum$ into a $\int$ in all this discussion.

The concept of a force function first appeared in hydrostatics. It was noticed that the local condition of equilibrium of a liquid is that it is subjected to a force obtained from a force function (the reader should try to deduce this law). It was then remarked, at the end on the 18th century, that the gravitational forces between a system of bodies problem are obtained from a force function, and that this remark is useful when studying the stability of the solar system.

EXAMPLE 1. The equation is $m\ddot{q} = f$. If $f = dU$ we make $T = m\|\dot{q}\|^2/2$ and $H = T - U$ is constant.

EXAMPLE 2. The law is: $m\ddot{q} - dU$ is orthogonal to the surface (it is remarkable that we don't need to say if this dU is the gradient of U in E or of U restricted to the surface.) We write for any velocity tangent to the surface $m\langle\ddot{q},\dot{q}\rangle + \langle dU,\dot{q}\rangle = 0$ and deduce that $H = T - U$ is constant.

EXAMPLE 3. As we decided to set $f = 0$ the energy is simply the kinetic energy. We have $2T = \int m\langle\dot{q},\dot{q}\rangle\, \mathrm{d}^n q$ where $\dot{q} = \dot{q}_G + \omega(q - q_G)$. With the usual computations we get $2T = M\langle\dot{q}_G,\dot{q}_G\rangle + \mathrm{trace}(\omega b {}^t\omega)$.

EXAMPLE 4. It is like Example 3 but $q_G = 0$ is the fixed point. Then $2T = -\mathrm{trace}(\omega b\omega)$. In order to check our Euler-Poisson equation $\dot{A} = [A,\Omega]$ we write T using the pull-back to the space where the particles are fixed: $2T = -\mathrm{trace}(\Omega B\Omega)$ or using $A = -B\Omega - \Omega B$, $4T = \mathrm{trace}(\Omega A)$. Then $4\dot{T} = \mathrm{trace}(\dot{\Omega}A) + \mathrm{trace}(\Omega\dot{A})$ where we check that the two terms are equal, and, according to Euler-Poisson equation, equal to $\mathrm{trace}(\Omega A\Omega - \Omega^2 A) = 0$.

EXAMPLE 5. We have $2T = \int m\langle\dot{q},\dot{q}\rangle\, \mathrm{d}^n q = -\mathrm{trace}(\omega b_K\omega)$, or, using the usual formula $a = -b_K\omega - \omega b_K$, $4T = \mathrm{trace}(\omega a)$. We have $U = M\langle g, q_G\rangle$ if the gravitation vector g is constant. To check our formulas, we compute $4\dot{T} = \mathrm{trace}(\dot{\omega}a) + \mathrm{trace}(\omega\dot{a})$. Both terms are different, only the second involves the derivative of b_K. If we don't want to compute this derivative, we can use both versions of Equation $(\star)$. By the first,

$$\mathrm{trace}(\dot{\omega}a) = -\mathrm{trace}(\dot{\omega}b_K\omega + \dot{\omega}\omega b_K) =$$

$$= -\mathrm{trace}\Big(\omega(b_K\omega^2 - \omega^2 b_K - M(q_G - K)\wedge(g + \omega\dot{K}))\Big).$$

The first two terms give opposite traces. By the second

$$\mathrm{trace}(\omega\dot{a}) = \mathrm{trace}\Big(\omega(M(q_G - K)\wedge g - M\dot{K}\wedge\dot{q}_G)\Big).$$

The sum is

$$4\dot{T} = M\mathrm{trace}\Big(\omega(2(q_G - K)\wedge g + (q_G - K)\wedge\omega\dot{K} - \dot{K}\wedge\dot{q}_G)\Big).$$

As ω is antisymmetric we can replace the $\wedge$'s by $\otimes$'s and double the expression.

$$2\dot{T} = M\mathrm{trace}(2\dot{q}_G \otimes g + \dot{q}_G \otimes \omega\dot{K} - \omega\dot{K} \otimes \dot{q}_G) = 2M\langle \dot{q}_G, g\rangle.$$

So $\dot{T} = \dot{U}$ and the energy $H = T - U$ is constant, as expected.

9. Symmetry and first integrals. This is chronologically the second way to "explain" the first integrals of linear and angular momentum. They are associated to the translational or the rotational symmetry of the system. This explanation is widely known as Emmy Noether's theorem. It is a subject, as we said, which began with Newton (see §6.2) and that Lagrange developed theoretically (œuvres v.4 p. 401) and put into practice (see the examples in his Mécanique Analytique).

At the level of differential geometry, a vector field on the phase space which commutes with the vector field defining the dynamics defines a symmetry. There is another object, which is also preserved by the latter vector field, called the symplectic form. Contracting the symmetry vector field with the symplectic form may give the differential of a first integral. Thus the symplectic form may associate a symmetry with a first integral.

But the symplectic form does not always exist. First, the energy should exist. There should be a force function. Then, the constraints should be holonomic. All this is satisfied, for example, in the case of a system of rigid bodies in gravitational interaction. In §7, we could already "explain" the first integrals in this case and now we have a second explanation, which is completely different! It is easy to show artificial situations where one explanation works and the other don't work. But in true mechanical systems where the first integrals exist, both explanation work, or, as we will see, they both fail to explain anything.

Noether's theorem should be presented in the more advanced parts of mechanics which would be the natural continuation of these lectures. The reader may consult Arnold's book. The main goal of these lectures is to recall that one should learn the concepts from the simplest to the most advanced, and not skip intermediate steps. We can distinguish three levels which correspond to the historical developments.

1) Newton's laws
2) Virtual velocities and d'Alembert principles
3) Lagrange and Hamilton equations

Our presentation does not follow the order. We did not explain the first level at first. We considered the second level and sometimes compared it with the first level, mentioning the reaction forces in several remarks. Our first explanation for the linear and angular momentum clearly pertains to the first level, while Noether's theorem is at the third level.

Both explanation fail when we consider the Chaplygin ball. The angular momentum is there but there is no corresponding symmetry. Anyway we don't know *a priori* a symplectic form that could associate symmetries and first integrals, as the system is non-holonomic.

We did not develop our explanation about central forces controlling the rigidity. At the moment we cannot treat the contact point. Anyway, what is constant in Chaplygin ball is not the angular momentum with respect to a fixed point, but with respect to the moving contact point.

One can always develop explanations but if they become too specific they do not present any predictive value. The physical reality is a permanent source of surprises for the scientist who believes in a universal explanation of everything... The computation in §6.5 proving that the angular momentum is constant in the case of the Chaplygin ball is quite simple and may be adapted to many cases. It belongs to the second level, which is also the one which better describes the rolling phenomenon.

9. Predicting integrability. Consider Example 4, the free motion of Euler's top. The configuration space has dimension 3: it is the space of isometries from $\vec{F} \to \vec{E}$ respecting orientation. If by a convention we call an element of this space "identity", the space becomes a group $SO(3)$. The phase space is the space whose points are "configuration+velocity". Its dimension is 6. On this space are defined the energy first integral and the 3 components of the angular momentum vector. They are independent functions. Fixing them we obtain submanifolds or subvarieties of the phase space of dimension $6-4=2$. Now consider a rotation around the angular momentum "vector". It sends points of the phase space to points of the phase space keeping the same value for the energy and the angular momentum vector. So the rotation group $SO(2)$ acts on the above subvarieties, and we can define a quotient space. This quotient space is one dimensional. Our reduction equation is:

$$6-1-3-1=1 \quad \longrightarrow \quad \text{3D Euler's top is integrable.}$$

We say that the system is reduced to dimension 1. After quotient we have a one-dimensional autonomous ordinary differential equation. Such equations are integrable. The reader may consult my *Lectures on the two-body problem* for an example of the classical process of integration when the reduction succeeds.

Consider now Euler's top in dimension 4. The reduction equation is:

$$12 - 1 - 6 - 2 = 3 \quad \longrightarrow \quad \text{not enough reduced.}$$

The last 2 corresponds to the $SO(2)\times SO(2)$ group which preserves a generic angular momentum bivector in dimension 4. It happens that there is another first integral, that was discovered by Schottky in 1891, by using a short computation due to Frahm. We predicted the integrability of Euler's top in dimension 3 without considering the symplectic form. Here we can use Schottky's first integral to decrease our dimension by one, but we also need to associate to it a symmetry vector field by using the symplectic form. Then we decrease by two and our reduction equation becomes

$$12 - 1 - 6 - 2 - 1 - 1 = 1 \quad \longrightarrow \quad \text{4D Euler's top is integrable.}$$

Still this is not quite correct because we don't know if the Hamiltonian flow associated to Schottky's integral generates an $SO(2)$ symmetry or if its orbits are only quasi-periodic, which is more likely to happen. We should rather deduce that there is an integrable 2-dimensional torus action.

Now we should consider the n-dimensional Euler top. Even if our reduction equation becomes quite bad, the system is still integrable, as discovered by Manakov in 1976. There are many first integrals. Here, for the first time, one should consult 20th century literature to try to understand what is happening. The Lax pair point of view is usually considered as illuminating.

Chaplygin ball. The configuration space is five dimensional: we choose a contact point on the table and an orientation of the ball. There is a three dimensional choice for the velocity, as the contact point does not slide. So the phase space has dimension 8. We can fix energy and angular momentum. Then, we can pass to the quotient by the translations. We cannot rotate the system without changing the angular momentum (except if this "vector" is vertical). So our reduction equation is:

$$8 - 1 - 3 - 2 = 2 \quad \longrightarrow \quad \text{quite good.}$$

We arrive at an autonomous ordinary differential equation on a two dimensional manifold. A classical non-integrable example is Van der Pol equation. The qualitative behavior cannot be very complicated, because the orbits cannot "cross" each other on the two-dimensional manifold. This explains our "quite good". But it is even better than that, because Chaplygin discovered a volume form (or measure) which is invariant by the dynamics. Then, according to remarks due to Jacobi, and called by him theory of the last multiplier, the invariant measure resists to the process of reduction, and gives an invariant measure for the two-dimensional dynamics. The measure can be counted in the reduction equation.

$$8-1-3-2-1=1 \quad \longrightarrow \quad \text{3D Chaplygin top is integrable.}$$

About 4D Chaplygin top, the question is not solved. The reduction equation is

$$15-1-6-3-1=3 \quad \longrightarrow \quad \text{not enough reduced.}$$

The reader should consult Y.N. Fedorov, V.V. Kozlov, *Various aspects of n-dimensional rigid body dynamics*, AMS Translations 168 (1995) pp. 141–171, A.V. Borisov, I.S. Mamaev, *Rolling of a rigid body on plane and sphere. Hierarchy of dynamics*, Regular and Chaotic Dynamics 7 (2002) pp. 177–200 and other texts by these authors.

10. Holonomic or not? We spoke a lot about non-holonomic systems without never explaining what they are. At our level of exposition, their distinction from holonomic systems is not important. But let us try to explain this briefly. In a holonomic system there is a configuration space M and the velocity space at a configuration x is the tangent space T_xM. The dimension of the phase space TM is twice the dimension of M. In the rolling ball case the configuration space is 5-dimensional and the velocity space only 3-dimensional. So the system is non-holonomic.

But the presentation above is ambiguous. We should say what is the configuration space. We consider the 2D rolling ball, i.e. the rolling disk on a line. One could say, the configuration space is 2-dimensional, because as above we can choose the contact point and the orientation of the disk. Then the velocity choice is only 1-dimensional and the system is non-holonomic. This conclusion is wrong. Indeed we *can* restrict the study to a 1-dimensional configuration space, and we *shall* do it. If a point A of the border of the disk is in contact with a point B of the line, it is obvious from the non-sliding hypothesis that during a motion controlled by any force field the point B will never be in contact with another point than A (and the corresponding 3D affirmation is wrong). So the configuration is completely determined by the contact point. It is 1-dimensional. The constraint is holonomic.

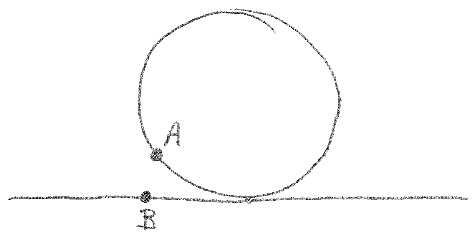

Figure 6: 2D rolling ball

To decide what is the true dimension of the configuration space, there is a nice test from differential geometry, related to Frobenius theorem. The interested reader will easily find references.

11. Note. It is obviously of first importance to know how to deduce the equations of motion of a given mechanical system. Curiously d'Alembert principle tends to disappear from mathematical textbooks in classical mechanics, and to be replaced by an exposition of less general principles, which involve more sophisticated mathematical concepts. We have nothing to say against the wonderful book "Mathematical Methods in Classical Mechanics" by V. Arnold. The author decided not to consider perfect non-holonomic constraints, but only holonomic constraints. He constructs the Lagrangian and deduces from it, at §19D, the laws of motion for this kind of systems. He explains clearly that this excludes the rolling ball in Remark 6, §21. Nevertheless he presents d'Alembert principle in several ways.

Another famous book of the 70's is R. Abraham and J. Marsden's "Foundations of Mechanics". Here the idea that one can deduce the equation of motion from a principle is absent. When concrete mechanical systems are presented their equation of motion or their Lagrangian are written without explanation. The book presents the mechanics as a chapter of symplectic geometry. Consequently the counter-example of non-holonomic constraints, which is well-known after the remarks of Hertz, is hidden.

Another result of similar ideologies is the modern presentation of the concept of integrability of a mechanical system. A definition is given from a set of first integrals in involution, in such a way that integrability is possible only for Hamiltonian systems. Nothing is said on the fact that integrability happens and may be described exactly in the same way when the notion of involution is not even defined, e.g. in the case of the usual rolling ball on a plane or for a generic ordinary linear differential system with constant coefficients. "What is a non-holonomic integrable system?" became a possible question, while it would be incomprehensible for Jacobi and other 19th century mathematicians. Again, Arnold was more reasonable. He presented the first integrals in involution as a sufficient condition for the integrability by quadratures. However, a more elementary and general point of view on integrability is presented and discussed in *A concept of Integrability of Dynamical Systems*, O. Bogoyavlenskij, C. R. Math. Rep. Acad. Sci. Canada 18 (1996) pp. 163–168.

For other such discussions see Sternberg's review of Abraham Marsden's book in Bulletin AMS, 2 (1980) or Borisov and Mamaev, "On the history of the development of nonholonomic dynamics", Regular and Chaotic Dynamics 7 (2001).

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