Contemporary Mathematics
Volume **198**, 1996

The symmetric central configurations of four equal masses

Alain Albouy

Abstract. In a relative equilibrium motion in the planar n-body problem, the configuration is called collinear central or planar central according to its dimension. We prove, by using a formal calculus program, that there are exactly three types of planar central configurations of four equal masses.

1. Introduction

We proved in [1] that every planar central configuration of four equal masses has at least one axis of symmetry. This conclusion was earlier conjectured by R. Moeckel; indeed, Moeckel even made a bet about this result based on his numerical experiments and even though the statement contradicted an earlier, opposing statement published by J. Palmore. This last statement has been contested later in [5].

Once the symmetry has been established, the next problem is to describe the set of symmetric configurations. There exist three types of such central configurations:
1) the four masses positioned at the vertices of a square,
2) three of the masses placed at the vertices of an equilateral triangle and the fourth one is located at the center of the triangle,
3) three of the masses are at the vertices of a particular isosceles triangle and the fourth one is located somewhere on the axis, inside the triangle.

It is clear that the configurations of the first two types are central. The only difficulty left is to prove that there is one and only one (type of) central configuration having just one axis of symmetry. The method that we present here is rigorous, but unfortunately much less satisfactory than the method used in [1] because it requires the use of a formal calculus program.

2. Preliminary remark on convex configurations

We can improve a result of [1] and prove that *the only planar convex central configuration of four equal masses is the square.* Let us recall that a configuration is called convex if none of the bodies is situated in the interior of the convex hull of the others. To prove our claim, it is sufficient to examine carefully the arguments of [1].

The key of that work is, without any doubt, the use of a certain system denoted (7). (See [1].) One can *a posteriori* notice that this system could be deduced very naturally from the equations of central configurations by forcing a certain invariance. Instead, we obtained it after [2], as the definition of the *configurations* *équilibrées* of four equal masses. These are the configurations of the motions of relative equilibrium in dimension six. There exist also motions of relative equilibrium in dimension four and of course two. In every case the configuration is "équilibrée".

The first proposition of [1] claims that *every planar convex central configuration of four equal masses is symmetric*. We obtained this result proving that the signs of the three terms of equation (7_1) are not compatible with the equation, when the configuration is not symmetrical. We make this argument more precise: (7_1) is only possible if its two members are zero, that is if $d = e$ and $b = c$, which also means $\Delta_3 = \Delta_4$. Now equation (7_4) cannot be realized neither, as one can check, when the configuration is not symmetrical, and not even if we set $\Delta_3 = \Delta_4$: we need $b = d$ and $e = c$, which also means $\Delta_1 = \Delta_2$. Equation (1) proves then, that $\Delta_1 = -\Delta_4$, and (6) leads to the conclusion: the configuration is a square.

3. Equations of central configurations

We consider n bodies in a Euclidean space of dimension $n-2$, given by the vectors $\vec{r}_1, \ldots, \vec{r}_n$. We express the constraint on the dimension of the *affine* space generated by the bodies by saying that there exist real numbers $\Delta_1, \ldots, \Delta_n$, not all zero, such that

$$\sum_{i=1}^{n} \Delta_i = 0, \tag{1}$$

$$\sum_{i=1}^{n} \Delta_i \vec{r}_i = 0. \tag{2}$$

These equations define the Δ_i *up to a factor* as soon as the $\vec{r}_i$ generate the space of dimension $n-2$. There exists a compatible definition of Δ_i as the oriented volume of the simplex generated by the points $1, \ldots, i-1, i+1, \ldots, n$, multiplied by $(-1)^i$. But this fixes an inadequate value of the free factor.

Let us use the squares of the mutual distances $s_{ij} = \|\vec{r}_i - \vec{r}_j\|^2$ and the Δ_i variables to write down the equations of central configurations. The geometrical relations between the s_{ij} and the Δ_i are:

$$\sum_{i=1}^{n} \Delta_i s_{ik} = \sum_{i=1}^{n} \Delta_i s_{il}, \qquad \text{for all } k \text{ and } l. \tag{3}$$

A configuration of n *equal* masses of dimension $n-2$ is central if and only if the s_{ij} and the Δ_i, which verify (1) and (3), also satisfy

$$s_{ij}^{-3/2} = \gamma + \nu \Delta_i \Delta_j. \tag{4}$$

for some real numbers γ and ν.

4. The case of a configuration of four equal masses

We know that such a configuration, if central, possesses an axis of symmetry which includes two of the bodies, for example 1 and 2. This symmetry is equivalent to the simple equality $\Delta_3 = \Delta_4$, as shown by equation (4). We denote

$$a = s_{12}, \quad f = s_{34}, \quad b = s_{13} = s_{14}, \quad d = s_{23} = s_{24}.$$

Equation (1) then becomes $\Delta_1 + \Delta_2 + 2\Delta_3 = 0$. We introduce, instead of the Δ_i, only one parameter t by writing

$$\Delta_1 = -t - 1, \quad \Delta_2 = t - 1, \quad \Delta_3 = 1.$$

System (3) then becomes

$$4b = f + (1-t)^2 a, \quad 4d = f + (1+t)^2 a,$$

and (4) becomes

$$\begin{aligned} a^{-3/2} &= \gamma + \nu(1-t^2), & b^{-3/2} &= \gamma - \nu(1+t), \\ f^{-3/2} &= \gamma + \nu, & d^{-3/2} &= \gamma - \nu(1-t). \end{aligned}$$

We put $a = 1$, which is a choice on the scaling of the configuration. We also put $f = z^2$, and we express γ and ν using both equations on the left side of the system above. We obtain

$$\nu = \frac{z^{-3} - 1}{t^2} \qquad \text{and} \qquad \gamma = z^{-3}\left(1 - \frac{1}{t^2}\right) + \frac{1}{t^2}.$$

We put

$$\begin{aligned} P(z,t) &= z^3 t^2 b^{-3/2} = (t-2)(t+1) + (2+t)z^3, \\ Q(z,t) &= 4b = z^2 + (1-t)^2. \end{aligned}$$

We must have

$$R(z,t) = P^2 Q^3 - 64 z^6 t^4 = 0.$$

An analogous manipulation with d shows that $R(z,-t) = 0$, so that the complete system is now written as

$$R(z,t) = R(z,-t) = 0,$$

or, forgetting the trivial solution with $t = 0$ (the square),

$$R_i(z,t) = R_p(z,t) = 0,$$

the polynomials R_i and R_p being defined by

$$R_i(z,t) = \frac{1}{2t}\big(R(z,t) - R(z,-t)\big), \quad R_p(z,t) = \frac{1}{2}\big(R(z,t) + R(z,-t)\big).$$

These two polynomials are, putting $u = t^2$,

$$\begin{aligned}
R_i =& 2(z^3-4)u^4 - 2(z^6-3z^5-5z^3+9z^2+14)u^3 \\
&+ 2(3z^7+8z^6-6z^5-6z^4-41z^3+30)u^2 \\
&+ 2(z^3-1)(3z^7+z^6+6z^5-12z^4-13z^3-27z^2+2)u \\
&+ 4(z^2-5)(z^3-1)^2(z^2+1)^2, \\
R_p =& u^5 + (z^6-10z^3+3z^2+24)u^4 \\
&+ (3z^8-5z^6-18z^5+3z^4+30z^3+33z^2-10)u^3 \\
&+ (3z^{10}-18z^8-6z^7-68z^6+84z^5+6z^4+58z^3-63z^2-52)u^2 \\
&+ (z^3-1)(z^2+1)(z^7-10z^5+3z^4+37z^3+18z^2-33)u \\
&+ 4(z^3-1)^2(z^2+1)^3.
\end{aligned}$$

We give now the decomposition as a product of irreducible factors of the resultant $S(z)$ in u of the two polynomials. We did not decompose z^3-1, to abbreviate.

$$\begin{aligned}
S =& -262144z^{12}(z^3-1)^4(z^2-3)(z^2+1)^3(z^{37}-61z^{34}+336z^{33}-240z^{32} \\
&+ 2052z^{31}-12120z^{30}+8400z^{29}-30456z^{28}+175113z^{27}-88548z^{26} \\
&+ 241040z^{25}-1364385z^{24}+338994z^{23}-1081984z^{22}+6241506z^{21} \\
&+ 642162z^{20}+2319507z^{19}-15790278z^{18}-12287376z^{17} \\
&+ 1386909z^{16}+11212992z^{15}+55894536z^{14}-19889496z^{13} \\
&+ 53738964z^{12}-128353329z^{11}+44215308z^{10}-172452240z^9 \\
&+ 160917273z^8-42764598z^7+217615248z^6-115440795z^5 \\
&+ 17124210z^4-139060395z^3+39858075z^2+39858075).
\end{aligned}$$

The factor z^2-3 corresponds to the equilateral triangle solution. Ilias Kotsireas noticed that only the factor $(z^2+1)^3$ disappears when we consider the analogous problem of 5 bodies in space. The last factor, of degree 37, gives the non-trivial solution, the one with only one axis of symmetry. Sturm algorithm proves that this factor has exactly three real roots, whose approximated values are:

$$z_1 = -1.41423178... \quad z_2 = 1.04689938... \quad z_3 = 1.71400032...$$

We exclude z_1, which is negative, and z_2, which gives as a common root u_2 of R_i and R_p the approximated value -4.18466433 which does not correspond to a real value of the t variable. Only z_3 remains, giving a value of t at about 2.11474891 and values of the sides

$$\begin{aligned}
\sqrt{a} &= 1 & \sqrt{b} &= 1.02230893... \\
\sqrt{f} &= 1.71400032... & \sqrt{d} &= 1.77760076...
\end{aligned}$$

Remark. Let us justify the order of elimination of the variables. The factor of degree 37 is intrinsically associated to the square root of the ratio f/a for the unique solution having just one axis of symmetry. The choice of this particular variable is not immaterial: all the other variables that we tried (u, f/a, γ...) are associated with polynomials of degree 37 or 74 with much bigger coefficients. Moreover, the choice of eliminating u rather than another variable at the last step of the computation seems to optimize the total degree of the resultant.

5. The problem of four equal vortices

The first studies of relative equilibria in the idealized problem of vortices are probably due to W. Thomson. Reference [6] discusses an astonishing method to obtain them, that we see at work in [3]. These relative equilibria are also (see [4]) the planar central configurations for the logarithmic potential: to define them, it is enough to replace by -1 the exponent $-3/2$ in the equations (4). The symmetry result [1] is still valid, and the research of configurations of four equal masses with just one symmetry is much simpler, becoming accessible to calculus by hand. One shows that such configuration does not exist. As a matter of fact, as the exponent moves along the interval $(-3/2, -1)$, the continuation of the non-trivial planar central configuration goes to the equilateral triangle, which becomes a "double root". The same phenomenon also appears with the $-1/2$ exponent. For the exponent -1, we get

$$\begin{aligned} R_i &= (f-3)(f-1+u), \\ R_p &= (2f-u-2)(f-u+1), \\ S &= 6(f-3)^2(f-1)f. \end{aligned}$$

We conjecture, with Carles Simó, that there is only one solution with just one axis of symmetry, for any negative value of the exponent, except for the two cases of degeneracy just mentioned.

I wish to acknowledge Robert Conte, for the precious information he gave me about the problem of vortices and Alain Chenciner for his participation on this work as well as on [1].

6. Bibliographical references

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IMCCE, 77, av. Denfert-Rochereau, 75014 Paris, France
E-mail address: albouy@imcce.fr