Note on the attraction of an ellipsoid in a spherical universe

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To Sergey Bolotin and Dmitry Treschev for their birthdays. Thanks for so many interesting discussions, including the ones which first informed me about the Newtonian mechanics in a spherical space.

Abstract. A classical theorem states that two confocal ellipsoids, each of them endowed with a surface distribution of mass which is called homeoidal, exert the same Newtonian force on an exterior point if they have the same total mass. We extend this theorem to the spherical geometry by adapting a forgotten proof by Chasles of the classical theorem. Compared to the existing results, this is a slightly stronger statement with a much shorter proof.

1. Introduction. In 1742, Maclaurin published this result (see [12], §641): *There are motions of uniform rotation of an incompressible liquid with constant density, only submitted to its own gravitation, where the surface of the liquid is a constant ellipsoid of revolution.*

The remarkable appearance of an exact ellipsoid excited the curiosity of the best mathematicians. New related statements and proofs were discovered during more than a century. In turn, these unexpected properties founded important branches of mathematical physics (potential theory, confocal ellipsoids, etc.).

Jacobi in 1834 extended the above statement to triaxial ellipsoids. These ellipsoidal relative equilibria are astonishing. However, they are a rather easy part of the theory. Ultimately, everything is deduced from this simple fact: A homogeneous layer delimited by two concentric homothetic ellipsoids does not attract points in its interior. This result, together with its simplest possible proof, was obtained by Newton for a "spheroid of revolution" ([14, 15], Book 1, Prop 91, Cor. C). The proof applies to any ellipsoid. Indeed, Newton's construction is essentially invariant by affine transformation.

The attraction of such a layer on the exterior domain is also computable, but with some difficulties. The following result was extended step by step from another corollary by Newton ([14], Prop 91, Cor. B), first generalized by Maclaurin, who was the first to define confocal ellipsoids ([12], §653). Legendre [10] extended the result. Soon after, Laplace [9] published a complete proof.

Two ellipsoids are said confocal if they have the same three planes of symmetry, and if their respective restrictions to these planes are confocal ellipses (that is, ellipses having both foci in common). We call a homogeneous ellipsoid the volume delimited by an ellipsoidal surface, endowed with a constant mass distribution.

Theorem 1. (Laplace, or Maclaurin, Legendre, Laplace) Two confocal homogeneous ellipsoids of the same total mass exert the same gravitational force on any exterior point (exterior to both of them).

This theorem extends the analogous statement about concentric spheres, discovered and proved by Newton (see [14], Prop. 74), and which is a main key in his deduction of the theory of the universal gravitation.

Legendre [11] reproved the result by adapting his previous argument. Poisson ([16], p. 499) considered this proof as a "démonstration plus directe, mais encore plus compliquée que celle que Laplace avait donnée auparavant"[1]. Let us recall that these three authors rank among the most skillful computers of their time, and that their time indeed produced the most extraordinary computational exploits. We will now tell how these complicated proofs became simple.

2. The proof by Chasles in 1838. A simplified proof by Ivory in 1809 was universally acclaimed. Ivory ([5], p. 355) discovered a reciprocity between interior and exterior attractions that Chasles clarified and simplified in 1838. The other deductions in his article, though not difficult, also required further simplifications. They concern the reduction of an integration on a volume to an integration on a surface. Poisson [16] in 1833 arrived at the correct conclusion that all the theory was indeed about the thin layers of the homogeneous ellipsoid, now called homeoids. *A homeoid is an ellipsoidal surface endowed with a surface distribution of mass. It is the limit of domains delimited by two ellipsoids, with a uniform density of mass inside, the second ellipsoid being homothetic and concentric to the first one, and tending to it while the total mass is kept constant*[2]. The following statement is the analogue of Theorem 1, with homogeneous ellipsoids replaced by homeoids.

Corollary 1. Two confocal homeoids of the same total mass exert the same gravitational force on any exterior point (exterior to both of them).

To deduce this statement from Theorem 1, we consider two homogeneous ellipsoids E and F as in Theorem 1, with confocal ellipsoidal boundaries $\bar{E}$ and $\bar{F}$. Let λ be a homothety factor, $0 < \lambda < 1$. Then $\lambda\bar{E}$ and $\lambda\bar{F}$ are confocal ellipsoids. The matter inside them, which belongs respectively to E and F, forms two homogeneous confocal ellipsoids that we call λE and λF. They exert two equal gravitational forces on any exterior point. As the force depends linearly on the mass, the "cavities" $E \setminus \lambda E$ and $F \setminus \lambda F$ exert equal gravitational forces on any exterior point. By passing to the limit $\lambda \to 1$, we see that the exterior forces are also the same in the case of two confocal homeoids. Reciprocally, to prove Theorem 1 from its homeoid version, it is enough to express the force exerted by a homogeneous ellipsoid as an integral in λ of forces exerted by homeoids. Theorem 1 and Corollary 1 are thus easily deduced one from the other.

Poisson [16] proved Theorem 1 through Corollary 1. He advertised Ivory's article but did not use it. He claimed to present a simple proof but yet his computation fills up 40 pages. His work was nevertheless influential. The geometers Steiner and Chasles soon reacted. Chasles proposed this new statement which concerns the Newtonian potential, which we always assume to be zero at infinity.

Corollary 2. ([3], §7) The gravitational potential of a homeoid is constant on any confocal ellipsoid.

[1] a more direct but even more complicated proof than the one Laplace had given earlier.

[2] This traditional definition of the homeoid may be simplified. We may say that the mass on any domain of the ellipsoidal surface is proportional to the volume of the cone constructed on this domain, with apex at the center. We may also say that the homeoid is the image by an affine transformation of the uniform spherical shell.

Corollary 2 is easy to deduce from Corollary 1. Indeed, the potential is constant inside the homeoid, since the force is zero. We know that the potential is continuous when crossing a surface distribution of mass. So, the potential exerted by a homeoid is constant on the homeoid. But this is also the potential of an inner confocal homeoid. QED

Can we deduce Theorem 1 or Corollary 1 from Corollary 2? Corollary 2 gives an incomplete description of the potential, which can be completed for example by using the Laplace equation, and either the outgoing flow of the force field or the potential at infinity. Such tools are well known but we will present a more elementary argument. We will see how the simplest deduction of Corollary 2 also gives Corollary 1. But let us mention another elegant argument. Corollary 2 could be proved by Steiner's argument [17]. Steiner deduced the direction of the force exerted by a homeoid from an astonishing observation: Given any exterior point P, there is a pairing of the small parts of the homeoid such that all the pairs exert at P a force in the same direction.

Chasles [2] proved Theorem 1 by adapting to homeoids Ivory's reciprocity. He discovered Lemma 1, that we can call the Ivory-Chasles algebraic Lemma, and deduced Lemma 2, that we can call the Ivory-Chasles integral Lemma.

Lemma 1. ([3], §2) Let $(a_0, b_0, c_0) \in (\mathbb{R}_+)^3$. Let two points $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $(X_0, Y_0, Z_0) \in \mathbb{R}^3$ satisfy

$$\frac{x_0^2}{a_0^2} + \frac{y_0^2}{b_0^2} + \frac{z_0^2}{c_0^2} = 1, \quad \frac{X_0^2}{a_0^2} + \frac{Y_0^2}{b_0^2} + \frac{Z_0^2}{c_0^2} = 1.$$

Let $(a_1, b_1, c_1) \in (\mathbb{R}_+)^3$. Consider the images of both points by the diagonal linear transformation $\mathcal{A} : (x, y, z) \mapsto (a_1 x/a_0, b_1 y/b_0, c_1 z/c_0)$. Their respective coordinates are $x_1 = a_1 x_0/a_0, \dots, Z_1 = c_1 Z_0/c_0$. We have

$$\frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} + \frac{z_1^2}{c_1^2} = 1, \quad \frac{X_1^2}{a_1^2} + \frac{Y_1^2}{b_1^2} + \frac{Z_1^2}{c_1^2} = 1.$$

If the confocality condition $a_0^2 - a_1^2 = b_0^2 - b_1^2 = c_0^2 - c_1^2$ is satisfied, then

$$(x_0 - X_1)^2 + (y_0 - Y_1)^2 + (z_0 - Z_1)^2 = (x_1 - X_0)^2 + (y_1 - Y_0)^2 + (z_1 - Z_0)^2.$$

Proof. ([3], note 1) We replace $X_1, Y_1, Z_1, x_1, y_1, z_1$ by their expression and expand. Clearly $x_0 X_1 = x_1 X_0$, etc. The difference between the left-hand side and the right-hand side is

$$(a_0^2 - a_1^2)\frac{x_0^2 - X_0^2}{a_0^2} + (b_0^2 - b_1^2)\frac{y_0^2 - Y_0^2}{b_0^2} + (c_0^2 - c_1^2)\frac{z_0^2 - Z_0^2}{c_0^2} = 0.$$

QED

The usual statement of Lemma 1 is in terms of confocal hyperboloids. But Chasles [2] remarked that the hyperboloids are useless for proving Theorem 1: "Je me propose seulement de présenter une nouvelle solution différente de la première, qui n'exige pas comme celle-ci la connaissance de plusieurs propriétés nouvelles des surfaces du second degré."[3]

The *corresponding* points were defined by Ivory as two points obtained from each other by the diagonal linear transformation $\mathcal{A}$ which sends one of the confocal ellipsoids onto the other one. What we call a diagonal linear transformation is a map of the form $(x, y, z) \mapsto (\alpha x, \beta y, \gamma z)$, with $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$.

[3] I propose only to present a new solution, different from the first, which does not require like the first the knowledge of several new properties of surfaces of the second degree.

Lemma 2. (Ivory, Chasles) The potential of a homeoid H_1 at a point P_0 is the same as the potential of the confocal homeoid H_0 of the same total mass as H_1, passing through P_0, evaluated at the point $P_1 \in H_1$ *corresponding* to P_0.

Proof. The gravitational potential is an integral of the form $\iint \sigma/r$. Lemma 1 shows that the distance r to the attracting point is the same in both integrals. Moreover, the surface distributions of mass σ on both homeoids "correspond" each other. This is due to the commutation of the homothety, which defines the homeoid, with the diagonal linear transformation, which defines the correspondence and sends constant density on constant density. More precisely, this argument shows the proportionality $k\sigma_0 = \mathcal{A}^*(\sigma_1)$, for some $k \in \mathbb{R}_+$, of the surface distribution σ_0 on the ellipsoid with parameters (a_0, b_0, c_0) and of the pull-back by the diagonal linear transformation $\mathcal{A}$ of the surface distribution σ_1 of the other ellipsoid. But the equality of the total masses $\iint \sigma_0 = \iint \sigma_1$ shows that $k = 1$. QED

Lemma 2 may be extended to other force laws. This was observed by Poisson in 1812 (see [16]) about the reciprocity which concerns homogeneous ellipsoids. The following elegant argument by Chasles, which introduces a third homeoid, is not well known.

Proof of Theorem 1. ([3], §8) As we said, it is enough to prove Corollary 1. We compare the attraction of two confocal homeoids H_0 and H_1 of the same total mass. Instead of the force each of them exerts at an exterior point P, we compare their potentials at P. Consider the unique homeoid H_2 passing through P, confocal to H_0 and to H_1, and of the same total mass. Call $P_0 \in H_0$ and $P_1 \in H_1$ the points corresponding to P. The potential of H_2 is the same at P_0 and at P_1 since this is the interior potential. Thus by Lemma 2 the potentials of H_0 and H_1 at P are equal. QED

Theorem 1 is a beautiful part of the theory of the attraction of ellipsoids. It raises a lot of questions, most of them of analytical nature. The most natural questions are answered in the parts of Chasles's paper which we did not reproduce. The literature is extremely rich, with thousands of pages of review papers and books. Except for the recent work of history [13], we do not know any publication which reproduces Chasles's argument.

3. Laplace's theorem in a spherical universe. The idea of a Newtonian gravitation where our familiar 3-dimensional Euclidean space is replaced by a 3-dimensional sphere inspired many interesting studies, already in the 19th century. Charles Graves [4] discovered in 1842 that "A material point may be made to describe a spherical conic if it be urged by a force, acting along the arc of a great circle drawn from the *focus* to the point, and varying inversely as the square of the sine of the vector arc."

Wilhelm Killing [7] published in 1885 a systematic study of the classical mechanics in an n-dimensional space of constant curvature. He obtained the analogue of Corollary 2 for the spherical law of force: Indem man auf zwei confocalen Ellipsoiden solche Punkte einander zuordnet, welche in $n-1$ elliptischen Coordinaten übereinstimmen, gelangt man in bekannter Weise zu dem Satze: "Zu einer unendlich dünnen, von ähnlichen Ellipsoiden begrenzten Schicht ist jedes confocale Ellipsoid eine Niveaufläche"[4]. We may remark that Killing ignores [3] in two ways. He generalizes Corollary 2 instead of the stronger Corollary 1, while, as we will confirm, the stronger form is as easy to prove. He defines the diagonal linear transformation through the elliptical coordinates, while Chasles advises to remove them from the proofs.

In 2000, V.V. Kozlov [8] considered the attraction of a spherical shell and of a segment in spherical geometry. He then stated the analogue of Corollary 2, giving a more accurate

[4]By assigning to two confocal ellipsoids those points with $n-1$ coinciding elliptical coordinates, one arrives in the known manner at the theorem: "For an infinitely thin layer bounded by similar ellipsoids, each confocal ellipsoid is a level surface".

description of the homeoids and of the question of the antipode of an attractor. The same year, C. Velpry [18] also discussed the spherical shell and the antipodes, in a different way.

The following analogue of the Chasles-Ivory algebraic lemma (Lemma 1) was published by Izmestiev and Tabachnikov. Here we simplify the statements and proofs in [6] (Lemma 4.10 and Remark 4.11) by removing the interesting facts which we do not use, which concern the $n-1$ confocal spherical quadrics. We omit the constant negative curvature space for which some remarks are needed (see [6], Lemma 4.12). We present the results in dimension $n=3$, but our argument works in any dimension.

Lemma 3. **[6]** Let $(a_0,b_0,c_0,h_0)\in(\mathbb{R}_+)^4$. Let two points $(x_0,y_0,z_0,w_0)\in\mathbb{R}^4$ and $(X_0,Y_0,Z_0,W_0)\in\mathbb{R}^4$ satisfy

$$\frac{x_0^2}{a_0^2}+\frac{y_0^2}{b_0^2}+\frac{z_0^2}{c_0^2}-\frac{w_0^2}{h_0^2}=0,\quad \frac{X_0^2}{a_0^2}+\frac{Y_0^2}{b_0^2}+\frac{Z_0^2}{c_0^2}-\frac{W_0^2}{h_0^2}=0,$$

$$x_0^2+y_0^2+z_0^2+w_0^2=1,\quad X_0^2+Y_0^2+Z_0^2+W_0^2=1.$$

Let $(a_1,b_1,c_1,h_1)\in(\mathbb{R}_+)^4$. Let the diagonal linear transformation $\mathbb{R}^4\to\mathbb{R}^4$, $(x,\dots,w)\mapsto(a_1x/a_0,\dots,h_1w/h_0)$ be applied to both points: $x_1=a_1x_0/a_0,\dots,W_1=h_1W_0/h_0$. We get

$$\frac{x_1^2}{a_1^2}+\frac{y_1^2}{b_1^2}+\frac{z_1^2}{c_1^2}-\frac{w_1^2}{h_1^2}=0,\quad \frac{X_1^2}{a_1^2}+\frac{Y_1^2}{b_1^2}+\frac{Z_1^2}{c_1^2}-\frac{W_1^2}{h_1^2}=0, \tag{A}$$

and, under the confocality condition $-h_1^2+h_0^2=a_1^2-a_0^2=b_1^2-b_0^2=c_1^2-c_0^2$,

$$x_1^2+y_1^2+z_1^2+w_1^2=1,\quad X_1^2+Y_1^2+Z_1^2+W_1^2=1, \tag{B}$$

$$(X_1-x_0)^2+\cdots+(W_1-w_0)^2=(x_1-X_0)^2+\cdots+(w_1-W_0)^2. \tag{C}$$

Proof. Let us check (B). There is a $\gamma\in\mathbb{R}$ with $a_1^2=\gamma+a_0^2$, $b_1^2=\gamma+b_0^2$, $c_1^2=\gamma+c_0^2$, $h_1^2=-\gamma+h_0^2$. So $a_1^2x_0^2/a_0^2+\cdots+h_1^2w_0^2/h_0^2=\gamma(x_0^2/a_0^2+y_0^2/b_0^2+z_0^2/c_0^2-w_0^2/h_0^2)+x_0^2+\cdots+w_0^2=1$. We get (C) by the obvious expansion. QED

Let us consider this other way of defining the surface distribution of mass on a homeoid in the Euclidean case. We may define it by a *field of 1-forms* which is tangent to the surface. Such a field, when contracted with a constant element of volume of $\mathbb{R}^3$ (a 3-vector), gives an element of area on the surface. Consequently, a *function f having the given surface as a level surface* defines such a field of 1-forms: the differential $\mathrm{d}f$ of the function. In this way, *for any $\lambda\in\mathbb{R}_+$, the quadratic form $\lambda(x^2/a^2+y^2/b^2+z^2/c^2)$ defines a distribution of mass that makes the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ a homeoid.*

In the spherical case, Kozlov [8] defined the homeoid with nonzero parameters (a,b,c,h) as follows. Consider the two functions of $(x,y,z,w)\in\mathbb{R}^4$, $f=x^2/a^2+y^2/b^2+z^2/c^2-w^2/h^2$, $g=x^2+y^2+z^2+w^2$. The equation of the ellipsoid is $f=0$, $g=1$. The surface distribution of mass is defined by $\mathrm{d}f\wedge\mathrm{d}g$. This 2-form contracted with a constant element of volume of $\mathbb{R}^4$ gives the element of area on the surface. The total mass of the homeoid may be changed for example by changing this element of volume. The proof of the following Lemma is extensively discussed in [6], §4.3. Newton's argument for the interior force works.

Lemma 4. **[7]** In spherical geometry, the spherical gravitational potential is constant inside a homeoid (inside means that the indefinite quadratic form above denoted by f is negative).

Lemma 5. **[6]** In spherical geometry, the surface distributions of mass of two confocal homeoids correspond each other through the diagonal linear transformation, if their total mass is the same.

Proof. Let $f_0 = x^2/a_0^2 + y^2/b_0^2 + z^2/c_0^2 - w^2/h_0^2$, $f_1 = x^2/a_1^2 + y^2/b_1^2 + z^2/c_1^2 - w^2/h_1^2$ and $g = x^2 + y^2 + z^2 + w^2$. The surface distribution on the homeoid $f_i = 0$, $g = 1$, $i = 0$ or $i = 1$, is defined, up to a constant factor, by $\mathrm{d}f_i \wedge \mathrm{d}g$. According to the proof of Lemma 3, the pull-back of f_1 is f_0, the pull-back of g is $\gamma f_0 + g$. Consequently the pull-back of $\mathrm{d}f_1 \wedge \mathrm{d}g$ is $\mathrm{d}f_0 \wedge \mathrm{d}g$. The total mass fixes the constant factor as in the Euclidean case. QED

Lemma 6. In spherical geometry, the potential of a homeoid H_1 at a point P_0 is the same as the potential of the confocal homeoid H_0 of the same total mass as H_1, passing through P_0, evaluated at the point $P_1 \in H_1$ corresponding to P_0.

Proof. The statement is the same as Lemma 2 and the deduction from Lemma 3 and Lemma 5 is the same. The potential is here $\iint \sigma \cot \rho$ where ρ is the angular distance to the attracting point, and σ the surface distribution of mass on the homeoid. The cotangent of ρ is a function of the quantity appearing in (C), that is, the square of the distance to the attracting point. QED

Theorem 2. In spherical geometry, two confocal homeoids of the same total mass exert the same spherical gravitational force on any exterior point (exterior to both of them; exterior means that the indefinite quadratic form above denoted by f is positive).

The proof is exactly the same as in the Euclidean case. For more information about this theory, we recommend the review part of [1], which presents old and recent results on classical mechanics in spaces of constant curvature.

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