INTEGRAL MANIFOLDS OF THE N-BODY PROBLEM

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Introduction

The *integral manifolds* of a differential system are the invariant manifolds obtained by fixing the value of the known first integrals. In the Newtonian problem of n bodies moving in a 3-dimensional space with their center of mass fixed at the origin, these integrals are the three components of angular momentum and energy.

Part D of this paper (which should be read after Parts A and B) gives a description of the integral manifold of the 3-body problem indicating that there are exactly 8 values of the energy for which its topology changes, assuming one fixes the angular momentum at an arbitrary nonzero value and decreases the energy from $+\infty$ (the first value encountered is zero, and the three masses are assumed distinct). But this description does not constitute a rigorous proof.

Part C (whose only prerequisite is the beginning of B) proves that there are at most 8 values. Two distinct phenomena are responsible for these changes in topology: the *critical points* of the energy function restricted to the chosen level manifold of angular momentum, and the *critical points at infinity* of the same function. The former are responsible for the last 4 values encountered as the energy decreases from $+\infty$. The critical points are the *relative equilibria* of Lagrange. Part C is devoted to the study of the latter phenomenon in the general setting of the spatial n-body problem. We show that this phenomenon is associated with a division of the system of particles into clusters, and that each cluster is a relative equilibrium. The critical points at infinity produce at most 26 changes of topology in the 4-body problem (counting zero), and it is only ignorance of the relative equilibria of more than three bodies that prevents an enumeration for five bodies.

Part B studies the level manifolds of angular momentum of a system of n bodies, and treats in a global way the question of *reduction or elimination of the node*. In Part B4 we sketch the explicit description of the manifolds for an ambient space of dimension greater than three.

The problem of the topology of integral manifolds is raised in Birkhoff [1], page 287, where the author ignores the phenomenon of critical points at infinity. Wintner [1] (§438 and p. 433) and Alexeyev [1] emphasize the importance of the problem. Smale [2], p. 50, points out the shortcomings of

Birkhoff's analysis, and proves that in the planar n-body problem, all changes in topology are due to critical points (except at zero energy). Smale also suggested to Cabral that he investigate the topology of integral manifolds in the spatial n-body problem. (Cabral [1] will be cited several times in this thesis; cf. his commentary on p. 61 of that reference.) Simó was the first to give the three additional values for the three-body problem, but his description, based on a projection onto the space of triangular configurations, has a singularity when the bodies are aligned, and it artificially introduces a ninth value which corresponds to a vertical triangle whose orthocenter is its (weighted) center of gravity (it is called "critical" or "singular" in Saari [1], who also raises questions about it). Finally, the existence of critical points at infinity is affirmed without explanation in Arnold, Kozlov, and Neishtadt [1], p. 104.

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A. Integral Manifolds of the Planar Problem

A1) Reduction of the Center of Mass and the Space of Dispositions

Consider n point-masses $m_1, \dots, m_n$ situated on the real line at $x_1, \dots, x_n$. The moment of inertia $I = \sum_{i=1}^n m_i x_i^2$ with respect to the origin defines a quadratic form which gives R^n the structure of a Euclidean vector space*.

We will denote by $\mathcal{D}$ the hyperplane in R^n defined by $\sum_{i=1}^n m_i x_i = 0$. A point of $\mathcal{D}$, called a "disposition," represents the projection onto an axis of a configuration of n bodies with center of gravity fixed at the origin, or the projection of their n velocities or accelerations. We will usually denote an element of $\mathcal{D}$ by a capital letter, or by its n coordinates (which include an "excess" coordinate). The symbol $\langle X, Y\rangle$ will denote the scalar product of the dispositions X and Y, and $\|X\|$ will denote the norm of X.

A2) First Integrals of the Planar Problem

We now select a system (O, x, y) of orthonormal axes in the plane. By placing the origin at the barycenter of the n bodies, we may describe the state of the system with 4 dispositions:

$$X = (x_1, \dots, x_n) \qquad Y = (y_1, \dots, y_n)$$

* We ignore the canonical structure here; in particular, I=1 defines a sphere, not an "ellipsoid."

$$P = (p_1, \dots, p_n) \qquad Q = (q_1, \dots, q_n)$$

where the i^{th} body has position coordinates x_i, y_i and velocity coordinates p_i, q_i. The energy h and the angular momentum C are then:

$$(e) \qquad \frac{1}{2}(\|P\|^2 + \|Q\|^2) - U(X,Y) = h,$$

$$(m) \qquad \langle X, Q\rangle - \langle Y, P\rangle = C,$$

where $U(X,Y)$ designates the potential

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \qquad \text{with} \qquad r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2.$$

A3) The Geometry of the Integral Manifolds (C nonzero)

The submanifolds of $\mathcal{D}^4$ defined by the system of equations $(e), (m)$ are called the integral manifolds of the problem.

It is worth noting that the energy is a simple function of the velocities, but not of the positions. Since Smale's work [1], an integral manifold is described by first fixing the positions *up to homothety* (since U is homogeneous), then finding the compatible size and velocities.

To see this, note that if C is strictly positive, equations (e) and (m) are equivalent to (en) and (m), where (en) is given by

$$(en) \qquad h = \frac{1}{2}(\|P\|^2 + \|Q\|^2) - \frac{U}{C}\big(\langle X, Q\rangle - \langle Y, P\rangle\big),$$

which may also be written

$$\Big\|P + \frac{U}{C}Y\Big\|^2 + \Big\|Q - \frac{U}{C}X\Big\|^2 = (\|X\|^2 + \|Y\|^2)\frac{U^2}{C^2} + 2h.$$

When X and Y are fixed, this equation defines a $2n-3$-dimensional sphere in velocity space (P,Q), of radius ρ and center $\mathrm{A} = (-YU/C, XU/C)$, such that

$$\rho^2 = \frac{IU^2}{C^2} + 2h, \qquad \text{where} \qquad I = \|X\|^2 + \|Y\|^2.$$

But (en) is homogeneous of degree zero in (X,Y).

Therefore, having chosen a nonzero couple (X,Y), we will interpret the intersection of the "sphere" (en) and the half-space

$$\big\{(P,Q)/\langle X, Q\rangle - \langle Y, P\rangle > 0\big\}$$

as the set of velocities (P,Q) compatible with the existence of a strictly positive real number λ such that the state $(\lambda X, \lambda Y, P, Q)$ solves the system $(e), (m)$.

The following remark clarifies the position of the sphere (en). The Lagrange-Jacobi equation (derived in a remark in Part C2) may be written

$$\ddot{I} = \|P\|^2 + \|Q\|^2 + 2h.$$

The condition $\ddot{I} = 0$ defines a second sphere of codimension one in velocity space, centered at the origin and of radius $\sqrt{-2h}$. This sphere is *perpendicular* to the sphere (en). To see this, let M be a point of their intersection. Then $\|\mathrm{OA}\|^2 = \|\mathrm{OM}\|^2 + \|\mathrm{MA}\|^2$, so that the two tangents MO and MA are orthogonal. M exists whenever the two spheres exist, in other words whenever $h \leq 0$ and $\rho \geq 0$. The sphere defined by $\ddot{I} = 0$ will be called the *virial sphere*, since the virial theorem of statistical mechanics assures that the relation $\ddot{I} = 0$ is satisfied on average by a stationary cluster of n bodies.

Figure 1 represents these spheres in the case $n = 2$, $h < 0$.

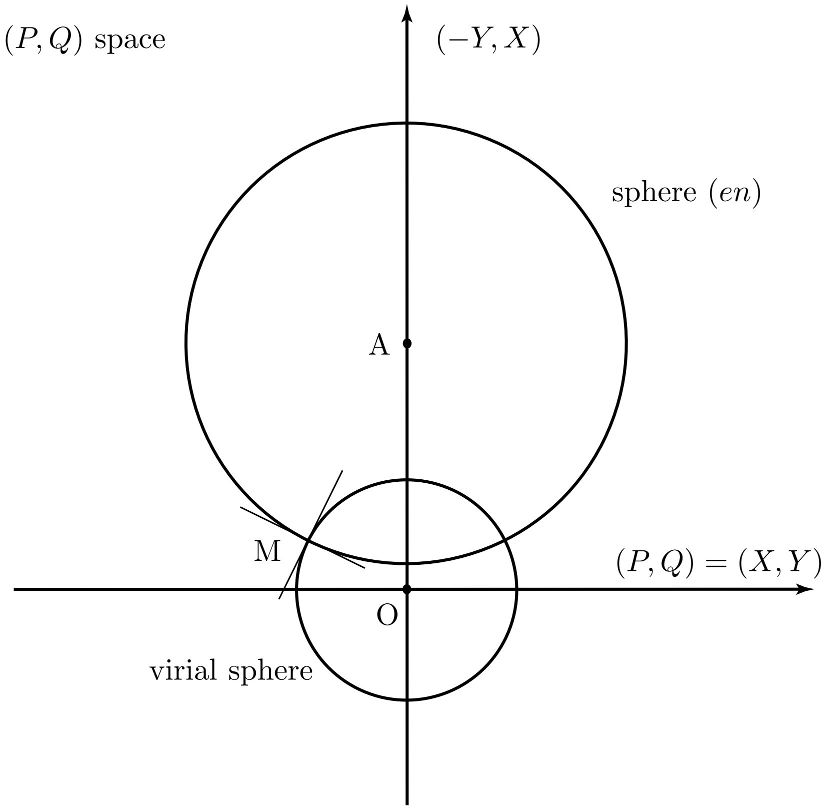

Figure 1

A4) The Topology of the Integral Manifolds (C nonzero, h negative)

If h is strictly negative, the sphere (en) is not truncated upon intersection with the half-space considered above. To describe an integral manifold, it suffices to study the condition $\rho \geq 0$, or

$$\sqrt{I}U \geq \sqrt{-2hC^2}.$$

We use the following terminology taken from the restricted problem. Given h and C,

— a connected component of (X, Y)-space, with the origin deleted, such that $0 \leq \rho < +\infty$ is called a *Hill's region*,

— the manifolds defined by the condition $\rho = 0$ are called the *zero velocity surfaces* (a slightly abusive terminology).

These objects, like the function ρ, are invariant under the group action of similarity transformations on configurations

$$(X, Y) \mapsto (aX + bY, -bX + aY), \text{ with } (a, b) \neq (0, 0).$$

The study of the Hill's regions is now classical. It consists in studying the function $\sqrt{I}U$ on $\mathcal{D}^2$ with the origin deleted (or on the sphere $I = 1$, or on the quotient of $\mathcal{D}^2$ by the group of similarity transformations, which is $P_{n-2}(\mathtt{C})$, the complex projective space), and also with the collision set ($x_i = x_j$ and $y_i = y_j$ for at least one pair i, j) deleted. The critical points are the unidimensional and bidimensional *central configurations*. One may also say that they are the configurations of the *relative equilibria* of the planar problem.

If $\sqrt{-2hC^2} \leq \min_{X,Y} \sqrt{I}U$, the Hill's region covers the whole space $\mathcal{D}^2$ with the collision set deleted.

The integral manifold is then "swept" as follows:

i) on the $2n-3$-dimensional sphere described by $\|X\|^2 + \|Y\|^2 = 1$, we choose a point in Hill's region, which represents a position up to homothety,

ii) "above" this point, there is a $2n-3$-dimensional velocity sphere (en), unless the point belongs to a zero velocity surface, in which case the sphere degenerates to a point.

From this description we infer that the integral manifold cannot change topology upon variation of h and C, unless $\sqrt{-2hC^2}$ passes through a critical point of the function $\sqrt{I}U$. Theorem E of Smale [2] makes this inference rigorous. Alternately, we could show that there are no critical points at infinity (which we shall do in part C3), then make use of the diffeomorphism from the introduction to part C.

The integral manifold is described in the simplest possible way in the case where $\sqrt{-2hC^2} < \min_{X,Y} \sqrt{I}U$. It is a sphere bundle above a sphere which is restricted to the complement, in the base, of a family of spheres of codimension 2 (the collisions). The quotient by the group $SO(2)$ of rotations about the origin is a sphere bundle above the complex projective space mentioned before, which is again restricted to the complement of a family of complex projective hyperplanes (collisions). In order to be more precise, one must characterize the sphere bundles considered. For this purpose it is useful to study separately the level manifolds of angular momentum, which are much more "geometric," and to consider the energy h as a function on these manifolds. Above the sphere, or above complex projective space, the angular momentum defines vector bundles whose associated unit sphere bundles are the bundles described above. We will see in Part B that the bundle above the sphere (before reduction) is trivial, but that the bundle above complex projective space (after reduction), previously described in Proposition 9.4 of Smale [1], is not, except in the 3-body case. The bundle restricted to the complement of the collisions is trivial even after reduction: by removing a projective hyperplane corresponding to the collision of two particles, the base is made contractible, and so one obtains a trivialization.

B. Angular Momentum

We are interested here in the submanifolds of the state space of a system of n bodies evolving in $\mathtt{R}^p$, obtained by fixing the angular momentum.

B1) Abstract Study

The space $\mathtt{R}^p$ is equipped with a system of orthonormal axes $(\mathrm{O}, x^1, \ldots, x^p)$. We again place the origin at the barycenter of the n bodies, and we describe the state of the system with $2p$ dispositions $X_1, \ldots, X_p$, $P_1, \ldots, P_p$. The first p are the projections of the configuration onto the axes, the second p are the projections of the velocities. We next choose an orthonormal basis for the space $\mathcal{D}$ of dispositions. The state of the system is then described by a matrix M of p rows and $2n-2$ columns. The i^{th} row is made up of the $n-1$ coordinates of X_i, followed by the $n-1$ coordinates of P_i. Symbolically, we write

$$M = \begin{pmatrix} X_1 & P_1 \\ \vdots & \vdots \\ X_p & P_p \end{pmatrix}.$$

Let O be a matrix of the group $O(p)$ of isometries of $\mathtt{R}^p$: ${}^tOO = I$, where I is the identity matrix. The transformation $M \mapsto OM$ describes the natural action of isometries on the system.

To each pair (i, j), we first associate the vector field:

$$\dot{X}_i = X_j, \quad \dot{P}_i = P_j, \quad \dot{X}_j = -X_i, \quad \dot{P}_j = -P_i, \quad \dot{X}_k = \dot{P}_k = 0$$

for k different from i and j. We also associate to (i, j) the first integral:

$$C_{ij} = \langle X_j, P_i \rangle - \langle X_i, P_j \rangle.$$

These fields generate the action of the group $SO(p)$ of rotations on $\mathtt{R}^p$, and the C_{ij} are the entries of the antisymmetric matrix

$$C = MJ{}^tM \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

We call the image of M the *motion space*, and denote it ImM. This space (and therefore its dimension rk$M$, the *rank* of the matrix M) is invariant with time. It is the vector subspace of $\mathbb{R}^p$ generated by the columns of M, or equivalently, by the n positions and n velocities of the bodies. We call the image of C the *fixed space*, thus generalizing the notion of the fixed plane of the spatial problem. It is immediate that Im$C \subset$ ImM.

Let us find the critical points of the map $M \mapsto C$. They are the points where a linear combination of the $\frac{1}{2}(p-1)p$ differentials of the functions C_{ij}, $1 \leq j < i \leq p$ vanish (we *assume* for simplicity that $2(n-1)p \geq \frac{1}{2}(p-1)p$). The same relation again exists among the vector fields associated above to the various pairs (i,j), which are the Hamiltonian fields associated to the fields C_{ij}. This linear combination generates the action of a one-parameter subgroup of $SO(p)$, and this action necessarily leaves fixed the state represented by M.

Proposition 1*. The point of the state space represented by the matrix M is a critical point of the angular momentum map if and only if rk$M \leq p-2$, in other words if the motion space is contained in a space of codimension 2 in $\mathbb{R}^p$.

Proof. The action of the one-parameter subgroup must leave the columns of M fixed: they all belong to the subspace of $\mathbb{R}^p$ invariant under the action, of codimension at least 2. Conversely, if the columns lie in a space of codimension 2, the group $SO(2)$ of rotations around this "axis" fixes M.

The angular momentum may be "reduced" by defining a "reduced" manifold, obtained from the level manifold of C by taking the quotient by the action of the subgroup of $SO(p)$ or of $O(p)$ which preserves the angular momentum ($OC^tO = C$). The quotient by $SO(p)$ is nonsingular (i.e., has no fixed points) when restricted to the intersection of the inverse image of the matrix C and the open set defined by rk$M \geq p-1$. For the quotient by $O(p)$, we must impose rk$M = p$.

But fixing the value of the angular momentum already imposes restrictions on the rank of M. For example, in a collinear motion of n bodies, the angular momentum vanishes. There is also Dziobek's Theorem (Wintner [1], p. 427): if the angular momentum vanishes, the 3-body problem is planar (rk$M$ is at most 2). More generally, we have

Proposition 2. In a level manifold of the angular momentum corresponding to a matrix C of rank $2k$, the rank of M satisfies the double inequality $2k \leq$ rk$M \leq k+n-1$.

Proof. The formula $C = MJ^tM$ shows that, on $\mathbb{R}^p$, C defines a skew bilinear form which is the *inverse image* of the canonical symplectic form of the space $\mathcal{D}^2$ of pairs (X,P) of dispositions associated to the matrix J. The quotient

* Cf. Cabral [1], p. 63 and p. 69, for an example of a situation where the critical level set is not a differentiable manifold.

$\mathbb{R}^p/\mathrm{Ker}^tM$ is bijectively related to Im^tM: the rank of C is also the rank of the restriction to this latter space, generated by the rows of M, of the 2-form J. In $\mathbb{R}^{2n-2}$, once the rank $2k$ of the symplectic form restricted to a subspace is given, the dimension of this subspace may take on all values between $2k$ and $k+n-1$: the argument is a classical use of symplectic orthogonality.

Example of an Application. A level manifold of the angular momentum in the vector space of matrices with p rows and $2n-2$ columns is regular (i.e., contains no critical points) if $p = 2k$ or $p = 2k+1$. The manifold reduced by the action of $SO(p)$, in other words the quotient of the level manifold by the action of the subgroup of $SO(p)$ which preserves the angular momentum, is also regular.

B2) The Normal Form

We have seen that an isometry O transforms M into OM, and consequently changes $C = MJ^tM$ into OC^tO. It is well known that in this way the anti-symmetric matrix C may be reduced to the normal form

$$(\star) \qquad \begin{pmatrix} 0 & -D & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where D designates a diagonal matrix with k strictly positive diagonal entries. To see this, one diagonalizes the symmetric matrix C^2 with respect to an orthonormal basis, then notices that if v is an eigenvector of this matrix, Cv is also an eigenvector corresponding to the same eigenvalue. In the degenerate case, one begins by decomposing $\mathbb{R}^p$ into the direct orthogonal sum of the kernel of C and its image, the fixed space.

Proposition 3. There exists a decomposition of the fixed space into the direct sum of m orthogonal subspaces $\sigma_1, \ldots, \sigma_m$ of respective dimensions $2k_1, \ldots, 2k_m$ and there exist m positive numbers $c_1 < \cdots < c_m$ such that the map $v \mapsto \frac{1}{c_j}Cv$ is a rotation by $\frac{\pi}{2}$ in the space σ_j (multiplication by i of a *complex structure*). The subgroup G of $O(p)$ consisting of isometries which leave angular momentum invariant is isomorphic to $U(k_1) \times \cdots \times U(k_m) \times O(p-2k)$.

Proof. The σ_j are the eigenspaces of C^2, and the $-c_j^2$ are the associated eigenvalues. An element O of G necessarily leaves invariant the decomposition of $\mathbb{R}^p$ into $m+1$ orthogonal subspaces: $OC^2{}^tO = C^2$. It belongs to $O(2k_1) \times \cdots \times O(2k_m) \times O(p-2k)$. But now $OC^tO = C$ means that O commutes with rotation by $\frac{\pi}{2}$ in each of the spaces σ_j. The complex structure is thus preserved by O: it is an element of the unitary group.

We note that the fixed space is equipped with a Hermitian structure induced by the Euclidean structure of $\mathbb{R}^p$ and the rotation by $\frac{\pi}{2}$ of the preceding proposition. We also point out that simple homotheties show that the topology of the level manifolds does not depend on the matrix D, whereas the dimension of the reduced manifold changes whenever two entries D_{ii} are equal.

We now choose a coordinate frame $(\mathrm{O}, x^1, \ldots, x^k, y^1, \ldots, y^k, z^1 \ldots, z^{p-2k})$ in $\mathbb{R}^p$ in which C takes the form $(\star)$, so that

$$M = \begin{pmatrix} X & P \\ Y & Q \\ Z & R \end{pmatrix},$$

where X, Y, P and Q now designate *block submatrices* with k rows and $n-1$ columns, and Z and R designate $(p-2k, n-1)$-blocks. The equation $MJ^tM = C$ may now be written

$$(m) \qquad \begin{pmatrix} X & P \\ Y & Q \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} {}^tX & {}^tY \\ {}^tP & {}^tQ \end{pmatrix} = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix},$$

$$(dm) \qquad \begin{pmatrix} X & P \\ Y & Q \end{pmatrix} \begin{pmatrix} -{}^tR \\ {}^tZ \end{pmatrix} = 0,$$

$$(mz) \qquad R^tZ - Z^tR = 0.$$

The equation (m) defines a regular real algebraic variety (hereafter referred to as a "manifold"; see B1). To show that it is nonempty, we consider its intersection with the subspace defined by the equations $X = Q$ and $Y = -P$. It is easy to see that it is a compact submanifold isomorphic to a complex Stiefel manifold. The blocks X and P, which suffice to determine the solution, satisfy $P^tX - X^tP = 0$ and $X^tX + P^tP = D$, in other words they describe a family of k vectors in $\mathcal{D}^2$, orthogonal with respect to the Hermitian structure (the k vectors in question are the first k rows of M). This compact submanifold is invariant under the action of the reducing group. In part B4, we will clarify its role in the topology of the manifold (m).

According to (m), the first $2k$ rows of M generate a $2k$-dimensional *symplectic* subspace of $\mathcal{D}^2$ (the restricted symplectic form is nondegenerate). The equation (dm) expresses the fact that each of the last $p-2k$ rows of M belongs to the orthogonal complement (with respect to the symplectic form) of this space, which is a $2(n-1-k)$-dimensional symplectic subspace. These rows generate an isotropic subspace according to (mz), but they are not always independent. These considerations allow us to clarify Proposition 2: the double inequality established there is optimal. A level manifold of the angular momentum map contains points corresponding to all the dimensions attained by the motion space, the largest dimension being achieved when the isotropic subspace under consideration is Lagrangian (i.e., of dimension $n-1-k$).

B3) The Plane and Space

We assume the angular momentum to be nonzero ($k = 1$, $p = 2$ or 3), and we represent the state by 6 dispositions, or rather by the matrix M:

$$\begin{pmatrix} X & P \\ Y & Q \\ Z & R \end{pmatrix}.$$

The three coordinates of angular momentum may be written

$$(dm) \qquad \begin{aligned} \langle Y, R\rangle - \langle Z, Q\rangle = 0, \\ \langle Z, P\rangle - \langle X, R\rangle = 0, \end{aligned}$$

$$(m) \qquad \langle X, Q\rangle - \langle Y, P\rangle = C.$$

We denote the coordinates of X in a chosen orthonormal basis of $\mathcal{D}$ by $\hat{x}_1, \ldots, \hat{x}_{n-1}$; those of Y, Z, etc. by $\hat{y}_j$, $\hat{z}_j$, etc: these real numbers are the entries of the matrix M.

We then set $2\xi_j = (\hat{x}_j + \hat{q}_j) + i(\hat{y}_j - \hat{p}_j)$, $2\eta_j = (\hat{x}_j - \hat{q}_j) + i(\hat{y}_j + \hat{p}_j)$ and $\zeta_j = \hat{z}_j + i\hat{r}_j$, in order to diagonalize the quadratic form (m) and represent (dm) in the form of a single complex equation:

$$(m) \qquad \sum |\xi_j|^2 - \sum |\eta_j|^2 = C,$$

$$(dm) \qquad \sum \xi_j\zeta_j - \sum \eta_j\bar{\zeta}_j = 0.$$

Proposition 4*. If equation (m) is satisfied, equation (dm) defines a subspace of real codimension 2 in the ζ-space $\mathbb{C}^{n-1} \sim \mathcal{D}^2$, which means that in the spatial problem, a level manifold of the angular momentum map is a vector bundle over the hyperboloid (m), the level manifold of angular momentum in the planar problem. Moreover, equation (dm) is invariant under the action of the "reduction group" $SO(2)$: $(\xi_j, \eta_j) \mapsto (e^{i\theta}\xi_j, e^{i\theta}\eta_j)$. The reduced manifold (i.e., the quotient of the level manifold by the action of the reduction group) in the spatial problem is again a vector bundle over the reduced manifold of the planar problem.

Proof. The two real equations (dm) are independent: this was demonstrated in Part B2, where the presentation in matrix form makes clear the more general result that (m) prohibits the degeneracy of the matrix used to write (dm). The remainder of the proof is clear.

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* Cabral [1], p. 70.

Proposition 5. We consider the new system

$$(m') \qquad \sum |\xi_j|^2 = C,$$

$$(dm') \qquad \sum \xi_j \zeta_j = 0,$$

defined on (ξ, η, ζ)-space and the reduction group of Proposition 4 (which introduces η). There exists a vector bundle isomorphism between (m), (dm) and (m'), (dm'), before and after reduction (i.e., the morphism commutes with the action of the reduction group).

Proof. First, note that the map $(\xi, \eta) \mapsto (\lambda\xi, \eta)$ transforms the hyperboloid into a cylinder if λ satisfies $\lambda^2(C + \sum |\eta_j|^2) = C$. It remains to establish an isomorphism between the fibers above the two points of the bases we have just shown to be in correspondence, which amounts to establishing a bijection between the subspaces of ζ-space defined by (dm) and (dm'). This is easily accomplished by orthogonally projecting (either) subspace onto the other. Using the complex parameter α, we parametrize the plane normal to (dm): $\alpha \mapsto (\alpha\bar{\xi}_1 - \bar{\alpha}\eta_1, \ldots, \alpha\bar{\xi}_{n-1} - \bar{\alpha}\eta_{n-1})$, or the plane normal to (dm'): $\alpha \mapsto (\alpha\bar{\xi}_1, \ldots, \alpha\bar{\xi}_{n-1})$. Substituting these parametrizations into (dm) and (dm') respectively, we obtain in either case the same endomorphism $\mathtt{C} \sim \mathtt{R}^2$: $\alpha \mapsto (\sum |\xi_j|^2)\alpha - (\sum \xi_j\eta_j)\bar{\alpha}$. It remains to check that its determinant is nonzero. An endomorphism of $\mathtt{C}$ of the form $\zeta \mapsto \alpha\zeta + \beta\bar{\zeta}$ has determinant $|\alpha|^2 - |\beta|^2$, as may be seen by calculating the image of $d\zeta \wedge d\bar{\zeta}$, or $(\alpha d\zeta + \beta d\bar{\zeta}) \wedge (\bar{\alpha} d\bar{\zeta} + \bar{\beta} d\zeta)$ or $(|\alpha|^2 - |\beta|^2) d\zeta \wedge d\bar{\zeta}$. In the present case this yields $(\sum |\xi_j|^2)^2 - |\sum \xi_j\eta_j|^2$. This determinant is positive if (m) is satisfied: the Cauchy-Schwarz inequality may be written $|\sum \xi_j\eta_j|^2 \le (\sum |\xi_j|^2)(\sum |\eta_j|^2)$.

Proposition 6. The manifold defined by the equations (m) and (dm') is isomorphic to the direct product of $\mathtt{C}^{n-1}$ with the cylinder given by $\sum_{j=1}^{n-1} |\xi_j|^2 = C$ in the space $\mathtt{C}^{2n-3}$ coordinatized by $\xi_1, \ldots, \xi_{n-1}$, $\eta_2, \ldots, \eta_{n-1}$. The isomorphism commutes with the action of the reduction group whose action on the cylinder just mentioned is defined by the formulas $\xi_j \mapsto e^{i\theta}\xi_j$, $\eta_j \mapsto e^{i\theta}\eta_j$.

Proof. When restricted to $\xi_1, \ldots, \xi_{n-1}$, $\eta_2, \ldots, \eta_{n-1}$, the isomorphism we wish to construct is the identity. Now, given η_1 and a point of the subspace (dm') of complex codimension 1, we associate to them first the point $(\eta_1\bar{\xi}_1, \ldots, \eta_1\bar{\xi}_{n-1})$ of the plane normal to (dm'), then the point of $\mathtt{C}^{n-1}$ which projects orthogonally onto (dm') and its normal plane at these two points.

Results

We may now describe the topology of the manifolds under study. For the planar n-body problem, a level manifold of the angular momentum map is isomorphic to the product of the sphere S^{2n-3} with $\mathtt{R}^{2n-2}$: this is the cylinder (m') of Proposition 5. For the spatial problem, this manifold may be identified with $S^{2n-3} \times \mathtt{R}^{4n-6}$, according to Proposition 6, which "stabilizes" stably trivial bundles above S^{2n-3}.

We now consider the manifolds obtained by passing to the quotient by the action of the reduction group S^1. The basic "object" is the quotient W_l^k of the cylinder given by $\sum_{j=1}^k |\alpha_j|^2 = 1$ in $\mathtt{C}^l$, coordinatized by $\alpha_1, \ldots, \alpha_l$, under the canonical action of the circle on $\mathtt{C}^l$: $\alpha_j \mapsto e^{i\theta}\alpha_j$. We see that W_l^k is the projective space $P_{l-1}(\mathtt{C})$ with a projective subspace $P_{l-k-1}(\mathtt{C})$ removed. Moreover, W_{l+1}^l, projective space with a point deleted, may be identified with the *tautological* line bundle above $P_{l-1}(\mathtt{C})$. Similarly, W_l^k is the *Whitney sum* of $l-k$ tautological bundles W_{k+1}^k.

In the planar problem, the reduced manifold is W_{2n-2}^{n-1} according to Proposition 5. The latter manifold appears in Proposition (9.4) of Smale [1]. It is known (e.g., Doubrovine, Novikov, Fomenko [1], p. 114) that this vector bundle is the Whitney sum of the tangent bundle of the projective space $P_{n-2}(\mathtt{C})$ and the trivial complex line bundle. This identification is interesting, and appears directly (and with more geometry) if one notices that the reduced manifold of *configurations* of n bodies in the plane is the product of $\mathtt{R}_+^*$ with $P_{n-2}(\mathtt{C})$, and the phase space is its tangent bundle (or cotangent bundle; cf. Arnold [1] Appendix 10, and Doubrovine, Novikov, Fomenko [1], p. 288).

For the spatial problem, Proposition 6 shows that the reduced manifold is $W_{2n-3}^{n-1} \times \mathtt{R}^{2n-2}$. The elementary calculation (cf. Milnor, Stasheff [1], p. 46) of the Stiefel-Whitney classes shows that for all $n \geq 3$ the bundle W_{2n-3}^{n-1} possesses no real section which is everywhere nonvanishing. For the planar problem, Pontryagin classes are required to establish the following weaker result (Milnor, Stasheff [1], p. 178): the bundle W_{2n-2}^{n-1} is (stably) nontrivial for $n \geq 4$. If there are only three bodies, the reduced manifold is the product of the tangent bundle of the Riemann sphere with $\mathtt{R}^2$, which is also $S^2 \times \mathtt{R}^4$ (the tangent bundle of a sphere is stably trivial). The same result may be obtained by noticing that the tautological bundle above $P_1(\mathtt{C})$ may be identified with its normal bundle through the quaternionic multiplication by j.

Before reduction, the level manifold of the spatial (but non-planar) motion space appears in Proposition 5 as the product of $\mathtt{R}_+^* \times \mathtt{C}^{n-1}$ with the Stiefel manifold of pairs of vectors $\xi, \bar{\zeta}$ in $\mathtt{C}^{n-1}$ which are orthogonal with respect to the Hermitian structure: $\zeta = 0$ has been excluded. This is $S^3 \times (S^1 \times \mathtt{R}_+^*) \times \mathtt{R}^4$ in the case of three bodies, since the bundle (dm') may be trivialized with quarternions. The reduction group acts in the standard way on the component S^1: the fibration is trivial and the base is given by the hypersurface of section obtained by fixing a point on this component. This is $S^3 \times \mathtt{R}^5$, or $\mathtt{R}^8$ with an $\mathtt{R}^4$-subspace removed. But the same result may be found with Proposition 5, which describes this manifold as the normal bundle of the tautological bundle (of Euler class 1), with the null section corresponding to planar motions removed, multiplied by $\mathtt{R}^4$ (triviality obtained as in the planar problem). Now the circle bundle on S^2 of Euler class 1 is S^3. A completely different approach would have led to the same result (only in this case): to carry out the reduction, rather than fix the three components of angular momentum and pass to the quotient by $SO(2)$, one could first fix the norm of C, then pass to the quotient by $SO(3)$.

B4) Remarks and Extensions

i) The matrix M introduced in Part B1 actually represents an intrinsic linear map. Let E be a p-dimensional vector space in which the n bodies r_i and their velocities v_i reside. To each linear form on E we associate first the list of its values on the n bodies, then the list of its values on the n velocities; these two lists are elements of $\mathcal{D}$. The matrix ${}^tM$ is then associated to this map from E^* to $\mathcal{D}^2$. The canonical symplectic form on $\mathcal{D} \times \mathcal{D}^*$ is transported by this map, since the moment of inertia identifies $\mathcal{D}$ with its dual. The result is a *bivector* belonging to $\bigwedge^2 E$, the angular momentum, which may be expressed as $\sum_{i=1}^n m_i r_i \wedge v_i$. How can its invariance express a rotation symmetry when we have not yet introduced a quadratic form on E? Because, it is only when there exist simultaneously a quadratic form on E and a potential function U on the space of maps from E to $\mathcal{D}$ such that the accelerations γ_i are given by the gradient ∇U for the induced metric, that the vanishing of the time derivative of the angular momentum is equivalent to the invariance of U under isometries for the quadratic form. More general "non-conservative" n-body problems also possess the first integral of angular momentum.

ii) We construct all the generalizations of *isosceles problems* of three bodies by means of a finite subgroup of the group G of Proposition 3. It is enough to set up an arbitrary number of equivalence classes of bodies, the bodies of each class all having the same mass and forming an "object" (positions and velocities) invariant under the chosen finite group.

iii) Let us examine the particular case $C = 0$. In Part B2, we saw that the rows of the matrix M generate an isotropic subspace. We posit the simple hypothesis $\mathrm{rk} M = p$, so that the level manifold is regular, and so that we may choose between the reduction groups $SO(p)$ and $O(p)$. We could show that in this case the reduced manifold may be identified with the Grassmann manifold of isotropic subspaces (oriented or not) equipped with a positive definite quadratic form.

iv) In order to study the topology in a case where C is degenerate, and where we have imposed $\mathrm{rk} M = p$, we orthogonally project (with respect to the Euclidean structure) the x- and y-rows of the matrix M onto $\mathcal{D}^2$, in the direction of the isotropic subspace generated by the z-rows. In this process, C remains unchanged, and the x- and y-rows end up in the orthogonal complement (with respect to the *Hermitian* structure) of the z-columns. If $p = 2k + 1$, the matrix M possesses only one z-row, and we use instead an analog of Proposition 5, which has the advantage of allowing $\mathrm{rk} M = p - 1$. We cannot, however, hope to obtain the analog of Proposition 6 for $k > 1$.

v) We must extend the study of the planar problem carried out in Part B3 to other even dimensions $p = 2k$. In this case the level manifold retracts onto the Stiefel manifold of Part B2. The method which ensures that this retraction takes place equivariantly with respect to the action of the rotation group generalizes what follows. We consider the particular case where M is a square matrix and $C = J$. The corresponding level manifold may be identified with the symplectic group (note that the configuration generates a Lagrangian subspace). We set $M = A + B$, and call $A = \frac{1}{2}(M - JMJ)$ the complex part and $B = \frac{1}{2}(M + JMJ)$ the anticomplex part. We have $AJ = JA$ and $BJ = -JB$, $I = A^tA - B^tB$ according to (m), which also ensures that $S = A^tB$ is symmetric. If H is the matrix $(A^tA)^{-1}$, $K = H^{\frac{1}{2}}A$ is unitary, and $M \mapsto K$ commutes with the (unitary) rotation group $M \mapsto OM$. We parametrize the level manifold by choosing an anticomplex matrix S and a unitary matrix K: H then satisfies $I = H^{-1} - SHS$, which ensures that H and S commute (multiply on the left by SH, on the right by HS), and determines H as the unique positive definite root of the latter equation. In this way we recover the well known topological identification of the symplectic group with $U(k) \times \mathrm{R}^{k(k+1)}$. Moreover, thanks to the equivariance with respect to the action of the rotation group, we also obtain the reduced manifold $\mathrm{R}^{k(k+1)}$. In this way, three bodies in four-dimensional space may be described by means of a point in R^6, when the angular momentum is chosen as prescribed. In the case of rectangular matrices, with the same particular angular momentum, one obtains reduced manifolds analogous to the tangent bundles of complex Grassmann manifolds. Before reduction, whatever the value of angular momentum, and once one imposes $\mathrm{rk} M = p$, the level manifold may be identified with the product of a complex Stiefel manifold with a contractible space.

C. Energy

In both the planar and spatial problem of n bodies, the study of the integral manifolds reduces to the study of the energy function h restricted to a level manifold of the angular momentum. In general, we begin with a real function f defined on a Riemannian manifold, and we denote its gradient vector field with respect to the metric by ∇f. The flow of the field $\frac{\nabla f}{\|\nabla f\|^2}$ furnishes a diffeomorphism between the two level manifolds of f, say $f = y_1$ and $f = y_2$, whenever $\frac{1}{\|\nabla f\|}$ is *bounded* on the closed set $f^{-1}[y_1, y_2]$, assumed to be complete. This so-called Palais-Smale condition [1] prohibits the "blow up" of orbits in finite time. When it is not satisfied, there exist sequences of points in $f^{-1}[y_1, y_2]$ such that $\|\nabla f\|$ tends to zero and f tends to a limit belonging to the interval $[y_1, y_2]$. Their limit points are critical points, and we say that there is a *critical point at infinity* when one of these sequences is not contained in a compact set. But $f^{-1}(y_1)$ may be diffeomorphic to $f^{-1}(y_2)$, and there may even exist a diffeomorphism of $f^{-1}[y_1, y_2]$ onto $f^{-1}(y_1) \times [y_1, y_2]$ which sends $f^{-1}(y)$ onto $f^{-1}(y_1) \times \{y\}$ without satisfying the Palais-Smale condition. In this case, it is sometimes possible to make the critical points at infinity disappear by choosing another Riemannian metric in such a way that $f^{-1}[y_1, y_2]$ remains complete, but such that the new norm of ∇f is bounded from below. The function $\frac{x_2}{1+x_1^2}$, defined on $\mathtt{R}^2$, provides an example where the natural norm is not the appropriate one.

We will begin by working with the natural state space metric, then we will show that it is necessary to define another metric which brings the level manifolds of h "closer together" at infinity.

C1) Further Results on the Space of Dispositions. Clusters.

We recall that we defined the hyperplane $\mathcal{D}_n$ of n-tuples in $\mathtt{R}^n$ such that $\sum_{i=1}^n m_i x_i = 0$, where the masses m_i also define the "moment of inertia" form $\sum_{i=1}^n m_i x_i^2$, which establishes a Euclidean structure on $\mathcal{D}_n$.

Let k be an integer, $1 \leq k \leq n$. To the partition of the set $\{1, \ldots, n\}$ into two subsets $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, we associate the orthogonal decomposition of the space $\mathcal{D}_n$ into the direct sum of the subspace of elements of the form $(x_1, \ldots, x_k, 0, \ldots, 0)$, the subspace of elements of the form $(0, \ldots, 0, x_{k+1}, \ldots, x_n)$, and the subspace of elements $(x_1, \ldots, x_n)$ such that $x_1 = \ldots = x_k$ and $x_{k+1} = \cdots = x_n$. These three subspaces, whose orthogonality is easily verified, may be identified respectively with $\mathcal{D}_k$ (masses $m_1, \ldots, m_k$), $\mathcal{D}_{n-k}$ (masses $m_{k+1}, \ldots, m_n$) and $\mathcal{D}_2$ (masses $m_1 + \cdots + m_k$ and $m_{k+1} + \cdots + m_n$). This construction immediately generalizes: to any partition of $\{1, \ldots, n\}$ into l *clusters* of respective cardinality $k_1, \ldots, k_l$, with $k_1 + \cdots + k_l = n$, we associate a decomposition of the space $\mathcal{D}_n$ into the direct sum of $l+1$ orthogonal subspaces naturally identified with $\mathcal{D}_{k_1}, \ldots, \mathcal{D}_{k_l}$ and $\mathcal{D}_l$. The last component is called the disposition of the *centers of gravity*.

This decomposition may be iterated each time there remain subspaces identified with $\mathcal{D}_k$, with $k \geq 3$. Continuing as far as possible, we obtain a decomposition of $\mathcal{D}_n$ into $n-1$ subspaces identified with $\mathcal{D}_2$, thus of dimension 1. By choosing from each $\mathcal{D}_2$ a vector (x_1, x_2) such that $x_2 - x_1 = 1$, we obtain a *Jacobi basis* of $\mathcal{D}_n$, which defines the classical Jacobi variables (one ordinarily chooses the decomposition associated to a partition into two subsets, one of which has cardinality 1, then iterates). The Jacobi basis is orthogonal, but not orthonormal, which has the effect of introducing disagreeable coefficients into subsequent formulas. We now consider an n-body problem on the line: states are represented by pairs of dispositions (X, P). The *canonical symplectic form* associates the real number $\langle X'', P' \rangle - \langle X', P'' \rangle$ to (X', P') and (X'', P''). If we want variables in which this form appears canonical, and if we use Jacobi variables to represent X, we must choose different variables to represent P, as is usually done. We rather choose an orthonormal basis for $\mathcal{D}$, so that the moment of inertia, the angular momentum, and the symplectic form may be written as simply as possible.

Consider now the n-body problem in p-dimensional space. A partition of the set of bodies into l clusters defines a decomposition of the state space into the direct sum of $l + 1$ orthogonal subspaces, each describing either the state of a cluster relative to its center of gravity, or the state of the set of centers of gravity (the decomposition is carried out for the configuration projected onto the axes, and for the velocities projected onto the same axes). The functions on state space representing the moment of inertia, kinetic energy, and angular momentum may be written as the sum of these same functions defined on a component of the decomposition. The potential U does not have this property.

We may however obtain something close to it.

Lemma 1. Given a sequence of configurations of n bodies in a space of dimension p, it is possible to extract a subsequence such that there exists a partition of the system into l clusters, and thus a decomposition of the space of configurations into $l + 1$ components having the following properties:

i) the configuration projected onto the component describing a cluster with respect to its center of gravity tends to a limit,

ii) the configuration projected onto the component describing the configuration of the centers of gravity is such that the distance between two arbitrary points tends to infinity.

Proof. We proceed by induction. Assume that a subsequence has been extracted such that there exists a partition of the first $n - 1$ bodies satisfying the above property. We consider the limit inferior of the distance of the n^{th} body from the center of gravity of the first cluster. If this limit is infinite, we proceed to the next cluster. Otherwise, we extract a subsequence such that this limit inferior is the limit. The sequence of positions of the body relative to the center of gravity remains in a compact set. From it we extract a convergent subsequence. If the limit inferior is infinite for all clusters, we create a cluster for the n^{th} body alone. The result follows. It is also possible to replace the limit inferior by the limit superior.

Lemma 2. Consider a sequence having the properties of the sequence extracted in Lemma 1. Let the function $\bar{U}$ be defined as the sum of the potentials U_i of each cluster, $1 \leq i \leq l$. Then $U - \bar{U}$ tends to zero and $\|\nabla U - \nabla \bar{U}\|$ tends to zero.

Proof. The function $U - \bar{U}$ may be written as $\sum \frac{m_i m_j}{r_{ij}}$, where the sum takes place only over pairs (i, j) such that the distance r_{ij} tends to infinity. It is easily verified that the norm of the gradient of this function also tends to zero (the components of the gradient are "forces," which tend to zero as $1/r^2$).

We have implicitly assumed that the function U is well defined, in other words that the configurations of the sequence are not collisional. We will, however, have to treat the case where there are collisions in the limit.

Lemma 3. Consider a sequence having the properties of the sequence extracted in Lemma 1. We consider the j^{th} cluster, with potential U_j, and we iterate the process of decomposition into clusters, regrouping into subclusters those bodies having the same limiting position. The function $\bar{U}_j$, defined as the sum of the potentials of each subcluster and of the potential of the configuration of the centers of gravity, is such that $U_j - \bar{U}_j$ and $\|\nabla U_j - \nabla \bar{U}_j\|$ tend to zero. The function $\bar{\bar{U}}$ defined as the sum of the $\bar{U}_j$ has the same property as $\bar{U}$ from Lemma 2.

Proof. The function $U_j - \bar{U}_j$ and its derivatives are continuous in a neighborhood of the collision limit. It suffices to find their values for the collisional configuration. For the function and its first derivatives, we find zero (the potential of the configuration of the centers of gravity is the potential of the projection onto the corresponding component: we have implicitly attributed the total masses of the clusters to their centers of gravity).

The Conley Matrix. We may write the gradient of the potential in a simple way by introducing the following positive definite symmetric bilinear form on the space of dispositions (cf. Pacella [1]):

$$\langle PA, P' \rangle = \sum_{i<j} \frac{m_i m_j}{r_{ij}^3} p_{ij} p'_{ij} \qquad \text{with} \qquad \begin{aligned} p_{ij} &= p_i - p_j, \\ p'_{ij} &= p'_i - p'_j. \end{aligned}$$

The right side of the first equation represents the value of the bilinear form on the two dispositions $P = (p_1, \ldots, p_n)$ and $P' = (p'_1, \ldots, p'_n)$, and depends, through the mutual distances, on a *configuration* of n bodies in a space of arbitrary dimension. The left side of the equation represents this bilinear form by means of the self-adjoint operator A, the "Conley matrix," on the space of dispositions.

In a space of arbitrary dimension, with a preferred axis (O, x), we consider a one-parameter family of configurations of n bodies whose orthogonal projection in the (O, x)-direction is fixed. Direct calculation gives $\dot{U} = -\langle XA, \dot{X} \rangle$, which means that

$$\nabla_x U = -XA,$$

where ∇_x designates the projection of the gradient onto x-space. Newton's equations for the problem in $\mathtt{R}^3$ with coordinate frame (O, x, y, z) are then

$$\begin{aligned} \ddot{X} &= -XA, \\ \ddot{Y} &= -YA, \\ \ddot{Z} &= -ZA. \end{aligned}$$

C2) Critical Sequences with Lagrange Multipliers

We describe the state of a system of n bodies in Euclidean space $\mathtt{R}^3$ with the matrix M

$$\begin{pmatrix} X & P \\ Y & Q \\ Z & R \end{pmatrix}$$

as in Part B1. The three components of angular momentum are then:

$$\langle Y, R\rangle - \langle Z, Q\rangle = C_x,$$
$$\langle Z, P\rangle - \langle X, R\rangle = C_y,$$
$$\langle X, Q\rangle - \langle Y, P\rangle = C_z.$$

We write

$$I = \|X\|^2 + \|Y\|^2 + \|Z\|^2 \quad \text{and} \quad K = \|P\|^2 + \|Q\|^2 + \|R\|^2$$

so that the energy appears in the form

$$h = \frac{1}{2}K - U.$$

We know (cf. Part B1) that a level set corresponding to a nonzero value of the angular momentum map is a smooth manifold, complete with respect to the natural metric, in the planar as well as the spatial problem. The vanishing of angular momentum does not give rise to a smooth level manifold: collinear motions (or motions confined to the origin in the planar problem) are critical points. This level set becomes smooth when one excludes motions contained in a space of codimension two, but it is not complete, so that the study of the restriction of the function h requires notions other than those of critical points and critical points at infinity discussed in the introduction to the present part. It can easily be seen that there is nothing "critical" at the "boundary" in the case of vanishing angular momentum, but simple homogeneity considerations (cf. Cabral [1]) show that $h = 0$ is then the only value corresponding to a change of topology.

Let us define the general notions to be used in our study of the function h restricted to a level set of C.

Consider a state of the system such that $\text{rk}M \geq 2$ (the motion is not collinear). According to Part B1, the three components of angular momentum are independent. The ordered triple $(\lambda_x^0, \lambda_y^0, \lambda_z^0)$ such that $\nabla h - \lambda_x^0 \nabla C_x - \lambda_y^0 \nabla C_y - \lambda_z^0 \nabla C_z$, denoted ϵ_0, is orthogonal to the gradient of angular momentum will be called an *optimal multiplier*. If ϵ_0 vanishes, then the state is a critical point for the function h restricted to the (locally regular) level manifold of C passing through the point.

Consider a sequence of states of the system. A sequence of triples $(\lambda_x, \lambda_y, \lambda_z)$ such that $\nabla h - \lambda_x \nabla C_x - \lambda_y \nabla C_y - \lambda_z \nabla C_z$, denoted ϵ, tends to zero in norm will be called a *compatible sequence (of multipliers)*. A sequence of states possessing a compatible sequence of multipliers will be called a *critical sequence*. If the states of such a sequence satisfy $\text{rk}M \geq 2$, then the sequence of optimal multipliers furnishes another compatible sequence. In fact, $\epsilon_1 = \epsilon - \epsilon_0$ is a linear combination of the gradients of C. Therefore $\|\epsilon\|^2 = \|\epsilon_0\|^2 + \|\epsilon_1\|^2$, and

$\|\epsilon_0\|$ tends to zero. A critical sequence such that h approaches a finite limit and C remains constant will be called a *horizontal critical sequence*. If it has no limit point, a horizontal critical sequence on a non-critical level manifold of C denotes the existence of a critical point at infinity for the function h restricted to this manifold: ϵ_0 is the gradient of this function. The limit points of such a critical sequence are evidently critical points (we note that since the energy is bounded from below, the limiting configuration is non-collisional).

We now undertake the study of the critical points and sequences.

First Tool: The Lagrange-Jacobi Derivative

A critical sequence is such that

$$(LJ) \qquad \frac{(K-U)^2}{I+K} \to 0.$$

Proof. The scalar product of ϵ with the following vector field, tangent to the level manifold of C,

$$\begin{aligned} \dot{X} &= X, & \dot{P} &= -P, \\ \dot{Y} &= Y, & \dot{Q} &= -Q, \\ \dot{Z} &= Z, & \dot{R} &= -R, \end{aligned}$$

is the derivative of h along the vector field, or $U-K$, because of homogeneities. This quantity, divided by the norm $\sqrt{K+I}$ of the vector field, therefore tends to zero.

Remark. The above vector field is associated with the Hamiltonian $\langle X,P\rangle + \langle Y,Q\rangle + \langle Z,R\rangle$, which is also the value of $\frac{1}{2}\dot{I}$. The expression $U-K$ is thus the value of the Poisson bracket $\frac{1}{2}\{\dot{I},h\}$, which also equals $-\frac{1}{2}\ddot{I}$. This is the Lagrange-Jacobi formula, previously cited in Part A3.

Exercise. Using the identity $K-U = \frac{1}{2}K + h$, show that a horizontal critical sequence such that the sequence of configurations remains bounded is bounded, and thus possesses a limit point. Thus the critical points at infinity have configurations of "infinite" size.

Second Tool: Multiplier Frames

In a multidimensional problem, a multiplier may be identified with an antisymmetric matrix Λ of the same size as C: we associate it with the vector field $-\frac{1}{2}\nabla(\mathrm{tr}\Lambda C)$. An orthonormal change of coordinates $M \mapsto OM$ transforms Λ to $O\Lambda{}^tO$. An orthonormal frame associated to a system of coordinates which brings the matrix Λ into normal form, as C was brought into normal form in Part B2, will be called a multiplier frame.

In the three-dimensional case, a multiplier frame is simply an orthonormal frame in $\mathtt{R}^3$ such that $\lambda_x = \lambda_y = 0$. In this case only the two following gradients are important (A is defined in Part C1):

$$
\begin{array}{c} \\ \dot X \\ \dot Y \\ \dot Z \\ \dot P \\ \dot Q \\ \dot R \end{array}
\begin{array}{c} \begin{array}{cc} \nabla C_z & \nabla h \end{array} \\ \begin{pmatrix} Q & XA \\ -P & YA \\ 0 & ZA \\ -Y & P \\ X & Q \\ 0 & R \end{pmatrix} \end{array}.
$$

Proposition 1. The rank of M equals 2 for a critical point of h in the spatial n-body problem: the positions and velocities are situated in the same plane. Conversely, any critical point for the planar problem, located in an arbitrary plane in $\mathbb{R}^3$, is a critical point for the spatial problem. These critical points are called relative equilibria.

Proof. Once an optimal multiplier frame has been chosen (we defined the notion of a critical point of h only in the case where this multiplier exists), it is clear that $R = 0$ and $ZA = 0$, which imposes $Z = 0$ (the projection of the "forces" onto the vertical axis vanishes). The remaining equations are those which define a critical point in the plane*.

Geometric Estimates. Consider a critical sequence equipped with a compatible sequence. We define the following quantities, associated to each point of the sequence, which are independent of the choice of multiplier frame, but which, contrary to the quantities appearing in (LJ), depend on the choice of multiplier:

$$
I_z = \|X\|^2 + \|Y\|^2, \qquad K_z = \|P\|^2 + \|Q\|^2.
$$

We have the following estimates, called "geometric" because they do not involve the potential:

$$
(E1) \qquad \|R\| = o(1),
$$

$$
(E2) \qquad \sqrt{K_z} = |\lambda_z|\sqrt{I_z} + o(1),
$$

$$
(E3) \qquad K_z = \frac{C_z^2}{I_z} + o(1).
$$

The first estimate is clear. The last two follow from the projection of the two gradients ∇C_z and ∇h onto (P, Q)-space. The norm γ of the projection of

* Another argument leads to the same conclusion assuming that the angular momentum is nonzero. The symplectic gradient of the energy must be a linear combination of the symplectic gradients of the components of angular momentum. But if we choose an angular momentum frame ("C lies along the vertical coordinate axis"), only the symplectic gradient of the vertical component is tangent to the level manifold of the angular momentum. The argument is concluded in the same way.

$\nabla h - \lambda_z \nabla C_z$ tends to zero, from which $(E2)$ follows by the triangle inequality. To obtain $(E3)$, we project the vector (P, Q), whose norm squared is K_z, first onto the vector $(-Y, X)$, and then onto its orthogonal complement in $\mathcal{D}^2$-space. The norm of the second projection is bounded by γ, which gives the $o(1)$.

Proposition 2. From a critical sequence with configurations that tend to a non-collisional limiting configuration, we may extract a subsequence converging to a relative equilibrium. The sequences of multipliers compatible with this subsequence are characterized by their limit, which is the value of the (nonzero) optimal multiplier of the limiting equilibrium.

Proof. We deduce from (LJ) that the sequence of Ks tends to the same limit as the sequence of Us. The compactness of the critical sequence implies the existence of a limit point. We must verify that $\mathrm{rk}M \geq 2$ at this limit point, by using for example the estimates above. First, $(E1)$ shows that K_z has the same limit as U. Thus if I_z tends to zero, $\mathrm{rk}M$ is not 1. Otherwise, $(E3)$ shows that C_z is bounded from below, so we know that the problem is also not collinear. The limit point is clearly a relative equilibrium (there are only two possible choices for the velocities, once a planar central configuration is given, and a circle of choices, given an aligned Moulton configuration). The conclusion pertaining to the multiplier is easily deduced from the independence of the gradients of the three components of angular momentum.

An Application to Small Critical Sequences

A small critical sequence is a critical sequence such that I tends to zero.

Proposition 3. Consider a small critical sequence and a corresponding compatible sequence. Then h tends to $-\infty$ (the sequence is not horizontal) and the norm of the multiplier tends to infinity.

Proof. It follows from (LJ) that K is equivalent to U, which tends to infinity. Therefore h is equivalent to $-\frac{1}{2}U$. Estimates $(E1)$ and $(E2)$ then show that $|\lambda_z|$ tends to infinity.

Application to Kinetic Energy

We now study the critical sequences of the function $\bar{h} = \frac{1}{2}K$. We first show that the horizontal critical sequences are the sequences such that the value of this function approaches zero.

Proposition 4. A critical sequence of the kinetic energy such that K is bounded from below by a strictly positive number is such that the norm of the angular momentum tends to infinity.

Proof. We may use the estimates $(E1)$, $(E2)$ and $(E3)$, constructed by looking at the lower half of the table of gradients preceding Proposition 1, which remains unchanged. Since $\bar{h}$ does not depend on the positions, the upper half gives:

$$(E4) \qquad |\lambda_z|\sqrt{K_z} = o(1).$$

Eliminating $|\lambda_z|$ from this equation and $(E2)$, we obtain

$$K_z = o(\sqrt{K_z}) + o(\sqrt{I_z}).$$

From $(E1)$ and the hypothesis of the proposition, we deduce that I_z tends to infinity. Equation $(E3)$ then suffices to finish the proof.

Proposition 5. Consider a critical sequence of the kinetic energy and a compatible multiplier. If the norm of the multiplier is bounded from below by a strictly positive number, then K and I_z tend to zero.

Proof. For the velocities, it suffices to read $(E1)$ and $(E4)$. For the positions, we eliminate K_z from $(E2)$ and $(E4)$:

$$\lambda_z^2 \sqrt{I_z} = o(1) + o(\lambda_z)$$

and I_z tends to zero.

C3) Results

We now study the most general critical sequences of h.

Principle of the Proofs

Each time we are able to replace the potential U by a modified potential $\bar{U}$ introduced in Part C1, we will be interested in a map $M \mapsto (C_x, C_y, C_z, \bar{h})$ having a particular form: it is defined on a product manifold, each factor of which describes a certain k-body problem, and it is the sum of its analogs on each factor of the decomposition. A sequence of multipliers compatible with a critical sequence of $\bar{h}$ is compatible with the sequences projected onto each of the factors, because the ϵ defined in C2 is transformed, by the projection onto one of the factors, into a vector of smaller norm. The projected sequences are thus critical. Conversely, if a sequence is such that its projection onto each of the factors is critical, then it is critical, provided there exists a multiplier simultaneously compatible with all the projected sequences.

Proposition 6. We consider a critical sequence of h such that the configuration approaches a limit. There are two possibilities: either it is a small critical sequence ($I \to 0$), or we may extract a subsequence converging to a relative equilibrium*.

Proof. Such a sequence satisfies the hypotheses of Lemma 3 (with a single cluster in the first "decomposition"). We carry out the decomposition into subclusters, then replace h by $\bar{\bar{h}} = \frac{1}{2}K - \bar{\bar{U}}$. To say that the sequence is critical for h is equivalent to saying that it is critical for $\bar{\bar{h}}$ and the compatible multipliers are the same: the norm of the difference of the gradients of the two functions tends to zero. We may apply the *principle*. The projection onto

* Shub's lemma [1] is very similar to this proposition. It may be deduced very simply by applying the "principle" to the function U restricted to the sphere I=1.

a component describing a subcluster is a small critical sequence. The norm of all compatible multipliers tends to infinity according to Proposition 3. The projection onto a component describing the centers of gravity is not collisional in the limit: it is possible to extract from it a subsequence converging to a relative equilibrium, according to Proposition 2. A compatible multiplier approaches a finite limit. Therefore, there is no multiplier compatible with the two projections, except in the cases described in the proposition.

Theorem. From every horizontal critical sequence (cf. C2), it is possible to extract a subsequence such that there is a partition of the system into l clusters having the following properties:

i) each cluster, translated to its center of gravity, approaches a relative equilibrium,

ii) the distance between two arbitrary clusters tends to infinity.

Proof. We begin by carrying out the extraction and the double decomposition into clusters of Lemma 3. Using the argument from the preceding proof concerning the incompatibility of multipliers, we deduce the following alternatives: either all the clusters (obtained after the first decomposition) behave as small critical sequences, or it is possible to extract a subsequence such that all the clusters translated to their centers of gravity approach a relative equilibrium. The first case is not compatible with the fact that the energy has a finite limit. In fact, Proposition 3 shows that the contribution of the clusters to the energy tends negatively to infinity, and Proposition 5 shows that if the multipliers are compatible, the overall contribution of the centers of gravity must approach zero. The result follows.

Model of an Unbounded Horizontal Critical Sequence

We recall that there exists a one-parameter family of relative equilibria whose configuration is, up to homothety, a given central configuration. We may parametrize such a family with the *multiplier* λ_z, and if its *weight* g, by definition equal to the homothetic invariant IU^2, is given, we obtain, using for example the estimates from Part C2,

$$I = \sqrt[3]{\frac{g}{\lambda_z^4}}, \qquad -2h = K = U = \sqrt[3]{\lambda_z^2 g}, \qquad C = \sqrt[3]{\frac{g}{\lambda_z}},$$

from which it follows that $g = -2hC^2$.

For a given value of λ_z, we choose l relative equilibria with multiplier λ_z (they all "turn" in the same direction and with the same speed) and we place them in l distinct horizontal planes in the space $\mathtt{R}^3$, in such a way that their centers of gravity are all situated on the vertical axis (O, z) with zero velocity. Once we have chosen an arbitrary sequence of positive numbers α approaching infinity, we obtain a model of a horizontal critical sequence by starting with the constructed configuration and by applying a sequence of "homotheties" of the

z-axis with respect to α, in other words by multiplying the vertical coordinates of the bodies by α while leaving the horizontal coordinates unchanged.

This model sequence is in fact a sequence of *critical points* of the function $\bar{h}$ (it is easy to see that the $\nabla\bar{h} - \lambda_z \nabla C_z$ corresponding to the centers of gravity is zero), with the same critical value, and thus *a fortiori* a horizontal critical sequence of h. The calculation of the critical value as a function of λ_z takes place as follows: to a sequence of the theorem where, for example, the system separates into two clusters of respective weights g_1 and g_2, we ascribe the weight $(\sqrt[3]{g_1} + \sqrt[3]{g_2})^3$. We then deduce the limits of the (additive) quantities I_z, $-2h$, K, U and C with the help of the formulas above for the relative equilibria.

Other Sequences

The most general horizontal critical sequences may differ considerably from the model sequences. However, we require a multiplier compatible with all the clusters, and simultaneously compatible with the component of the centers of gravity. If there are no clusters, Proposition 4 shows that $\bar{h}$ and thus h tend to zero. If there is at least one cluster, we place ourselves once and for all in a limiting multiplier frame: according to Proposition 2, the limiting cluster is situated in the horizontal plane of this frame. All the other limiting clusters are horizontal relative equilibria, with the same multiplier. Proposition 5 tells us in particular that the orthogonal projection of the configuration of the centers of gravity along the vertical axis of a compatible multiplier frame tends to the origin. But the same projection along the vertical axis of the limiting multiplier (limit of the previous axis) does not necessarily tend to the origin, as the cluster may "escape" very quickly along the vertical axis. What is much worse, the velocities of the centers of gravity of the clusters, forced by the conclusions of the same proposition to tend to zero, may nevertheless contribute to the *horizontal* component of the limiting angular momentum, for the same reason. The planar case is less problematic:

Corollary 1. The critical points at infinity of the *planar* n-body problem are such that the energy tends to zero.

Proof. If there is a cluster, the configuration of the centers of gravity must approach the multiple collision at the origin (cf. Proposition 5), which contradicts property ii) of the theorem.

Very Critical Sequences

We equip the tangent bundle of the state space with the metric which is a multiple of the preceding metric and which defines the norm

$$\|.\|_s = (1+I)^{-\frac{1}{2}} \|.\|$$

and we call a sequence critical with respect to this new metric a *very critical sequence*. This means that $(1+I)^{\frac{1}{2}} \|\epsilon\|$ tends to zero, using the notation from

Part C2, since we must also change the gradient. A very critical sequence is therefore critical, but we also have the following properties.

i) The state space is complete in the new metric. In fact, the usual norm of the velocity on a new geodesic is of order $(1+I)^{\frac{1}{2}}$. It is thus possible to bound the distance from the point to the origin as a function of the time t by a function such as e^t, which shows that the geodesic flow is complete, so that the space is complete by the Hopf-Rinow theorem.

ii) We may replace the conclusion of Proposition 5 by: $(1+I)K$ and $(1+I)I_z$ approach zero, by noticing that the $o(1)$ of $(E1)$, $(E2)$ and $(E4)$ becomes $o(1/\sqrt{1+I})$. We must note that here I designates the total size of the system, whereas I_z and K describe only the component of the centers of gravity. But the clusters contribute only a finite amount to I. Since IK bounds the components of the angular momentum, the flaw just described is corrected: in a very critical sequence, the contribution of the centers of gravity to the angular momentum tends to zero. We note that the minor defect mentioned just before this one is also corrected, because II_z tends to zero.

iii) The model sequences are very critical. Since they are composed of critical points of $\bar{h}$, to see this it suffices to estimate $\|\nabla_s U - \nabla_s \bar{U}\|_s$. This quantity does tend to zero, as shown for example by the estimate sketched in the proof of Lemma 2, or, more simply, by the homogeneity of U and its derivatives, which allows the calculation to be carried out on a bounded sequence of configurations, similar to those of Lemma 3. Sequences more general than those of the model, where one simply requires the distance between two clusters to diverge to infinity, may be not very critical.

Characterization of the Singular Values

We call the possible limits of h on a very critical horizontal sequence the *singular values*. If h_1 and h_2 are two values of the energy such that there is no singular value between them, and if the value chosen for C is nonzero, then there exists a diffeomorphism between the two corresponding level manifolds: namely, the one constructed in the introduction to this part.

Corollary 2. Assume that the total angular momentum C of the system is nonzero. The singular values h_s of the energy may be deduced from the *singular weights* $-2h_sC^2$. The cube roots of the singular weights are obtained by calculating the cube roots of the weights of each central configuration obtained from an arbitrary subset of bodies, and by adding them in all possible ways corresponding to a partition of the system into such subsets.

Proof. This is the calculation carried out on the model sequences.

There are three singular values in the three-body problem which correspond to critical points at infinity of nonzero energy. They are obtained by separating the system into a cluster of one body (of weight zero) and a cluster of two bodies. The details will be presented in Part D.

D. Description in the Spatial Problem

We propose here to “sweep” a level manifold of the energy function defined on a non-critical level manifold of the angular momentum map ($C \neq 0$) by following the strategy suggested by Easton [1] (an error unfortunately found its way into that article). We will use the description given in Part A for the planar problem, and we will quickly restrict ourselves to the 3-body problem with negative energy and nonzero angular momentum.

As usual, we describe the state of the system with 6 dispositions:

$$\begin{pmatrix} X & P \\ Y & Q \\ Z & R \end{pmatrix}.$$

The equations

$$(dm) \qquad \begin{aligned} \langle Y, R\rangle - \langle Z, Q\rangle &= 0, \\ \langle Z, P\rangle - \langle X, R\rangle &= 0, \end{aligned}$$

$$(m) \qquad \langle X, Q\rangle - \langle Y, P\rangle = C.$$

define a level manifold of the angular momentum as a vector bundle above the level hyperboloid (m) of the angular momentum in the planar problem: (dm) is the equation of a codimension 2 subspace of (Z, R)-space (cf. B3, Proposition 4). The “sweeping” process will be carried out as follows: given four dispositions X, Y, P, Q satisfying (m), and Z_0 and R_0, two dispositions belonging to the subspace (dm) of (Z, R)-space such that $\|Z_0\|^2 + \|R_0\|^2 = 1$, we describe the straight line contained in the level manifold of angular momentum

$$(Dr) \qquad \begin{pmatrix} X & P \\ Y & Q \\ \lambda Z_0 & \lambda R_0 \end{pmatrix},$$

by allowing λ to vary from $-\infty$ to $+\infty$.

It is easy to see that the energy

$$(e) \qquad h = \frac{1}{2}(\|P\|^2 + \|Q\|^2 + \|R\|^2) - U$$

increases with $|\lambda|$. In fact, $\|R\|$ increases and U decreases, since all distances increase. We choose a strictly negative value of the energy h_0. We say that the straight line (Dr) is *exceptional* if h remains less than h_0 for all λ.

Proposition. If the straight line (Dr) is exceptional, then either the configuration (X, Y, Z_0) is collisional, or Z_0 is a collisional disposition and R_0 vanishes.

Proof. In the first case, U is infinite and the straight line is exceptional. We therefore suppose that U is finite, at least for nonzero λ; R_0 must then vanish so that the limit of h along the radius will remain negative, and Z_0 must be collisional so that this limit will be strictly negative (otherwise all mutual distances tend to infinity).

The Exceptional Lines in the 3-Body Problem

We first remark that Z vanishes whenever a configuration of n bodies is in alignment. In fact, in this case there exists a coordinate frame such that $Y = 0$: (m) becomes $\langle X, Q\rangle = C$, and (dm) gives $\langle Z, Q\rangle = 0$, which is absurd if Z is a nonzero multiple of X.

Remark. This interesting property has analogs when we consider spatial dimensions greater than 3. These analogs are obtained by writing the matrix M from Part B1 as $\bigl(X\ P\bigr)$, where X and P are block submatrices: C is then $X^tP - P^tX$, and one argues using the image of this matrix.

The only straight lines such that U remains equal to $-\infty$ are therefore the lines above the collision states of the planar problem such that $Z_0 = 0$ and thus $\langle X, R\rangle = \langle Y, R\rangle = 0$. This last condition means that the vertical velocity of the isolated body vanishes.

There remains the case where Z_0 is collisional and R vanishes. We fix a configuration (X, Y), up to homothety, and use the representation from Part A. The half-space

$$\bigl\{(P, Q)/\langle X, Q\rangle - \langle Y, P\rangle > 0\bigr\}$$

contains the sphere (en). The ball thus delimited contains in general three "exceptional" closed disks, in other words, disks with exceptional lines above their points. Each of these is associated with a choice of a "collisional" Z_0; it is situated in the plane $\langle P, Z_0\rangle = \langle Q, Z_0\rangle = 0$, according to equations (dm); its radius, if Z_0 corresponds to the collision of the pair $\{i, j\}$, is given by the formula

$$\rho_{ij}^2 = \frac{1}{C^2}\frac{(m_i m_j)^3}{m_i + m_j} + 2h_0.$$

In fact, the exceptional disk $\{i, j\}$ is the set of points such that the limit of h on the line parametrized by $(Z_0, 0)$ is less than h_0. The fact that h increases along the straight line implies that (P, Q) is in the ball (en). Calculation of the radius ρ_{ij} may be carried out as follows. To deduce equation (en) from (e) in Part A, we multiplied U by $C^{-1}(\langle X, Q\rangle - \langle Y, P\rangle)$. But if we wish to restrict ourselves to velocities such that $\langle P, Z_0\rangle = \langle Q, Z_0\rangle = 0$, we may replace this expression with the quotient by C of the expression for the angular momentum of the *pair* $\{i,j\}$: this quotient equals 1 for the velocities in question, and has the same homogeneity properties as the previous one. We thus obtain a variant of equation (en) which yields

$$\rho_{ij}^2 = \frac{\bar{I}U^2}{C^2} + 2h_0,$$

where $\bar{I}$ designates $\frac{m_i m_j}{m_i+m_j} r_{ij}^2$, the moment of inertia of the pair. We must replace U by its limit $\bar{U} = \frac{m_i m_j}{r_{ij}}$ to obtain the desired expression. It is instructive to compare this with the method from Part A for the problem of two bodies of masses m_i and m_j.

But the ρ_{ij} do not always have positive squares, and it is precisely this condition that determines the appearance of the exceptional disks, which very likely change the topology of the level manifolds.

We now proceed to the negative values of energy, starting from zero. To start with, for every non-collisional planar configuration, there exist three disjoint exceptional disks, above which there is an exceptional straight line. For a collisional planar configuration, we have the exceptional direction $Z_0 = 0$ above all the velocities of the half-space, and only two exceptional disks (the third is sent to the boundary). Next, an exceptional disk disappears each time the energy passes through a *singular* value $h = -\frac{(m_i m_j)^3}{2(m_i+m_j)C^2}$. After three passages, there is a sphere S^5 above all the non-collisional planar configurations up to homothety, and two disks D^5 above the collisional configurations. Next comes the critical value $h = -\frac{(m_2 m_3 + m_3 m_1 + m_1 m_2)^3}{2(m_1+m_2+m_3)C^2}$ due to Lagrange's configuration, which changes the Hill's regions, and the three values due to the collinear equilibria of Euler.

Remark on an Isosceles Problem

We consider the spatial isosceles problem of three bodies symmetric under rotation by π around the vertical axis. It is clear that it possesses one of the three critical points at infinity described above, along with one of Euler's critical points. It is therefore the simplest case where a change of topology takes place for one of our *singular* values. It is easy to see, for example by applying the method of Part A, that the reduced manifold is $S^2 \times \mathtt{R}$ for h greater than or equal to the singular value, and S^3 for h between the singular value and the critical value.

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