Introduction to the N-body problem

Ravello, september 1997

Alain Chenciner

UFR de Mathématique, Université Paris 7,
2, place Jussieu, 75251 Paris cedex 05
et
Astronomie et Systèmes Dynamiques,
Bureau des Longitudes, 3, rue Mazarine, 75006 Paris

These notes contain almost no proofs. Their purpose is simply to be a guide to the oral lectures and the reading of the quoted papers. I choosed to stress the more algebraic aspects of Celestial Mechanics : reduction, homographic motions, comparisons to two-body problems, collisions, $\cdots$ a sort of sketchy rewriting of a substantial part of Wintner's classical book [W1]. This is based on a common paper with Alain Albouy [AC1] and many more ideas from Albouy's beautiful works. I end with a description of Xia's proof of the existence of non-collision singularities for the five-body problem in three-space [X1]. Many fundamental aspects of the N-body problem are not even mentioned, for instance the topology of integral manifolds, the proofs of non-integrability, existence of various types of solutions, stability $\cdots$ Neither are the more astronomical aspects of Celestial Mechanics – periodic and quasi-periodic motions, secular systems and long term evolution of planetary systems.

To the title of each paragraph is appended the main reference used to write it (not necessarily the main historical one). Other references are quoted in the text [between brackets] or just given in the bibliography.

1-1. The shape of N points in a normed space [AC1]

The equivalent formulas

$$\beta\big((\xi_1,\cdots,\xi_n),(\eta_1,\cdots,\eta_n)\big) := \sum_{i,j} < \vec{r}_i, \vec{r}_j >_E \xi_i\eta_j = \sum_{i,j} (-\frac{1}{2}r_{ij}^2)\xi_i\eta_j$$

define a quadratic form on the hyperplane

$$\mathcal{D}^* := \Big\{(\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \xi_i = 0\Big\}$$

of $\mathbb{R}^n$. The vectors $\vec{r}_1,\cdots,\vec{r}_n$ are elements of a finite dimensional euclidean space E, whose scalar product and norm are written $< >_E$ and $\| \|_E$, and the positive numbers $r_{ij} = \|\vec{r}_i - \vec{r}_j\|_E$ are the mutual distances.

This form defines the configuration of the n points up to a common rigid motion (translation and rotation) in E. The hyperplane $\mathcal{D}^*$ is to be considered as the dual of the space $\mathcal{D} = \mathbb{R}^n/(1,\cdots,1)$ of *dispositions*, that is the space of n-uples of points on a line up to translation. The $n(n-1)/2$ numbers r_{ij}^2 are the coordinates of β in a natural basis of the space of quadratic forms on $\mathcal{D}^*$. Considered as a homomorphism of $\mathcal{D}^*$ to $\mathcal{D}$, $\beta = {}^t x{\circ}\epsilon{\circ}x$, where the isomorphism $\epsilon : E \to E^*$ defines the euclidean structure of E and $x : \mathcal{D}^* \to E,\ x(\xi_1,\cdots,\xi_n) = \sum_{i=1}^n \xi_i\vec{r}_i$, defines the configuration up to translation. This is nothing but the Gram construction. One checks that $\mathrm{Ker}\beta = \mathrm{Ker}x$ and $\mathrm{Im}\,\beta = \mathrm{Im}\,{}^t x$. A necessary and sufficient condition that the r_{ij}^2 be indeed the squares of the mutual distances of n elements of a normed space E is that β be positive (compare to Blumenthal [Bl1]).

1-2. Masses as an euclidean structure on the dispositions [AC1]

¿From the euclidean structure $\|(x_1,\ldots,x_n)\|^2 = \sum_{i=1}^n m_i x_i^2$ on $\mathbb{R}^n$ associated with n positive masses m_i, we define an euclidean structure on $\mathcal{D}$ by $\|(x_1,\ldots,x_n)\|_\mu = \frac{1}{M}\sum_{i<j} m_i m_j (x_i - x_j)^2$. One checks that the associated isomorphism $\mu : \mathcal{D} \to \mathcal{D}^*$ is defined by $\mu^{-1}(\xi_1,\ldots,\xi_n) = \left(\frac{\xi_1}{m_1},\ldots,\frac{\xi_n}{m_n}\right)$ and $\mu(x_1,\ldots,x_n) = \big(m_1(x_1 - x_G),\ldots,m_n(x_n - x_G)\big)$, where $x_G = (m_1x_1 + \cdots + m_nx_n)/M$ is the *center of mass* of the x_i ($M = m_1 + \cdots + m_n$ is the total mass).

Notice that one can now represent the quotient space $\mathcal{D}$ by the section $x_G = 0$ orthogonal to $(1,\cdots,1)$, with the induced metric : reducing the translational symmetry amounts to fixing the center of mass. All this is in germ in Jacobi; for example, the classical *Jacobi coordinates* amount to choosing a particular orthogonal basis of $\mathcal{D}^*$. The quadratic form on E^* associated to $b := x{\circ}\mu{\circ}{}^t x$ is called the *inertia form* of the configuration. The forms β and b are closely related : b being on the side of ambient space turns with the bodies under an isometry of E, while β being on the side of the bodies is invariant under isometries of E. The common trace I of the endomorphisms $\mu{\circ}\beta$ and $b{\circ}\epsilon$ is the *moment of inertia* of the configuration with respect to its center of mass :

$$I = \frac{1}{M}\sum_{i<j} m_i m_j \|\vec{r}_i - \vec{r}_j\|^2 = \sum_{i=1}^n m_i \|\vec{r}_i - \vec{r}_G\|^2.$$

Let $\mathrm{vol}_{i_1\cdots i_k}$ be the volume of the $(k-1)$-dimensional parallélotope generated in E by the vectors $\vec{i_1i_2},\ldots,\vec{i_1i_k}$, that is $(k-1)!$ times the volume of the simplex defined by the points $i_1,\ldots,i_k$. One checks that

$$\det(\mathcal{I}d_{\mathcal{D}^*} - \lambda\mu{\circ}\beta) = \det(\mathcal{I}d_E - \lambda b{\circ}\epsilon) = 1 - \eta_1\lambda + \cdots + (-1)^{n-1}\eta_{n-1}\lambda^{n-1},$$

where $\eta_{k-1} = \frac{1}{M}\sum_{i_1<\cdots<i_k} m_{i_1}\cdots m_{i_k}\mathrm{vol}^2_{i_1\cdots i_k}$.

2-1. Lagrange equations : absolute motions [AC1]

Since Lagrange[L2], the equations of the newtonian n-body problem are written

$$m_i\ddot{\vec{r}}_i = \frac{\partial U}{\partial \vec{r}_i},\;\; U = \sum_{i<j} m_i m_j \Phi(r_{ij}^2),\;\; \Phi(s) = \mathcal{G}s^\kappa,$$

with $\kappa = -\frac{1}{2}$ in the newtonian case. U is the *force function*, opposite to the potential energy. "Reducing" the translations one can write this equation as the following equality of homomorphisms from $\mathcal{D}$ to E^* :

$$(N) \qquad \epsilon\circ\ddot{x}\circ\mu = dU(x)\,.$$

We have identified U with a real function on the space $\mathrm{Hom}(\mathcal{D}^*, E)$ and $\mathrm{Hom}(\mathcal{D}, E^*)$ with the dual of this space via the bilinear mapping $(\phi,\psi)\mapsto \mathrm{trace}\,({}^t\phi\circ\psi)$. The isomorphism $x\mapsto \epsilon\circ x\circ\mu$ defines an euclidean structure on $\mathrm{Hom}(\mathcal{D}^*, E)$ in terms of which (N) becomes

$$\ddot{x} = \nabla U(x),$$

where ∇ is the gradient. We shall denote by a dot $\cdot$ the scalar product :

$$x\cdot y = \mathrm{trace}\,(\mu\circ{}^t x\circ\epsilon\circ y) = \sum_{i=1}^n m_i < \vec{r}_i - \vec{r}_G, \vec{s}_i - \vec{s}_G >_E,$$

if x and y are respectively represented by $(\vec{r}_1,\cdots\vec{r}_n)$ and $(\vec{s}_1,\cdots\vec{s}_n)$. For instance, $I = x\cdot x$ is just the squared norm of x.
Writing $U(x) = \hat{U}(\beta)$, one obtains $dU = 2\epsilon\circ x\circ d\hat{U}$, which puts the equations of motion in the form

$$(N) \qquad \ddot{x} = 2x\circ A,$$

where the *Wintner-Conley endomorphism* $A = d\hat{U}\circ\mu^{-1} : \mathcal{D}^* \to \mathcal{D}^*$ depends linearly on the masses. We call "absolute" the corresponding motions, which are usually obtained by fixing the center of mass.

2-2. Lagrange equations : relative motions [AC1]

After reduction of the translations, the phase space of the n-body problem can be taken as the tangent space to the $Hom(\mathcal{D}^*, E)$ and identified with $\mathrm{Hom}(2\mathcal{D}^*, E)$ where $2F$ means the product of two copies of F. The elements will be written $z = (x, y)$, x for the positions $(\vec{r}_1,\cdots,\vec{r}_n)$, y for the velocities $(\dot{\vec{r}}_1,\cdots,\dot{\vec{r}}_n)$. The *space of motion* is the image $\mathrm{Im}\,z$ of z and the equations of motion are

$$(N) \qquad \dot{x} = y, \qquad \dot{y} = 2x\circ A.$$

By the Gram construction we get $\mathcal{E} \in Hom_+(2\mathcal{D}^*, 2\mathcal{D})$ (the + sign means symmetric positive)

$$\mathcal{E} := {}^t z{\circ}\epsilon{\circ}z = \begin{pmatrix} {}^t x{\circ}\epsilon{\circ}x & {}^t x{\circ}\epsilon{\circ}y \\ {}^t y{\circ}\epsilon{\circ}x & {}^t y{\circ}\epsilon{\circ}y \end{pmatrix} := \begin{pmatrix} \beta & \gamma - \rho \\ \gamma + \rho & \delta \end{pmatrix}.$$

So, after reduction of both translations and rotations, the state of the system is described by four elements $\beta, \gamma, \delta, \rho$ of $\mathrm{Hom}(\mathcal{D}^*, \mathcal{D})$, whose first three are symetric (β and δ are moreover positive) and the last antisymetric. Computing $\dot{\mathcal{E}}$ with the help of (N) we get the reduced equations which generalize systems obtained by Lagrange[L1] and Betti[Be1] and whose solutions are called "relative" motions :

$$(NRel) \qquad \begin{aligned} \dot\beta &= 2\gamma, \\ \dot\gamma &= {}^t A{\circ}\beta + \beta{\circ}A + \delta, \\ \dot\delta &= 2({}^t A{\circ}\gamma + \gamma{\circ}A) - 2[A, \rho), \\ \dot\rho &= [A, \beta). \end{aligned}$$

Given a bilinear form θ on $\mathcal{D}^*$, we have written $[A, \theta) = {}^t A{\circ}\theta - \theta{\circ}A$. For instance, due to the symmetry of $A{\circ}\mu$, the endomorphism $\mu{\circ}[A, \beta)$ is the usual commutator $A{\circ}B - B{\circ}A$ of the endomorphisms A and $B = \mu{\circ}\beta$.

2-3 Invariants and first integrals [AC1]

The traces I, J et K of endomorphisms $B = \mu{\circ}\beta$, $C = \mu{\circ}\gamma$ and $D = \mu{\circ}\delta$ of $\mathcal{D}^*$ can be written

$$I = x \cdot x, \quad J = x \cdot y, \quad K = y \cdot y.$$

On the level of traces, what is left of the equations of motion is

$$\dot J = \frac{\ddot I}{2} = K + 2\kappa U, \quad \dot H = 0,$$

where $H = \frac{1}{2}K - U$ is the total energy, sum of the kinetic energy in a galilean frame fixing the center of mass and of the potential energy. The first is the *Lagrange-Jacobi relation* (or *virial* relation, see Jacobi[J1], Poincaré[Po1] p. 90,91) and the second is the *conservation of energy*. As the Lagrange-Jacobi relation can also be written $\frac{\ddot I}{2} = 2H + 2(\kappa + 1)U$, we see that, *as I controls the size of the system, that is* $\sup_{i,j} r_{ij}$, *its second derivative $\ddot I$ or equivalently the potential function U, controls the clustering* $\inf_{i,j} r_{ij}$.

One already notices the particular case of the potential of Jacobi-Banachiewicz ($\kappa = -1$) [Ba1,W2] for which $\ddot I = 2H$ is constant, which implies the new first integral $G = 2IH - J^2$. We shall describe later the symmetry due to the homogeneity of the potential function. The Lagrange-Jacobi relation is basic to our understanding of the global behaviour of solutions of the n-body problem : if we write it $\dot{J} = 2H + 2(\kappa+1)U$, we see that, in the newtonian case or more generally if $\kappa > -1$, the positivity of U implies that J is increasing along any solution whose total energy H is ≥ 0. The existence of such a *Lyapunov function* precludes any non trivial recurrence, in particular it forbids any periodic motion. It is well known that things are much more complicated in negative energy. The basic tool replacing J will then be *Sundman's function* (see 6-2).

Finally, we discuss the angular momentum $\sum_{i=1}^n m_i \vec{r_i} \wedge \dot{\vec{r_i}}$, which is a bivector of E. Considered as an antisymetric form on E^*, that is an antisymetric homomorphism from E^* to E, it can be written

$$\mathcal{C} = z\circ\omega_\mu\circ{}^t z = -x\circ\mu\circ{}^t y + y\circ\mu\circ{}^t x,$$

where $\omega_\mu : 2\mathcal{D} \to 2\mathcal{D}^*$ is defined by $\omega_\mu(u,v) = \big(-\mu(v), \mu(u)\big)$. One readily computes its derivative $\dot{\mathcal{C}} = 2x\circ(-\mu\circ{}^t A + A\circ\mu)\circ{}^t x = 0$. This proves $\mathcal{C}$ is invariant. The *support* of the bivector $\mathcal{C}$, that is the image of $\mathcal{C} \in Hom(E^*, E)$, is called the *fixed space*. Its dimension, always even, is the rank of $\mathcal{C}$. One finds in Albouy[A1] the proof by elementary symplectic geometry of the estimates

$$\operatorname{rank}\mathcal{C} \leq \operatorname{rank}\mathcal{E} \leq \frac{1}{2}\operatorname{rank}\mathcal{C} + n - 1$$

which generalize a theorem of Dziobek saying that for three bodies with zero angular momentum, the motion necessarily takes place in a fixed plane. We set

$$[\mathcal{C}] = \sqrt{-(\mathcal{C}\circ\epsilon)^2},\ \mathcal{J}_\mathcal{C} = [\mathcal{C}]^{-1}\circ\mathcal{C}\circ\epsilon,\ \Omega_\mathcal{C} = \epsilon\circ\mathcal{J}_\mathcal{C},\ |\mathcal{C}| = \frac{1}{2}\operatorname{tr}[\mathcal{C}] = \frac{1}{2}\langle\mathcal{C}, \Omega_\mathcal{C}\rangle$$

(of course, in the definition of $\mathcal{J}_\mathcal{C}$, we invert $[\mathcal{C}]$ only on the fixed space). As $\Omega_\mathcal{C} = \epsilon\circ\mathcal{J}_\mathcal{C}$ and $\mathcal{J}_\mathcal{C}^2 = -\mathrm{Id}$ on the fixed space, the triple $(\epsilon, \Omega_\mathcal{C}, \mathcal{J}_\mathcal{C})$ endows the fixed space with a *hermitian* structure (compatible euclidean, symplectic and complex structures). A similar structure $(\kappa, \Omega, \mathcal{J})$ is obtained on $Hom(\mathcal{D}^*, E)$ by setting $\kappa(x) = \epsilon\circ x\circ\mu$ (compare to 2-1), $\mathcal{J}(x) = \mathcal{J}_\mathcal{C}\circ x$ and $\Omega = \kappa\circ\mathcal{J}$. It defines a hermitian structure on the subspace of $x = (\vec{r_1}, \cdots, \vec{r_n}) \in Hom(\mathcal{D}^*, E)$ such that each $\vec{r_i}$ belongs to the fixed space. When $\dim E = 2$, $\mathcal{C}$ can be thought of as a real number c whose norm is $|\mathcal{C}|$. If $c \neq 0$, the said subspace is the whole space and the complex structure is defined by the rotation of each $\vec{r_i}$ by $\pm\frac{\pi}{2}$ according to whether c is positive or negative. When $\dim E = 3$, $\mathcal{C}$ can be thought of as a vector $\vec{\mathcal{C}}$ of length $|\mathcal{C}|$, the operator being the "vectorial product" by this vector.

If $\mathcal{C} \neq 0$, the fixed space is the plane orthogonal to this vector and the complex structure is defined by $\mathcal{J}\big((\vec{r}_1, \cdots, \vec{r}_n)\big) = (\frac{\vec{\mathcal{C}}}{|\mathcal{C}|} \wedge \vec{r}_1, \cdots, \frac{\vec{\mathcal{C}}}{|\mathcal{C}|} \wedge \vec{r}_n)$. Even without restricting to the fixed space, we shall consider $\mathcal{J}$ as a (eventually non invertible : $\|\mathcal{J}(x)\| \leq \|x\|$) complex structure and call the set of all elements of the form $\lambda_1 x + \lambda_2 \mathcal{J}(x),\ \lambda_1, \lambda_2 \in \mathbb{R}$, the *complex line generated by* x.

The space $Hom_+(2\mathcal{D}^*, 2\mathcal{D})$ of relative states is endowed with a Poisson structure whose symplectic leaves are the intersections of the submanifolds obtained by fixing the rank of $\mathcal{E}$ and of those obtained by fixing the rotation invariants of the angular momentum. One fixes these invariants by fixing the traces of the iterates (of even order, those of odd order are equal to zero) of $\omega_\mu \circ \mathcal{E}$, which are equal to those of the iterates of $\mathcal{C} \circ \epsilon$.

3-0. The newtonian two-body problem with $\mathcal{C} = 0$ [MS1]

As $\mathcal{C} = 0$, y is proportional to x and every motion takes place on a fixed line, so that one can suppose $E = R$ and $x, y \in Hom(\mathcal{D}^*, R) = \mathcal{D}$. Moreover, as x won't change its sign, we shall suppose that $x = r > 0$. The total energy is $H = \frac{1}{2}\|y\|^2 - U(x) = \frac{1}{2}\|\dot{r}\|^2 - \frac{k}{r}$, where $k = \mathcal{G} m_1 m_2$. The actual integration of the differential equations of motion is well known. A solution either ends with a collision at time t_0 – in which case $r = O\big((t_0 - t)^{\frac{2}{3}}\big)$ independently of the value h of H – or it escapes to infinity (in infinite time). This last possibility occurs only in the case when the energy is non negative. When $t \to +\infty$, one then has $r = 0\big(t^{\frac{2}{3}}\big)$ if $h = 0$ (*parabolic* motion) or $r = 0(t)$ if $h > 0$ (*hyperbolic* motion). To characterize the motions which lead to escape, it is convenient to introduce the function

$$G(r, t) = r^{\frac{3}{2}} - \frac{3}{2}\sqrt{2k}\, t.$$

Its time-derivative along a solution is $\frac{d}{dt} G\big(r(t), t\big) = \frac{3}{2} r^{\frac{1}{2}} \left(\dot{r} - \sqrt{\frac{2k}{r}}\right)$, so that G is non-decreasing if and only if $h > 0$ AND $\dot{r} > 0$. For easy visualisation, especially of limit velocities, it is good to use coordinates $(\dot{r}, \frac{1}{r})$ (compare 3-1) in the phase plane, so that constant energy curves become parallel parabola. Finally, let us recall how one regularizes the collision "à la Levi Civita" [LC1], replacing it by an elastic bounce after reducing the velocity to keep it finite : one sets $r = z^2$ and $dt = 2z^2 d\tau$. In the new time τ, the differential equation of regularized motion becomes

$$\left(\frac{dz}{d\tau}\right)^2 - 2hz^2 = 2k.$$

When the energy h is negative, this is just a harmonic oscillator.

3-1. The newtonian two-body problem with $\mathcal{C} \neq 0$: absolute motions [A1]

One can suppose that $E = \mathbb{R}^2$ and denote by $x = (x', x'')$ an element of the configuration space $Hom(\mathcal{D}^*, \mathbb{R}^2) = \mathcal{D}^2$. In terms of the euclidean structure on $\mathcal{D}$, the energy and angular momentum integrals become

$$\frac{1}{2}(\|y'\|^2 + \|y''\|^2) - U(x', x'') = h; \quad < x', y'' > - < x'', y' >= c.$$

Following Smale and Albouy, one understands the topology of the sets defined by these equations by fixing x *up to homothety* and determining compatible velocities and sizes. In the case when $c \neq 0$, Albouy noticed that this determination is most conveniently done by replacing the equations by the following ones, homogeneous of degree 0 in x :

$$\begin{aligned}\frac{1}{2}(\|y'\|^2 + \|y''\|^2) - \frac{U(x', x'')}{c} < x', y'' > - < x'', y' >= h,\\ < x', y'' > - < x'', y' >= c,\end{aligned}$$

which he writes

$$\begin{aligned}\|y' + \frac{x''U}{c}\|^2 + \|y'' - \frac{x'U}{c}\|^2 = \frac{IU^2}{c^2} + 2h,\\ < x', y'' > - < x'', y' >= c.\end{aligned}$$

After fixing x up to homothety, the velocities satisfying these equations belong to the intersection of the sphere S in $\mathcal{D}^2$ of center $\left(-\frac{x''U}{c}, \frac{x'U}{c}\right)$ and radius $\frac{IU^2}{c^2} + 2h$ and of the half-space $(< x', y'' > - < x'', y' >) > 0$ (if $c > 0$). Albouy also noticed that the sphere S is orthogonal to the *virial sphere* of equation $\|y'\|^2 + \|y''\|^2 = -2h$, so called because the Lagrange-Jacobi relation implies that its equation can be written $\ddot{I} = 0$. All this holds for the general case of n newtonian bodies in the plane. But only in the case of two bodies is the radius of the sphere S independant of the configuration. In this case, the invariant sets are tori ($h < 0$) or cylinders ($h \geq 0$). The determination of the actual integral curves (conics) is obtained by fixing the *Laplace vector* $\vec{\mathcal{L}} = -U\vec{r}_{12} + \dot{\vec{r}}_{12} \wedge \vec{\mathcal{C}}$, defining the direction of perihelium of the orbit and its excentricity, which, in this setting, becomes after a rotation of $\frac{\pi}{2}$ and a division by c the vector

$$\vec{L} = \left(y' + \frac{x''U}{c}, y'' - \frac{x'U}{c}\right)$$

joining the center of the sphere S to the point $y = (y', y'')$. To actually solve the equations of motion it is convenient to work with the *size* $r = I^{\frac{1}{2}}$. In the basis of the plane $\mathcal{D}^2$ defined by the vectors (x', x'') and $(-x'', x')$, the coordinates of y are $\dot{r}$ and $\frac{c}{r}$, so that $K = \dot{r}^2 + \frac{c^2}{r^2}$, or equivalently $IK - J^2 = c^2$, and the Lagrange-Jacobi equation becomes

$$\ddot{r} = \frac{\partial \tilde{U}}{\partial r}, \quad \text{where} \quad \tilde{U} = U - \frac{c^2}{2r^2}.$$

Considered as a function of r, $\tilde{U}$ is the *amended potential function.* To this equation should be added the *law of areas*

$$\dot{\theta} = \frac{c}{r^2},$$

where $(x', x'') = (r\cos\theta, r\sin\theta)$. From now on, one can perform the complete integration as usual by taking $\frac{c}{r}$ as a new variable.

Remark. Exactly as we did on the line, we can regularize the collision by setting $x = z^2$ and $dt = |z^2|d\tau$. Of course, the variables z and $x = x' + ix''$ are now complex numbers (see also [Mi1]).

3-2. The two-body problem : relative motions [AC1]

This is the only case where a configuration is determined by its size, i.e. by $I = \frac{m_1 m_2}{m_1+m_2} r_{12}^2$. In particular, $U = \mathcal{G}(m_1 m_2)^{1-\kappa}(m_1+m_2)^\kappa I^\kappa = kI^\kappa$. The endomorphisms B, C, D of the 1-dimensional space $\mathcal{D}^*$ are homotheties of respective ratios their traces I, J, K and there are no antisymetric forms, so that the equations of relative motion reduce to the ones on the traces. The integrals of the reduced system are the energy $H = \frac{1}{2}K - kI^\kappa$ and the squared norm $c^2 = |\mathcal{C}|^2 = IK - J^2$ of the angular momentum. Fixing these, one determines an integral curve. The corresponding motion takes place on a line if and only if $IK - J^2 = 0$ (this is just the case of equality in Cauchy-Schwarz).

Note the two ways of reducing the problem : one can first go the quotient by the symmetry group and then fix the invariants of the first integrals, or first fix the integrals as we did in 3-1 and then go to the quotient by the subgroup fixing these integrals (elimination of θ). Of course the difference is really meaningful if the dimension of E is strictly bigger than 2. In dimension 3, the last way leads to the *reduction of the node*.

Remark. There are other cases where the potential function U depends only on the moment of inertia I. Apart from the trivial case of harmonic oscillators ($\kappa = 1$) where U is proportional to I, there is the curious case of the colinear three-body problem with equal masses and $\kappa = 2$. The reader will check the algebraic identity

$$\frac{\big((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\big)^2}{2} = (x_1-x_2)^4 + (x_2-x_3)^4 + (x_3-x_1)^4$$

which implies that U is proportional to I^2. Actually, apart from these cases, the colinear three-body problem is "integrable" only when $\kappa = -1$ (Jacobi-Banachiewitz) or $\kappa = \frac{1}{2}$ (constant force).

4. Generalized two-body motion : homographic solutions [AC1]

One calls *homographic* a solution $z(t) = \big(x(t), y(t)\big)$ of equations (N) such that there exists a real function ν of the time and a relative configuration β_0 with $\beta(t) = \nu(t)^2\beta_0$, where $\beta(t) = {}^t x(t) \circ \epsilon \circ x(t)$. Intimately linked with the symetries of newtonian-like hamiltonians, the homographic solutions are the only solutions one is able to explicitely compute for the general n-body problem. They comprise two important particular cases :

– *homothetic solutions solutions* $z(t) = \big(x(t), y(t)\big)$ such that there exists a real function ν of time and an absolute configuration x_0 with $x(t) = \nu(t)x_0$.

–*rigid solutions* $z(t)$ such that the relative configuration $\beta(t)$ does not depend on time.

The setting introduced in the first two chapters allows giving nice and short proofs of all the affirmations and propositions which follow :

–The configuration x of a homothetic solution at any time is characterized by the fact that it is *central* : there exists a real number λ such that $dU(x) = 2\lambda\epsilon \circ x \circ \mu$ (this number is necessarily equal to $\frac{\kappa U}{I}$).

–A motion is rigid if and only if it is a motion of *relative equilibrium*, defining an equilibrium of the relative equations $(NRel)$: it is characterized by the equation $\dot{\mathcal{E}} = 0$, that is

$$\gamma = 0, \quad \delta + {}^t A \circ \beta + \beta \circ A = 0, \quad [A, \rho) = [A, \beta) = 0.$$

In particular, the relative configuration of a rigid motion satisfies the equation

$$[A, \beta) = 0.$$

Such configurations are said to be *equilibrated* (in French *équilibrées*). Their name comes from the fact that they are exactly the configurations which admit a relative equilibrium motion in a space of big enough dimension ($2n - 2$ is of course sufficient for n bodies) : the attracting forces can be exactly "equilibrated" by the centrifugal forces. A central configuration is equilibrated because one deduces from the definition that $\beta \circ A = \lambda\beta = {}^t(\lambda\beta) = {}^t A \circ \beta$.

Proposition. *The configuration of a homographic motion is equilibrated. It is even central, except possibly in dimension higher than two if the degree of homogeneity 2κ of U is -2, or if the motion is rigid.*

One can give a fairly good description of what is a homographic motion but describing the possible configurations, for example the central ones, is a very difficult problem as soon as there are more than three bodies (see 5-2). Let us start with solutions of relative equilibrium :

Proposition. *A motion of relative equilibrium is a uniform rotation of the absolute state z (not periodic in general as soon as the dimension of E is at least 4) i.e. there exists a constant antisymetric form Ω on the space of motion such that $\dot{z} = \epsilon^{-1} \circ \Omega \circ z$. In the newtonian case the motion takes place in a space of even dimension.*

If we put aside the Jacobi-Banachiewitz potential, the remaining cases have central configuration. The description of homothetic solutions is easy : for any normalized ($I = 1$) central configuration x_0 and any real solution $\zeta(t)$ of the differential equation $\ddot{\zeta} = 2\kappa U(x_0)|\zeta|^{2\kappa-2}\zeta$ (essentially (N) in the one-dimensional case), $x(t) = \zeta(t)x$ is a homothetic solution. Moreover, every homothetic solution is of this type.

The following proposition shows that non homothetic ones are indeed *complex homothetic* :

Proposition. *The space of motion* $\operatorname{Im} z$ *of a homographic, non homothetic, solution with central configuration, coincides with the fixed space. For the complex structure on $Hom(\mathcal{D}^*, \operatorname{Im} z)$ induced by the angular momentum, y is at any time a complex multiple of x : if $x_0 = \|x(0)\|^{-1}x(0)$ is the normalized initial configuration, $x(t) = \zeta(t)x_0$ where ζ is a complex function of the time satisfying $\ddot{\zeta} = 2\kappa U(x_0)|\zeta|^{2\kappa-2}\zeta$. Inversely, any complex solution of this differential equation gives rise to a complex homothetic solution.*

One deduces from this proposition that a homographic motion with central configuration, in particular any non rigid homographic motion, is a generalized Keplerian motion : all bodies describe similar conics around the center of mass.

5-1. Equilibrated and central configurations : equations and easy results [AC1]

To write down usable equations of these configurations it is convenient to represent bilinear forms on $\mathcal{D}$ or on $\mathcal{D}^*$ by $n \times n$ matrices. Such a representation is unique in the case of $\mathcal{D}$ but ambiguous in the case of $\mathcal{D}^*$ where we have to chose an extension of the form to $\mathbb{R}^n$. Representing β by the matrix of general term $-\frac{1}{2}s_{ij} := -\frac{1}{2}r_{ij}^2$, $d\hat{U}$ by the matrix of general term $-\frac{\partial \hat{U}}{\partial s_{ij}}$ and μ^{-1} by the diagonal matrix of the m_i^{-1}, one finds that $\Pi = \beta \circ d\hat{U} \circ \mu^{-1} = \beta \circ A \in Hom(\mathcal{D}^*, \mathcal{D})$ is represented by the matrix whose coefficients are the

$$P_{ij} = \frac{1}{2m_j} \sum_{l \neq j} (s_{il} - s_{ij}) \frac{\partial \hat{U}}{\partial s_{lj}}.$$

To get convenient coordinates for the antisymetric part of Π, one notices that the exterior product by $(1, \cdots, 1)$ factorizes through an embedding of

$\bigwedge^2 \mathcal{D}$ in $\bigwedge^3 \mathbb{R}^n$. The coordinates of the image of the bivector $\Pi - {}^t\Pi$ by this injection are the

$$\begin{aligned} P_{ijk} &= P_{ij} + P_{jk} + P_{ki} - P_{ik} - P_{kj} - P_{ji} \\ &= -\frac{1}{2}\nabla_{ijk} + \frac{1}{2}\sum_{l\neq ijk} Y^l_{ijk}, \end{aligned}$$

with $i < j < k$, where

$$\nabla_{ijk} = \begin{vmatrix} 1/m_i & 1/m_j & 1/m_k \\ s_{jk} - s_{ki} - s_{ij} & s_{ki} - s_{ij} - s_{jk} & s_{ij} - s_{jk} - s_{ki} \\ \partial\hat{U}/\partial s_{jk} & \partial\hat{U}/\partial s_{ki} & \partial\hat{U}/\partial s_{ij} \end{vmatrix},$$

and

$$Y^l_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ s_{jk} + s_{il} & s_{ki} + s_{jl} & s_{ij} + s_{kl} \\ \dfrac{1}{m_i}\dfrac{\partial\hat{U}}{\partial s_{il}} & \dfrac{1}{m_j}\dfrac{\partial\hat{U}}{\partial s_{jl}} & \dfrac{1}{m_k}\dfrac{\partial\hat{U}}{\partial s_{kl}} \end{vmatrix}.$$

The equations $P_{ijk} = 0$, $i < j < k$, define the equilibrated configurations. Of course, if n is strictly greater than three 3, we get too many equations but this is nevertheless the best way of writing down the equations.
In the newtonian case, $\hat{U} = \sum_{1\leq i<j\leq n} m_i m_j \Phi(s_{ij})$, where Φ is defined by $\Phi(s) = \mathcal{G}s^{-1/2}$, and the above equations become *linear in the masses* ! A most important property of Φ is the concavity of its derivative φ. For three bodies, it implies immediately the following proposition :

Proposition. *In the newtonian case, a configuration of three bodies of equal masses is equilibrated if and only if it is isoceles.*

On the other hand, *iii*) of the following proposition shows that the equations of central configurations depend only on the symetric part of Π :

Proposition. *The following conditions are equivalent and they characterize the relative central configurations (i.e. the relative configurations of central configurations).*

i) there exists a real number λ such that $({}^tA - \lambda\mathcal{I}d)|_{\mathrm{Im}\,\beta} = 0$,
ii) there exists a real number λ such that $\beta\circ A - \lambda\beta = 0$,
iii) there exists a real number λ such that ${}^tA\circ\beta + \beta\circ A - 2\lambda\beta = 0$,
These properties imply that $\lambda I = \mathrm{tr}(\mu\circ\beta\circ A) = \kappa U$.

The coordinates of the symetric part of Π on the natural basis of the space $Hom_{\mathrm{sym}}(\mathcal{D}^*, \mathcal{D})$ of symetric bilinear forms on $\mathcal{D}^*$, are the invariant combinations

$$P_{ii} + P_{jj} - P_{ij} - P_{ji} = s_{ij}\Sigma_{ij} + \sum_{h\neq i,j} m_h(S_{ih} - S_{jh}),$$

where

$$\Sigma_{ij} = 2(m_i + m_j)S_{ij} + \sum_{h\neq i,j} m_h(S_{ih} + S_{jh}), \quad S_{ij} = \varphi(s_{ij}).$$

(in the same way, the $r_{ij}^2 = -\frac{1}{2}s_{ii} - \frac{1}{2}s_{jj} + \frac{1}{2}s_{ij} + \frac{1}{2}s_{ji}$ are the coordinates of β as asserted in paragraph 1-1). Following Albouy [A4], we shall write the equations of central configurations setting $\eta = 0$, where the definition of η is the same as the one of Π with A replaced by $A - \lambda\,\mathrm{Id}$, that is U replaced by $U_\lambda = U - \lambda I$ (one can fix λ; the only effect of this will be to determine the "size" I of the configuration).

We give now a variational characterization of the equilibrated and central configurations : For a relative configuration β, let us define its *isospectral manifold* as the set

$$\mathcal{S}_\beta = \big\{\theta \in \mathrm{Hom}_+(\mathcal{D}^*, \mathcal{D}) \mid \mathrm{spectrum}\,(\mu\circ\theta) = \mathrm{spectrum}\,(\mu\circ\beta)\big\}.$$

It is also the set of relative configurations defined by configurations having, *up to rotation*, the same inertia form b.

Proposition. *1) A relative configuration β is équilibrated if and only if it is a critical point of the restriction of $\hat{U}$ to its isospectral manifold. 2) Let E be an euclidean space. A configuration $x \in \mathrm{Hom}(\mathcal{D}^*, E)$ is central if and only if $\|x\|^{-1}x$ is a critical point of the restriction of U to the sphere of equation $I = 1$. 3) A relative configuration β of rank p is central if and only if β is a critical point of the restriction of $\hat{U}$ to relative configurations of the same rank p and moment of inertia $I = \mathrm{trace}\,(\mu\circ\beta)$.*

Comparing *1*) and *3*), one checks again that central configurations are equilibrated : indeed, fixing I is just fixing the spectral invariant η_1, and fixing to p the rank is equivalent to setting $0 = \eta_{p+1} = \eta_{p+2} = \cdots$.

Central configurations are well understood in only two cases : for n bodies on a line or for 3 bodies. In the first case, say for Newton's potential, *Moulton's theorem* asserts that for a given order of the bodies on the line there exists exactly one central configuration [Mou1, A4]. This generalizes a result of Euler for 3 bodies [E1, E2] and is based on an argument of convexity. In the second, *Lagrange's theorem* asserts that whatever be the masses, the equilateral triangle is the only non colinear central configuration of 3 bodies [L1]. This last result is easy to prove in our setting : we have just to look for a critical point of the potential function $\hat{U}(\beta) = m_1m_2\Phi(r_{12}^2) + m_1m_3\Phi(r_{13}^2) + m_2m_3\Phi(r_{23}^2)$ on the set of normalized (say $I = \frac{1}{M}(m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2) = 1$) non flat triangles. Taking the derivatives with respect to the r_{ij}^2's we get that the three quantities $\varphi(r_{ij}^2)$ must be equal, where $\varphi = \Phi'$. This gives the result as soon as φ is injective, which is true in the newtonian case.

5-2. Albouy's theorem on central configurations of four equal masses in the plane [A2, A3, A4]

This case is dual to Moulton's case : in place of $\dim \mathrm{Im}\,\beta = 1$, one has $\dim \mathrm{Ker}\beta = 1$. It is called the Dziobek's case by Albouy.

Let $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0$, represent an element of $\mathcal{D}^*$ which generates $\mathrm{Ker}\beta = \mathrm{Ker}x$ (see 1-1) : $\sum_{i=1}^4 \Delta_j \vec{r}_j = 0$. In other words, the $-\frac{\Delta_j}{\Delta_i}$ are the barycentric coordinates of the i-th body with respect to th three others. According to Mœbius, who by the way was an astronomer, one can normalise the Δ_i's so that $(-1)^i \Delta_i$ be the oriented area of the triangle obtained by forgetting the i-th body. Interpretating the square of the volume of a simplex as we did in 1-2, one can get the following identity, originally used by Dziobek in his studies of central configurations :

$$\frac{\partial\, \mathrm{vol}_{1234}^2}{\partial r_{ij}^2} = k \Delta_i \Delta_j,$$

where k is a constant. From the homogeneity of vol^2, this transforms the coplanarity condition $\mathrm{vol} = 0$ into the condition

$$\sum_{1 \le i < j \le 4} \Delta_i \Delta_j r_{ij}^2 = 0.$$

According to the variational characterization given in 5-1, the relative central configurations are the critical points β of the potential $\hat{U}$ considered as a function on the set of normalised ($I = 1$) configurations such that $\mathrm{vol}_{1234}^2 = 0$:

$$\frac{\partial \hat{U}}{\partial r_{ij}^2} = \nu \frac{\partial\, \mathrm{vol}_{1234}^2}{\partial r_{ij}^2} + \xi \frac{\partial I}{\partial r_{ij}^2},$$

that is for any couple $i < j$:

$$m_i m_j \left(\varphi(r_{ij}^2) - \frac{\xi}{M} \right) = k \nu \Delta_i \Delta_j.$$

Recall from 5-1 that one can always adjust the size of the configurations we are looking for so that $\frac{\xi}{M} = 1$. We shall then set $\tilde{\varphi} = \varphi - 1$ and $\Delta_i' = \frac{\Delta_i}{m_i}$, $k\nu = \nu'$, so that the equations become $\tilde{\varphi}(r_{ij}^2) = \nu' \Delta_i' \Delta_j'$. Recall also from 5-1 the equations of equilibrated configurations, which are satisfied by central configurations, and notice that replacing φ by $\tilde{\varphi}$ in these equations makes no harm.

Theorem. *If φ is increasing ($\varphi' > 0$) and concave ($\varphi'' < 0$), any central configuration of four equal masses in the plane has a symmetry.*

Corollary. *Up to homothety, there are exactly three central configurations of four equal masses in the plane : one convex, the square, and two non convex, the equilateral triangle with a mass at the center of mass and a certain isoceles triangle with a mass somewhere on the axis of symmetry inside the triangle.*

The main idea of the proof of the theorem is that the order of the real numbers $\Delta_i\Delta_j$ dictates the order of the mutual distances r_{ij}^2. If there is no symmetry, we can rename the bodies and eventually change the orientation of the plane so that

$$\Delta_1 < \Delta_2 < \Delta_3 < \Delta_4 \quad \text{and} \quad \Delta_1 + \Delta_4 < 0 < \Delta_2 + \Delta_3,$$

and obtain a sign contradiction in the equations

$$P_{123} = P_{124} = P_{134} = P_{234} = 0$$

of equilibrated configurations given in 5-1. The convex case, defined by $\Delta_1 < \Delta_2 < 0 < \Delta_3 < \Delta_4$, is easier to handle than the non convex one where $\Delta_1 < 0 < \Delta_2 < \Delta_3 < \Delta_4$. For the corollary one needs the assistance of a formal calculus program. The shape of the isoceles triangle is determined by the unique real root of a real polynomial of degree 37. It may be worth recalling that before this work appeared one did not even know that the number of central configurations of four equal masses in the plane was finite ! The reader should now give a look at Albouy's paper [A4] which puts in a particularly nice conceptual setting most of the significant results on central configurations, in particular those of Dziobek, Moulton, Conley, Mœckel and Albouy himself.

6-1. Comparison of the general problem to a one dimensional two-body problem : asymptotic estimations of the size via cluster decompositions [MS1]

The only general method to get asymptotic estimates for the solutions of an n-body problem is to compare it to the only case which is fully understood, that is the two-body problem. The most famous such comparison goes back to Sundman and is decribed in the following paragraph. We start here with a simpler comparison, to a one-dimensional problem, where, contrarily to Sundman's, neither the total energy nor the angular momentum of the two problems are comparable. This method in described in detail in the paper of Marchal and Saari.

To a partition of the set $\{1, 2, \cdots, n\}$ into two components, there corresponds a partition of an n-body configuration into two disjoint *clusters*. It is not hard to prove that the maximum Λ, among all these partitions, of the distances of the centers of masses of two clusters, possesses the following properties (newtonian case):

1) Along a motion, the moments where Λ is not analytic are isolated.

2) There exists a positive constant k, depending only on the masses, such that at each point where Λ is not analytic one has

$$\ddot{\Lambda} \geq -k/\Lambda^2.$$

3) If t_0 is a value of non analyticity of Λ,

$$\lim_{t\to t_0^-} \dot\Lambda(t) \leq \lim_{t\to t_0^+} \dot\Lambda(t).$$

These estimates amount to comparing a given n-body problem to a colinear two-body problem whose Hamiltonian is $f = \dot\Lambda^2 - 2k/\Lambda$. One deduces from 2) and 3) that along a motion, the function $f(\Lambda, \dot\Lambda)$ increases along the parametrized curve $t \mapsto \big(\Lambda(t), \dot\Lambda(t)\big)$ if $\dot\Lambda > 0$, decreases if $\dot\Lambda < 0$. The possible asymptotic behaviours of Λ follow easily from this comparison which is better understood in the plane $(\dot\Lambda, 1/\Lambda)$:

1) Bounded motion : Λ and $\dot\Lambda$ stay bounded,

2) Parabolic motion : Λ goes to ∞ and $\dot\Lambda$ goes to 0,

3) Hyperbolic motion : Λ goes to ∞ and $\dot\Lambda$ goes to a non zero limit,

4) Super-hyperbolic motion : Λ and $\dot\Lambda$ go to ∞.

The last possibility cannot occur for less than four bodies (see 7-1) but it does indeed occur for more than four bodies (see 8 and [SX1]).

Caution. When $t \to \infty$ the asymptotic behaviours of I and Λ^2 are the same but this is definitely not true of their derivatives.

6-2. Comparison of the general problem to a planar two-body problem : Sundman's inequality and Sundman's function [Ch1]

After fixing the center of mass, the motion of a two-body problem takes place on a fixed line if $\mathcal{C} = 0$, in a fixed plane otherwise. In the first case x and y are proportional and Cauchy-Schwarz inequality $IK - J^2 \geq 0$ becomes an equality. In the second case, y is always a complex multiple of x for the complex structure defined by the angular momentum (see 2-3). One easily deduces from this the identity $IK - J^2 = |\mathcal{C}|^2$ as a complex Schwarz equality (compare 3-1).

For more bodies, this equality becomes *Sundman's inequality*

$$IK - J^2 \geq |\mathcal{C}|^2,$$

which one obtains by replacing the norm $\|y\|$ of the velocities configuration y by the norm of its orthogonal projection on the complex line (i.e. real plane) generated by x. Eliminating K with the help of Lagrange-Jacobi relation in case $\kappa \neq -1$, one transforms Sundman's inequality into the differential inequality

$$I\ddot{I} - 2IH - \frac{1}{4}\dot{I}^2 - |\mathcal{C}|^2 \geq 0$$

which expresses that the second derivative of I is always greater or equal to the value it would have for a two-body problem in the plane with same energy, same angular momentum and same values of I and $\dot{I}$. Once H and $|\mathcal{C}|$ given, the integral curves of the corresponding differential equation are the level curves in the plane of coordinates $(I, J = \dot{I}/2)$ of *Sundman's function* S, defined on the phase space by

$$S = I^{-\frac{1}{2}}(J^2 + |\mathcal{C}|^2) - 2I^{\frac{1}{2}}H,$$

and Sundman's inequality is equivalent to saying that the derivative

$$\dot{S} = I^{-\frac{3}{2}}J(IK - J^2 - |\mathcal{C}|^2)$$

of S along a solution has the same sign as $J = \frac{1}{2}\dot{I}$, which means that *I and S are at the same time increasing or decreasing !* In the case of two bodies, Sundman's function is but a constant which depends only on the two masses. As we already noticed in 3-1, it is natural to replace as coordinates I and J by $\frac{|\mathcal{C}|}{r}$ and $\dot{r}$ when $|\mathcal{C}| \neq 0$: the integral curves of Sundman's function become circles orthogonal to the circle of centre $(0,0)$ and radius $\sqrt{-2H}$, each one being characterized by its center, with coordinates $(0, \sqrt{I}U/|\mathcal{C}|)$. Contemplating $\dot{S}$ one sees that the n-body motions along which S remains constant are on the one hand those for which the moment of inertia I remains constant, on the other hand those such that the equality $IK - J^2 = |\mathcal{C}|^2$ remains constantly satisfied, that is the complex homothetic ones (see 4). Notice that in the homographic motions with non central configuration, $IK - J^2$ remains equal to a constant strictly greater than $|\mathcal{C}|^2$.

Problem. Does the constancy of I imply that the motion is rigid ?

Let us give a simple but fondamental consequence of the existence of Sundman's function : in the case of two bodies, a collision can occur only if the motion takes place on a line, that is if the angular momentum is zero. Let us say that a motion of n bodies undergoes a *total collision* (or total collapse) at time t_0 if $\lim_{t\to t_0} I(t) = 0$.

Lemme (Sundman). *A total collision of n bodies can occur only if the angular momentum $\mathcal{C}$ is equal to* 0.

The proof is very simple : one checks that close enough to the total collision, the function I is decreasing, which implies that Sundman's S function is also decreasing. But if $\mathcal{C}$ does not vanish, the term $I^{-\frac{1}{2}}|\mathcal{C}|^2$ forces S to go to $+\infty$.

For Sundman, triple collisions were just an obstacle to the analytical continuation of the solutions of the three-body problem. According to Painlevé, the only "singularities" of the three-body problem are collisions (compare 8); as double collisions can be regularized as branch points (generalize 3-0 and 3-1), it only remained to Sundman to prove the non-accumulation of double collisions to get his famous description of each solution of the three-body problem with non vanishing angular momentum by a convergent series in a new time. By the way, it is well known that this decription is of no use at all for understanding the problem.

Remark. At each point (x, y) of the phase space, the velocities configuration y is the orthogonal sum of a component y_h, proportional to x, which causes a purely homothetic deformation of the configuration, a component y_r of pure rotation and a component y_d which is the only one to induce a deformation of the normalized relative configuration $I^{-1}\beta$. One checks immediately that $\|y_h\|^2 = I^{-1}J^2$, so that in terms of this *Saari's decomposition* of y, Sundman's inequality amounts to replacing in $K = \|y\|^2 = \|y_h\|^2 + \|y_r\|^2 + \|y_d\|^2$, the rotation term $||y_r||^2$ (indeed the squared norm of its projection on the complex line generated by x) by $I^{-1}|\mathcal{C}|^2$, and ignoring the deformation term $||y_d||^2$.

7-1. Non-zero angular momentum : the theorems of Sundman, Birkhoff and Marchal-Saari [Bi1, MS1]

The signication of Sundman's theorem can be precised as follows : supposing $\mathcal{C} \neq 0$ and $H < 0$, we consider the time evolution of the couple $\left(\dot{r}, \frac{|\mathcal{C}|}{r}\right)$ in the upper half-plane foliated by the level circles of Sundman's function (see 6-2 and recall that $r = \sqrt{I}$ measures the size of the system). The evolution curve follows one of these circles if there are two bodies; otherwise, it goes towards bigger circles if $\dot{r} > 0$ and towards smaller ones if $\dot{r} < 0$. This behaviour leaves open two possibilities : either after changing sign a finite number of times, $\dot{r}$ finally stays positive, either it oscillates indefinitely. In the first case, I cannot go to 0. In the second, it could only after an infinite number of oscillations which, due to Sundman's theorem would necessarily take infinite time. Actually, this last possibility was already ruled out in 1912 by Sundman [Su2] in the case of three bodies : he proved that along any motion with non vanishing angular momentum, the size of the system stays bounded away from zero. Fifteen years later, Birkhoff [Bi1] showed more precisely that whenever the system reaches a size smaller than a certain limit, one body must escape : the system asymptotically decouples into two clusters, a single body and a close couple, having each either a parabolic ($r = O(t^{\frac{2}{3}})$) or a hyperbolic ($r = O(t)$) motion with respect to the center of mass of the system. Actually, the fact that superhyperbolic motion is impossible with only three bodies was already proved by Chazy in 1922 [Cha1]. This is intuitive, as the two clusters asymptotically behave as two bodies on a line (compare to 8). Finally a similar but necessarily less precise result was proved in 1974 by Marchal and Saari :

Theorem. *For given values of the total energy $H < 0$ and angular momentum $\mathcal{C} \neq 0$ of an n-body problem, there exists a lower bound $r_{\min}$ of the size of any bounded motion : as soon as the size of the system becomes smaller, one of the bodies at least escapes to infinity.*

The proof consists in controlling the quantity Λ introduced in 6-1 and proving it eventually goes to infinity. The simplest way woud be to get control on its derivative and show it gets bigger than the escape velocity for the one dimensional two body problem, but this turns out to be possible directly only in the case of three bodies where the system eventually decomposes stably into two clusters, so that $\dot{\Lambda}$ is equivalent to $\dot{r}$. When the number of bodies is greater than three, Marchal and Saari succeed in proving that if, after reaching at time t_1 a too small value, the size of the system grows again to a maximum at time t_2, the function G introduced in 3-0 satisfies $G(t_2) - G(t_1) > 0$. This integral estimation implies that there exists at least one value of the time t for which $\frac{d}{dt}G(t)$ is positive. Using 6-1 this proves that Λ goes to infinity.

7-2. Zero angular momentum : the symmetry of homothety and the collision manifold [Ch1]

We saw in 6-2 that $\mathcal{C} = 0$ is a necessary condition for a total collision to occur. The simplest example of such solutions are the homothetic ones which exist as soon as the potential function is homogeneous. The symmetry due to this homogeneity is actually the key to the analysis of total collisions ($I \to 0$) and also of *totally parabolic* solutions ($K \to 0$) which play a dual role. If the potential function $U(x)$ is homogeneous of degree 2κ, this symmetry is materialized on the phase space by the vector-field

$$Y = (x, \kappa y),$$

that is by the differential equation $\dot{x} = x$, $\dot{y} = \kappa y$ (compare to Elie Cartan [Ca1] par. 93), whose flow defines a homothety of the configuration and a cleverly scaled one of the configuration of velocities. This is indeed quite a poor symmetry : it does not preserve the symplectic form ω except when $\kappa = -1$ ($\mathcal{L}_Y\omega = (\kappa+1)\omega$), it does not commute with the integrals of motion ($\mathcal{L}_Y H = \partial_Y H = 2\kappa H$, $\mathcal{L}_Y\mathcal{C} = \partial_Y\mathcal{C} = (\kappa+1)\mathcal{C}$) unless these are equal to zero and even worse it does not commute with the vector-field $X_H = (y, \nabla U(x))$ wich defines the motions except when $\kappa = 1$ ($\mathcal{L}_Y X_H = [Y, X_H] = (\kappa - 1)X_H$). Nevertheless, this last equation implies by Frobenius theorem that the phase space is foliated by integral manifolds of the field of planes generated by Y and X_H. The singular leaves of this foliation are precisely the homothetic solutions. To be able to go to the quotient by the symmetry field Y, it is wiser to generate this foliation by Y and a vector field $\tilde{X}_H$ which commutes with it. This is the case of $\tilde{X}_H = \phi X_H$ as soon as the "integrating factor" $\phi(x,y)$ satisfies $\mathcal{L}_Y\phi = (1-\kappa)\phi$. One possible choice is $\phi = I^{\frac{1-\kappa}{2}}$ and we shall make it. The integral curves of $\tilde{X}_H$ are those of X_H with a different parametrization (a solution ending in finite time for one vector-field may continue indefinitely for the other). Notice that only the components of the normalised angular momentum $|H|^{-\frac{\kappa+1}{2\kappa}}\mathcal{C}$ are invariant by Y. It is only when $\mathcal{C}=0$ and $H=0$, both invariant under Y, that we really reduce the dimension of the problem by going to the quotient. Euler [E1] was the first to use this symmetry to reduce the three-body problem on the line with $H=0$. In the sequel it will be convenient to work with the following invariant functions :

$$\tilde{J} = I^{-\frac{1+\kappa}{2}}J, \quad \tilde{K} = I^{-\kappa}K, \quad \tilde{U} = I^{-\kappa}U, \quad \tilde{H} = I^{-\kappa}H, \quad \tilde{\mathcal{C}} = I^{-\frac{1+\kappa}{2}}\mathcal{C}.$$

The hypersurface $I=1$ turns out to be a good representative of the quotient by Y of the phase space. Let

$$\tilde{Z} = \tilde{X}_H - \tilde{J}Y = I^{\frac{1-\kappa}{2}}X_H - I^{-\frac{1+\kappa}{2}}JY$$

be the unique vector field which has the same image as $\tilde{X}_H$ in the quotient and is tangent to the level hypersurfaces of I. We denote by Z and call *the reduced vector-field* the restriction of $\tilde{Z}$ to $I=1$. Going to the quotient can be done by replacing $\tilde{X}_H$ by Z and $\tilde{J},\tilde{K},\tilde{U},\tilde{H},\tilde{\mathcal{C}}$ by $J,K,U,H,\mathcal{C}$. The flow of Z in the region $I=1$, $H<0$ (resp. $I=1$, $H>0$) reproduces the dynamics of any one of the hypersurfaces of negative (resp. positive) constant energy.

Definition. *The* collision manifold *is the quotient by Y of the set of states with zero energy and zero angular momentum. It can be identified with the set of (x,y) such that $I=1$, $H=0$, $\mathcal{C}=0$.*

It follows from the Lagrange-Jacobi relation (see 2-3) that the function $\tilde{J}$ is a Liapunov function for the restriction to the collision manifold of the reduced vector field Z. As a consequence, *there is no recurrence in the collision manifold.*

Remark. The flow of Y commutes with the natural action $(x,y)\mapsto(Ax,Ay)$ of the isometries A of E. The vector-field Z and the collision manifold go to the quotient by this action as do the functions $I,J,K,U,H,|\mathcal{C}|$. We shall use the same notations after this quotient.

The following lemma and its corollary are easy to prove :

Lemme. *The singularities of the reduced vector-field Z belong to the collision manifold : they are the states (x_0,y_0) which define a homothetic motion of zero energy and verify $J_0 = x_0\cdot y_0 \neq 0$. The integral curves of*

Z which are asymptotic (positively or negatively) to these are contained in the union of the subsets $\mathcal{C}=0$ and $H=0$. They correspond to motions of the n bodies along which either $I\to 0,\ K\to\infty$ in finite time : total collision), or $K\to 0,\ I\to\infty$ (in infinite time : completely parabolic motion). Moreover, if $\mathcal{C}=0$ and $H\neq 0$ (resp. $H=0$ and $\mathcal{C}\neq 0$) it is I (resp. K) which goes to zero.

Corollary. *Under the hypotheses of the above lemma, if I goes to 0 when t goes to t_0, one has $J_0(t-t_0)>0$ and, near t_0, I is equivalent to $\big[(1-\kappa)J_0(t-t_0)\big]^{\frac{2}{1-\kappa}}$, J is equivalent to $J_0\big[(1-\kappa)J_0(t-t_0)\big]^{\frac{1+\kappa}{1-\kappa}}$, K is equivalent to $J_0^2\big[(1-\kappa)J_0(t-t_0)\big]^{\frac{2\kappa}{1-\kappa}}$ and U to $\frac{1}{2}J_0^2\big[(1-\kappa)J_0(t-t_0)\big]^{\frac{2\kappa}{1-\kappa}}$; in the same way, if I goes to infinity when t goes to infinity, one has $J_0t>0$ and, near infinity, I is equivalent to $\big[(1-\kappa)J_0t\big]^{\frac{2}{1-\kappa}}$, J to $J_0\big[(1-\kappa)J_0t\big]^{\frac{1+\kappa}{1-\kappa}}$, K to $J_0^2\big[(1-\kappa)J_0t\big]^{\frac{2\kappa}{1-\kappa}}$ and U to $\frac{1}{2}J_0^2\big[(1-\kappa)J_0t\big]^{\frac{2\kappa}{1-\kappa}}$. This implies that no subclusters are formed : each mutual distance between bodies is of the order of $|t-t_0|^{\frac{1}{1-\kappa}}$ if I goes to 0 at time t_0, and $|t|^{\frac{1}{1-\kappa}}$ if I goes to infinity.*

The following converse of the above results puts together results of Sundman, McGehee, Saari [Su1, Su2, Mc1, Sa2].

Theorem. *A total collision solution (resp. a completely parabolic solution) can exist only if the angular momentum (resp. the energy) vanishes. In both cases, the corresponding integral curve of Z converges to the set of singularities of this vector-field. In particular, the normalized configuration $s=I^{-\frac{1}{2}}x$ tends to the set of central configurations and all the estimates of the above corollary are valid.*

In both cases, the key technical points are on the one side the *existence of a finite non zero limit J_0 of $\tilde{J}$*, which gives time estimates, on the other hand the *existence of a finite non zero limit U_0 of $\tilde{U}$*, a compacity result which insures one stays far from partial collisions and allows proving the existence of a limit set towards which the orbit of Z converges. These two points correspond to the classical asymptotic estimates of $\dot{I}$ and $\ddot{I}$ which are found in Wintner's classical book : in case a total collision occurs at time t_0, I is of the order of $|t-t_0|^{\frac{4}{3}}$, because the temporal derivative of $I^{\frac{3}{4}}$, which is equal to $\frac{3}{2}\tilde{J}$, tends to $\frac{3}{2}J_0\neq 0$; the estimations of $\dot{I}$ and $\ddot{I}$ are the ones one would obtain by formal differentiation but no Tauberian theorem is needed. The only true difference between the two cases is the necessity in the completely parabolic case to *a priori* estimate the asymptotic behaviour of I using 6-1.

8. Going to infinity in finite time [Ch2]

The qualitative study of motions getting close to the simultaneous collision of three or more bodies is one key to unveiling the extraordinary complexity which can be displayed by solutions of the n-body problem. This is not surprising in view of the Thom's principle that to get hold of a global topology or a global dynamics one should first understand the singularities. The BASIC FACT is the *existence of solutions of the three-body problem which after avoiding a total collision, eject one of the bodies at a velocity arbitrarily higher than any former velocity in the system.* It is this phenomenon which was used by Mather and Mc Gehee (for four bodies on the line [MM1]), Xia (for five bodies in space [X1]), and Gerver (for $3N$ bodies in the plane, N big enough [G1]) to prove the existence of (non physical : one needs really punctual bodies) solutions of the n-body problem along which some bodies "go to infinity" in finite time. In the first case, the solution has an infinite number of regularized double collisions; in the two last ones it has no collision at all but passes closer and closer to triple collisions an infinite number of times. This answers a question asked by Painlevé at the end of last century in his famous *Leçons de Stockholm* [Pa1], about the existence of "non collision singularities" which could obstruct the prolongation of a solution. Painlevé's question has given rise to many important works : those of Painlevé himself, who proved on the one hand that the *lim inf* of the minimal distance of two bodies must go to zero in a system tending to a singularity, on the other hand that the singularities of the three-body problem are all due to collisions; those of Von Zeipel [Z1, Mc3], who proved that the size of a system tending to a "non collision singularity" goes necessarily to infinity; those of Mc Gehee [Mc1, Mc2] at last, who unveiled the BASIC FACT. The late discovery of this fact is somewhat amazing when one sees how obvious it is, at least in the colinear case : the conservation of energy implies that at a double collision, the velocity of each body with respect to their center of mass be infinite; if the third mass m_3 hardly misses the triple collision with m_1 and m_2, and collides with m_2 immediately after the collision of m_2 with m_1, it will take advantage of the arbitrarily high velocity of m_2 to rebound itself with a arbitrarily high velocity. Technically, one studies the flow of the reduced vector-field Z (see 7-2) in the neighborhood of the collision manifold. In the Mather-McGehee solutions, m_3 rebounds on a fourth mass m_4 and the same scenario repeats itself indefinitely in an arbitrarily small span of time. The three first bodies are each time closer to a triple collision and, when far from m_3, the couple m_1, m_2 has an essentially elliptic (regularized) motion (with excentricity 1, of course) each time faster and with semi-axis smaller. What makes possible this repetition is that *1)* provided the mass m_3 is small enough, it does come back near m_2 after rebouncing on m_4; *2)* the number of collisions between m_1, m_2 (or the frequency of the essentially elliptic motion) between two returns of m_3 tends to infinity when one starts closer and closer to a triple collision of the first three masses. This last fact allows to actually get closer and closer to this triple collision at each return for a Cantor set of initial conditions.

To get rid of the collisions it seems natural to seek for solutions of the four-body problem in the plane close to the Mather-McGehee ones. But till now, nobody has succeeded in doing that. In the spatial solutions of Xia, the isolated mass m_4 is replaced by a second couple m_4, m_5 and the maximum symmetry is supposed, to keep the dimension of the phase space reasonable (12 after fixing the center of mass) and to take advantage of the very precise studies by C. Simó [Si1] of the flow on the collision manifold of the isoceles three-body problem : the messenger m_3 moves on a fixed line which is a symmetry axis of the system and each one of the triplets (m_1, m_2, m_3) and (m_3, m_4, m_5) remains all the time isoceles. When far from m_3, each couple has around its own center of mass an essentially elliptic motion whose excentricity is bigger and bigger and semi-axis smaller and smaller after each interaction. At each step one gets closer to a simultaneous double and triple collision and finally the centers of mass of these two couples go to infinity in finite time. Checking the possibility of such motions without collision is not easy because, as one gets closer and closer to a simultaneous triple and double collision, the total angular momentum must be equal to zero. A clever argument shows this is garanteed asymptotically provided the limit directions (one has to show they do exist !) of the axis of the two couples are neither parallel nor orthogonal. Another difficulty is that the simple reboncing on m_4 must be replaced by a careful shooting close to a triple collision : if the synchronisation is not good enough, m_3 may just go through the couple m_4, m_5 and get definitively lost ! Finally, let us mention Gerver's solutions [G1] where a planar regular polygone, the N (big) vertices of which are the centers of mass of N couples, all of the same mass, explodes in finite time under the influence of N small and quick messengers which visit each couple in turns in a synchronized way.

9 Reading : the three-body problem in the plane

The reader is urged to open the wonderful review article [Mo1] by Rick Mœckel. He will recognize many of the topics studied in this course and discover how weird already is the simplest non-integrable N-body problem.

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