Minicourse CIM, Nankai University, May 2014: The planar circular restricted three body problem in the lunar case

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Abstract

The course is a short introduction to some aspects of the simplest non-integrable three body problem, the study of which goes back to the seminal works of Hill, Poincaré and Birkhoff. After Goursat (or Levi Civita) regularisation we are led to study a conservative twist map of an annulus whose boundaries are the so-called Hill's solutions. We shall show in particular the existence of "quasi-collision solutions" that is of collision-free solutions which come arbitrarily close to collisions. In the phase space, such solutions belong to KAM invariant tori of the regularized system. The main references are [Co, CL]. The extensions to the full three-body problem of the existence of quasi-collision solutions, is done in [F, Z]. The works of Poincaré on the restricted problem are described in [C3].

1 The Kepler problem as an oscillator[1]

The (normalized) motions in a plane of a particle submitted to the Newtonian attraction of a fixed center – the so called *Kepler problem* – are the solutions of the equation

$$\ddot{x} = -x/|x|^3,$$

where $x = x_1 + ix_2 \in \mathbb{C} = \mathbb{R}^2$ is identified with a complex number and the dot denotes the time derivative. These equations are the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial \bar{y}},\ \dot{y} = -\frac{\partial H}{\partial \bar{x}}$$

associated to the Hamiltonian $H : (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \to \mathbb{R}$ and the symplectic form ω respectively defined by

$$H(x, y) = |y|^2 - 2/|x|, \quad \omega = 2\mathrm{Re}\,(dy \wedge d\bar{x}) = 2(dy_1 \wedge dx_1 + dy_2 \wedge dx_2).$$

[1] A large part of these notes is taken from [C2]

The Levi-Civita mapping $(z,w) \mapsto \big(x = 2z^2,\ y = w/\epsilon\bar{z}\big)$ defines a two-fold covering

$$(L.C.) \qquad K^{-1}(0) \setminus \{z=0\} \to \Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$$

from the complement of the plane $z=0$ in the 0-energy 3-sphere $K^{-1}(0)$ of the harmonic oscillator

$$K(z,w) = |z|^2 + |w|^2 - \epsilon^2 = \epsilon^2|z|^2\left[H\left(2z^2, w/\epsilon\bar{z}\right) + 1/\epsilon^2\right],$$

to the energy hypersurface $\Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$ of the Kepler problem (both diffeomorphic to $S^1 \times \mathbb{R}^2$).

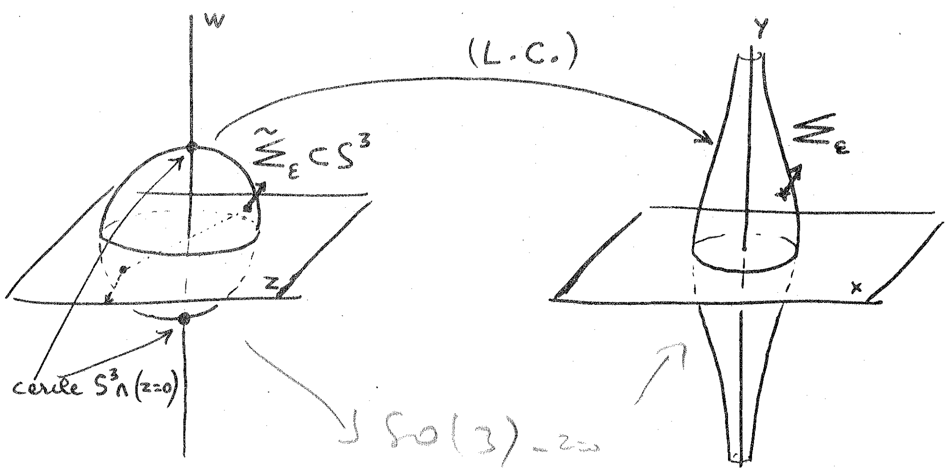

Figure 1: The Levi Civita transformation

It is conformally symplectic and sends integral curves of the harmonic oscillator with energy ϵ^2 to those of the Kepler problem with energy $-1/\epsilon^2$ after the change of time $dt = 2\epsilon|x|dt'$ which prevents the velocity to become infinite at collision. In the coordinates $u_1 = w + iz$, $u_2 = \bar{w} + i\bar{z}$ these integral curves are $u_1(t) = c_1e^{it},\ u_2(t) = c_2e^{it},\ |c_1|^2 + |c_2|^2 = 2\epsilon^2$, that is the intersections of the 3-sphere with the complex lines $u_1/u_2 = cste$, or in other words the fibers of the *Hopf fibration* $(u_1,u_2) \mapsto u_1/u_2 : S^3 \to P_1(\mathbb{C})$. The closest approximation to a section of the Hopf map, the annulus

$$\arg u_1 + \arg u_2 = 0 \pmod{2\pi}$$

is a global surface of section of the flow of the Harmonic oscillator in a sphere of constant energy: with the exception of the two fibers which form its boundary, all the fibers cut this annulus transversally in two points; hence, the second return map is the identity. Thus perturbations of the Kepler problem with negative energy are essentially perturbations of the identity map. This is one of the main sources of degeneracies in celestial mechanics.

2 The simplest non-integrable Hamiltonian: the restricted problem in the lunar case

The equations of the n-body problem

$$\frac{d^2\vec{r}_i}{dt^2} = g\sum_{j\neq i}\frac{m_j(\vec{r}_j-\vec{r}_i)}{||\vec{r}_i-\vec{r}_j||^3}$$

make sense even if some of the masses vanish. Such masses are influenced by the non-zero masses but do not influence them. We shall consider two primaries, say the Sun (mass μ) and the Earth (mass ν) which have a uniform circular motion around their center of mass and a 0-mass third body, say the Moon, which stays close to the Earth. We identify the inertial plane with $\mathbb{C}$ (coordinate $X = X_1 + iX_2$ centered on the center of mass of the couple Sun-Earth) and introduce rotating *(synodic)* complex coordinates (ζ, u) by setting

$$X = \zeta e^{i\omega t}, \quad Y = \dot{X} = ue^{i\omega t}, \quad \text{that is} \quad u = \dot{\zeta} + i\omega\zeta.$$

The equations become

$$\ddot{\zeta} + 2i\omega\dot{\zeta} - \omega^2\zeta = g\mu\frac{\zeta_S-\zeta}{|\zeta_S-\zeta|^3} + g\mu\frac{\zeta_E-\zeta}{|\zeta_E-\zeta|^3},$$

where $\zeta_S = -\frac{\nu}{\mu+\nu}r_0$ and $\zeta_E = \frac{\mu}{\mu+\nu}r_0$ are the respective (fixed) positions of the Sun and the Earth in the rotating frame. They take the following Hamiltonian form (independant of t because of rotational invariance):

$$\frac{d\zeta}{dt} = \frac{\partial H_{syn}}{\partial\bar{u}}, \quad \frac{du}{dt} = -\frac{\partial H_{syn}}{\partial\bar{\zeta}}, \quad \text{where}$$

the Hamiltonian and the symplectic form are respectively

$$\begin{cases} H_{syn}(\zeta,u) = |u|^2 + 2\omega\mathrm{Im}(\zeta\bar{u}) - 2\dfrac{g\mu}{|\zeta_S-\zeta|} - 2\dfrac{g\nu}{|\zeta_E-\zeta|}, \\ \qquad \omega_{syn} = 2\mathrm{Re}\,(du\wedge d\bar{\zeta}). \end{cases}$$

Explanation. In the inertial frame, the equations of motion have the (time-depending) Hamiltonian form

$$\frac{dX}{dt} = \frac{\partial H_{in}}{\partial\bar{Y}}, \quad \frac{dY}{dt} = -\frac{\partial H_{in}}{\partial\bar{X}}, \quad \text{where}$$

the Hamiltonian and the symplectic form are respectively

$$\begin{cases} H_{in}(X,Y,t) = |Y|^2 - 2g\mu\dfrac{\zeta_S e^{i\omega t}-X}{|\zeta_S e^{i\omega t}-X|} - 2g\nu\dfrac{\zeta_E e^{i\omega t}-X}{|\zeta_E e^{i\omega t}-X|}, \\ \qquad \omega_{in} = 2\mathrm{Re}\,(dY\wedge d\bar{X}) = 2(dY_1\wedge dX_1 + dY_2\wedge dX_2). \end{cases}$$

Embedding the extended phase space (coordinates (X, Y, t) as the zero energy surface of the autonomous Hamiltonian $K_{in}(X, t, Y, E) = H_{in}(X, Y, t) + E$, with symplectic form $\Omega_{in} = 2\left(\mathrm{Re}\,(dY \wedge d\bar{X}) + dE \wedge dt\right)$, we check that the transformation

$$(\zeta, u, t, F) \mapsto \left(X = \zeta e^{i\omega t}, Y = u e^{i\omega t}, t, E = F + +2\omega \mathrm{Im}(\zeta \bar{u})\right)$$

is symplectic when the left hand space is endowed with the symplectic form $\Omega_{syn} = 2\left(\mathrm{Re}\,(du \wedge d\bar{\zeta}) + dF \wedge dt\right)$. It follows that in the rotating coordinates, the Hamiltonian H_{syn} is given by the sum

$$H_{syn}(\zeta, u, t) = H_{in}(\zeta e^{i\omega t}, u e^{i\omega t}, t) + 2\omega \mathrm{Im}(\zeta \bar{u}),$$

whose second term is proportional to the *angular momentum*

$$= (X_1 Y_2 - X_2 Y_1) = -\mathrm{Im}(\zeta \bar{u}) = -\mathrm{Im}(X \bar{Y}).$$

Due to the invariance under rotation of the problem, H_{syn} is independant of time. We shall use slightly different coordinates, centered on the earth:

$$x = \zeta - \frac{\mu}{\mu + \nu} r_0, \quad y = u - i\omega \frac{\mu}{\mu + \nu} r_0.$$

Moreover we shall normalize the equations by setting

$$g = 1, \quad \mu + \nu = 1, \quad r_0 = 1, \quad \text{so that} \quad \omega = \sqrt{g(\mu + \nu)}/r_0^{\frac{3}{2}} = 1.$$

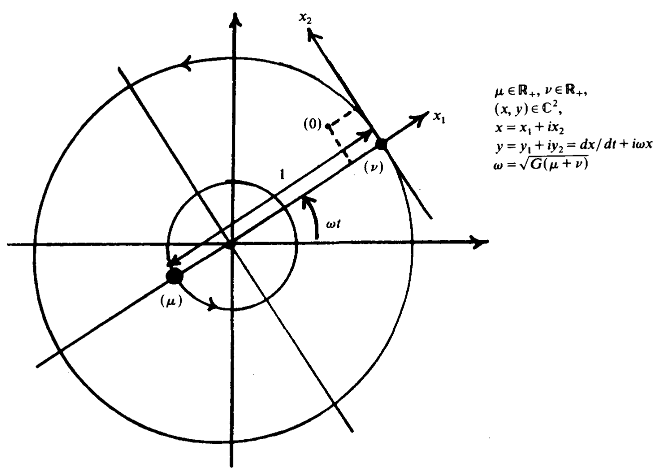

Figure 2: Rotating coordinates

The equations of motion of the Moon become

$$\dot{x} = \frac{\partial H}{\partial \bar{y}}, \; \dot{y} = -\frac{\partial H}{\partial \bar{x}},$$

where the Hamiltonian (up to the constant term which we have changed) and the symplectic form are respectively

$$\begin{cases} H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \dfrac{2\nu}{|x|} - \dfrac{2\mu}{|x+1|} - \mu(x+\bar{x}) + 2\mu, \\ \omega = 2\mathrm{Re}\,(dy \wedge d\bar{x}) = 2(dy_1 \wedge dx_1 + dy_2 \wedge dx_2). \end{cases}$$

As in the first section, we consider the energy hypersurface $H^{-1}(1/\epsilon^2)$, with ϵ a small parameter. Its projection on the x plane is made of three connected components: a neighborhood of the Sun, a neighborhood of the Earth and a neighborhood of infinity (the so-called Hill's regions, which imply Hill's stability result, praised by Poincaré).

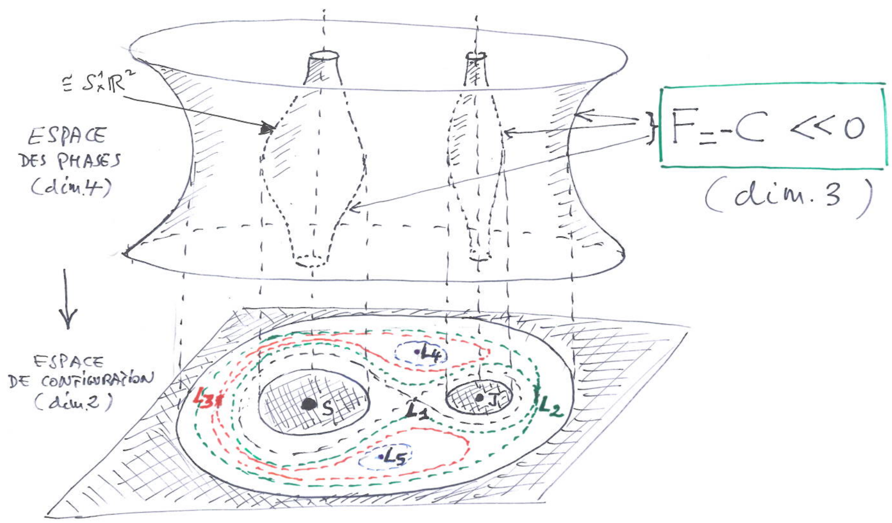

Figure 3: Hill's regions

We shall be interested in the connected component of $H^{-1}(1/\epsilon^2)$ where $|x|$ stays small. Then

$$H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \frac{2\nu}{|x|} - 2\mu\left[\frac{1}{4}|x|^2 + \frac{3}{8}(x^2+\bar{x}^2) + O_3(x)\right].$$

We see that the influence of the Sun on the Moon becomes negligible with respect to the one of the Earth and that at the collision limit, it disappears and one is left with a Kepler problem. To make this apparent, we again apply the Levi-Civita transformation $(z,w) \mapsto \big(x = 2z^2,\ y = w/\epsilon\bar{z}\big)$. We get

$$K(z,w) = \epsilon^2|z|^2\left[H\left(2z^2, \frac{w}{\epsilon\bar{z}}\right) + \frac{1}{\epsilon^2}\right] = f^2(z,w)|z|^2 + |w|^2 - \nu\epsilon^2 - \epsilon^2\mu g(z),$$

where

$$f(z,w) = \sqrt{1 + 2i\epsilon(\bar{z}w - z\bar{w})}, \quad g(z) = 2|z|^2\left(\frac{1}{|2z^2+1|} - 1 + z^2 + \bar{z}^2\right).$$

As in the Kepler case, the direct image of the restriction to $K^{-1}(0)\setminus\{z=0\}$ of the Hamiltonian flow $\dot z=\frac{\partial K}{\partial \bar w}$, $\dot w=-\frac{\partial K}{\partial \bar z}$ becomes the flow of the restricted problem with Jacobi constant $-1/\epsilon^2$ after the change of time $dt=2\epsilon|x|dt'$.

Each truncation of the Taylor expansion of $K(z,w)$ at the origin,

$$K(z,w)=-\nu\epsilon^2+|z|^2+|w|^2+2i\epsilon|z|^2(\bar z w-\bar w z)-\epsilon^2\mu(2|z|^6+3|z|^2(z^4+\bar z^4)+0_8(z)),$$

makes sense dynamically when restricted to $K^{-1}(0)$: we get

at order 2, the harmonic oscillator, which regularizes the Kepler problem;

at order 4, the regularization of the Kepler problem in a rotating frame ;

at order 6, *Hill's problem*. This is the highest order of interest to us.

Remark: Euler's two fixed centers problem and Lagrange's problem. Another classical way of writing down the equations in a rotating frame is to use the variables ζ and $\dot\zeta$ which lead to the following expression for the synodic Hamiltonian:

$$H_{syn}(\zeta,\dot\zeta+i\omega\zeta)=|\dot\zeta|^2-\omega^2|\zeta|^2-2\frac{g\mu}{|\zeta_S-\zeta|}-2\frac{g\nu}{|\zeta_E-\zeta|}.$$

As $\mu|\zeta_S-\zeta|^2+\nu|\zeta_E-\zeta|^2=(\mu+\nu)|\zeta|^2+\mu|\zeta_S|^2+\nu|\zeta_E|^2$ and $\omega^2=g(\mu+\nu)/r_0^3$, this can be written

$$H_{syn}(\zeta,\dot\zeta+i\omega\zeta)=|\dot\zeta|^2-2\Omega(\zeta)+\text{cst},\quad \text{where}$$

$$\Omega(\zeta)=g\mu\left(\frac{|\zeta_S-\zeta|^2}{2r_0^3}+\frac{1}{|\zeta_S-\zeta|}\right)+g\nu\left(\frac{|\zeta_E-\zeta|^2}{2r_0^3}+\frac{1}{|\zeta_E-\zeta|}\right).$$

The relation between the two sets of variables corresponds to the addition of a *magnetic term* to the symplectic form:

$$\text{Re}\,(du\wedge d\bar\zeta)=\text{Re}\,(d\dot\zeta\wedge d\bar\zeta)+i\omega(d\zeta\wedge d\bar\zeta).$$

In other words, the magnetic term in the symplectic form absorbs the Coriolis term in the Lagangian. Forgetting it leads to the Hamiltonian H_L and symplectic form ω_L respectively

$$H_L(\zeta,\dot\zeta)=|\dot\zeta|^2-2\Omega(\zeta),\quad \omega_L=2\text{Re}\,(d\dot\zeta\wedge d\bar\zeta).$$

The Coriolis force is taken out but the repulsive centrifugal force is preserved. If both are taken out, one finds the *two fixed centers problem*, which was shown by Euler to be completely integrable. In case $\mu=\nu$, the system governed by H_L coincides with the so-called *Lagrange problem* which consists in adding a repulsive center located at the middle of the segment joining the two primaries ; it was shown by Lagrange to be also completely integrable (a good reference is [A]). Unfortunately, it was soon realized that these integrable were of no real use in understanding the restricted problem.

3 Hill's solutions

The truncation $\hat{K}(z,w) = -\nu\epsilon^2 + f^2(z,w)|z|^2 + w^2$ of K at fourth order is a completely integrable Hamiltonian, a first integral being the angular momentum or, what is equivalent, the function $f^2(z,w)$. This is not surprising as we already knew that the restriction to $K^{-1}(0)$ corresponds to the completely integrable Kepler problem in a rotating frame. The intersection of level hypersurfaces of K and f^2 defines in general a two-dimensional torus, except when the two hypersurfaces are tangent, that is when $w = \pm if(z,w)z$. In this case the intersection degenerates to a circle; in $K^{-1}(0)$, this defines two solutions which project (by a 2-1 map) onto the two circular solutions (one direct, one retrograde) of the rotating Kepler problem with the given value $-1/\epsilon^2$ of the Jacobi constant.

From now on, two roads may be followed: one can, along with Kummer [K], stick to symplectic coordinates or one can, as did Conley, use the simpler but not symplectic coordinates

$$\xi_1 = w + if(z,w)z, \quad \xi_2 = \bar{w} + if(z,w)\bar{z}.$$

We shall follow Conley. The equations $\dot{z} = \frac{\partial K}{\partial \bar{w}}$, $\dot{w} = -\frac{\partial K}{\partial \bar{z}}$ take the form

$$\dot{\xi}_1 = i\xi_1\left(1 - \frac{\epsilon}{2}|\xi_1 - \bar{\xi}_2|^2\right) + \epsilon^2 O_5(\xi_1,\xi_2),\ \dot{\xi}_2 = i\xi_2\left(1 + \frac{\epsilon}{2}|\xi_1 - \bar{\xi}_2|^2\right) + \epsilon^2 O_5(\xi_1,\xi_2).$$

For this section, we do not need the exact expression of the terms of order 5.

We shall show that the energy hypersurface $K^{-1}(0)$ contains two periodic solutions of minimal periods close to 2π, corresponding to the so-called *Hill's lunar orbits*, direct and retrograde, which are almost circular periodic motions of the Moon around the Earth in the rotating frame. The value 0 of the energy does not play a special role and it is in fact possible to prove the existence of two "Lyapunov" families of periodic solutions stemming from the origin and foliating two smooth (even analytical) germs of invariant surfaces in the (z,w) four dimensional phase space. This is a degenerate version of Lyapunov' theorem, the degeneracy being the double eigenvalues $\pm i$ of the linearization $\dot{\xi}_1 = i\xi_1$, $\dot{\xi}_2 = i\xi_2$, of the vector-field at $\xi_1 = \xi_2 = 0$. Recall that this degeneracy comes from the fact that all solutions of the Kepler problem with a given energy are periodic with the same period. Here are the main steps of the proof of the existence of Hill's orbits.

i) Putting the vector-field into normal form at order 3: the idea, which goes back to Poincaré's thesis and was much developped by Birkhoff, is to simplify as much as possible a finite part of the vector-field's Taylor expansion at the origin by means of local change of variables tangent to Identity. It relies on the fact that replacing $X = (x_1, \cdots, x_n)$ by $Y = X + h(X)$, where the components of $h(X)$ start with terms homogeneous in X of degree r, transforms the equation $\dot{X} = AX + F(X)$ into the equation $\dot{Y} = AY + [A,h](Y) + O_{r+1}$, where $[,]$ is the Lie bracket of the two vector-fields. If $A = diag(\lambda_1, \cdots, \lambda_n)$ and $h = (h_1, \ldots, h_n)$ with $h_s(Y) = y_1^{i_1} \cdots y_n^{i_n}$ and $h_j = 0$ if $j \neq s$, one checks that $[A,h] = k$ with $k_s(Y) = (i_1\lambda_1 + \cdots + i_n\lambda_n - \lambda_s)y_1^{i_1} \cdots y_n^{i_n}$ and $k_j = 0$ if $j \neq s$.

It follows that one can suppress only *non-resonant* terms, i.e. those for which no *resonance relation* $i_1\lambda_1+\cdots+i_n\lambda_n-\lambda_s=$ is satisfied.

In our case, this allows to replace the equations by the following (we kept the same name for the variables):

$$\dot\xi_1 = i\xi_1\left(1+\alpha|\xi_1|^2+\beta|\xi_2|^2\right)+\epsilon^2\varphi_1(\xi_1,\xi_2),$$

$$\dot\xi_2 = i\xi_2\left(1+a|\xi_1|^2+b|\xi_2|^2)\right)+\epsilon^2\varphi_2(\xi_1,\xi_2),$$

with $\alpha=\beta=-\frac{\epsilon}{2}$, $a=b=+\frac{\epsilon}{2}$, φ_1 and φ_2 of order 5 in $\xi_1,\xi_2,\bar\xi_1,\bar\xi_2$. In the neighborhood of the origin, the flow $\Phi_t(\xi_1,\xi_2)=(\xi_1(t),\xi_2(t))$ can be written

$$\xi_1(t)=e^{it}\left[\xi_1\big(1+i(\alpha|\xi_1|^2+\beta|\xi_2|^2)t\big)+\epsilon^2\alpha_1(\xi_1,\xi_2,t)\right],$$

$$\xi_2(t)=e^{it}\left[\xi_2\big(1+i(a|\xi_1|^2+b|\xi_2|^2)t\big)+\epsilon^2\alpha_2(\xi_1,\xi_2,t)\right],$$

with α_1,α_2 of order 5 in $\xi_1,\xi_2,\bar\xi_1,\bar\xi_2$ uniformly in t belonging to a compact.

ii) Regularizing the equations for a periodic solution by means of a blow-up: We look for a periodic solution whose period T is close to the period 2π of the solution $\xi_2=0$ of the rotating Kepler problem approximation (an analogous reasoning can be made for a solution close to $\xi_1=0$). Because of the existence of the energy first integral, the equations which define a periodic solution of period T, that is $\xi_1(T)=\xi_1$, $\xi_2(T)=\xi_2$, are consequence of the equations

$$Arg\,\xi_1(T)-Arg\,\xi_1=2\pi,\quad \xi_2(T)-\xi_2=0.$$

Writing down directly these equations would lead to possibly non differentiable terms like $\alpha_1(\xi_1,\xi_2)/\xi_1$. Indeed, they read

$$2\pi=T+\arg\left[1+i(\alpha|\xi_1|^2+\beta|\xi_2|^2)T+\epsilon^2\frac{\alpha_1(\xi_1,\xi_2,T)}{\xi_1}\right],$$

$$\left[e^{iT}\left(1+i(a|\xi_1|^2+b|\xi_2|^2)T\right)-1\right]\xi_2+\epsilon^2e^{iT}\alpha_2(\xi_1,\xi_2,T)=0.$$

We solve this problem by a further localization in a domain of the form $|\xi_2|\le|\xi_1|$ by means of a complex blow-up

$$\xi_1=z_1,\quad \xi_2=z_1z_2$$

which replaces such a term by $\alpha_1(z_1,z_1z_2)/z_1$ which is now differentiable. The first equation determines T as a C^3 function of $z_1,\bar z_1,z_2,\bar z_2$,

$$T=2\pi-2\pi|z_1|^2(\alpha+\beta|z_2|^2)+o_3,$$

where o_3 vanishes at order 3 along $z_1=0$. The second one becomes

$$2\pi i|z_1|^2\big(a-\alpha+(b-\beta)|z_2|^2\big)z_2+o_3=0_3.$$

As $a-\alpha=\epsilon\neq0$, solving this equation leads to a C^1 surface tangent to the plane $z_2=0$, that is in the (ξ_1,ξ_2) space to a C^2 surface N_1 tangent at order 2 to the plane $\xi_2 = 0$. Intersecting with the energy hypersurface $K = 0$ gives the seeked for periodic solution. In the same way, one proves the existence of N_2 tangent to $\xi_1 = 0$.

iii) Proving the analyticity of N_1 and N_2: this is done in Conley's thesis by closely following the proof given in the non resonant case by Siegel and Moser. To understand the formulas, one suppresses the resonant terms of any order by means of a formal (not convergent !) transformation. One gets new (formal coordinates) ζ_1, ζ_2 such that $\dot{\zeta}_1$ and $\dot{\zeta}_2$ become formal series in the resonant terms $\zeta_i|\zeta_j|^2$ and $\zeta_i(\zeta_j\bar{\zeta}_k)$. Rewriting the computation of periodic solutions as above leads to formal surfaces N_1 and N_2 where, for example, N_1 is defined by a (formal) equation of the form $\zeta_2 = \gamma(|\zeta_1|^2)\zeta_1$, the restriction of the vector-field being of the form $\dot{\zeta}_1 = \alpha(|\zeta_1|^2)\zeta_1$ where α has purely imaginary values (this corresponds to the fact that N_1 is foliated by periodic solutions surrounding the origin). One proves the convergence of γ and α by writing down majorant series.

4 The annulus twist map

Replacing the boundaries $\xi_1 = 0$ and $\xi_2 = 0$ of the Kepler annulus by the two Hill orbits, one can now construct a global annulus of section of the flow in the 3-sphere $K^{-1}(0)$ and analyze the first return map. Such an annulus is of course not unique and it will be convenient to chose it so as to contain the "collision circle" of equation $z = 0$.

In order to get precise enough information on the first return map, one must analyze the equations up to the 5th order where the influence of the Sun comes into play. Writing down a normal form up to this order implies first computing the effect on terms of order five of the change of variables leading to a normal form at order 3. In fact, one can dispense with this: it is enough to suppress only the non resonant terms of order 5, keeping the terms of order 3 as they stood initially. Moreover, the above analysis of the submanifolds N_1 and N_2 whose intersection with $K = 0$ defines Hill's orbits, shows that there exists an analytic change of variables which transforms them into coordinate planes. A finer analysis shows that such a straightening change of variables differs from Id only by terms $\epsilon A + \epsilon^2 B$, where A is resonant of order 5 and B is of order 7. One deduces that such a straightening of N_1 and N_2 does not bring any new change to the differential equation up to order 5. Finally, we get new coordinates (ζ_1, ζ_2) such that N_1 and N_2 are respectively defined by $\zeta_1 = 0$ and $\zeta_2 = 0$, and the energy hypersurface $K^{-1}(0)$ and the collision circle $z = 0$ by

$$\frac{1}{2}\big(|\zeta_1|^2 + |\zeta_2|^2\big) - \nu\epsilon^2 + \epsilon O_6(\zeta) = 0, \quad \text{and} \quad \zeta_1 - \bar{\zeta}_2 + \epsilon O_5(\zeta) = 0.$$

It follows that an annulus of section in $K^{-1}(0)$ containing the collision circle and bounded by the Hill orbits can be defined by the equation (see figure 76)

$$Arg\,\zeta_1 + Arg\,\zeta_2 + \epsilon O_4(\zeta) = 0 \pmod{2\pi}.$$

Computing a little more, one can find coordinates (φ,ρ) on this annulus, such that the two boundaries are close to $\rho=\pm 1$ and the first return map takes the form

$$P_\epsilon(\varphi,\rho)=\left(\varphi+\frac{1}{2}-\frac{\nu}{2}\epsilon^3-\frac{3\nu^2}{2}(1-\frac{\mu}{4})\epsilon^6\rho+0(\epsilon^7),\ \rho+O(\epsilon^7)\right).$$

Coming back to the definition of this annulus, one checks that the return map corresponds essentially to the passages of the orbit of the Moon through aphelium in the rotating frame.

Remark. For writing down formulas, working in the 2-fold covering $K^{-1}(0)$ of the energy hypersurface diffeomorphic to S^3 is convenient but one can prefer to state the results downstairs in the compactification (regularization), diffeomorphic to $SO(3)$ (that is to the real projective space of dimension 3), of the original energy hypersurface $H^{-1}(-\frac{1}{\epsilon^2})$. The first return map then becomes a perturbation of the Identity (the Kepler case) of the form

$$\mathcal{P}_\epsilon(\tilde\varphi,\rho)=\left(\tilde\varphi-\nu\epsilon^3-3\nu^2(1-\frac{\mu}{4})\epsilon^6\rho+0(\epsilon^7),\ \rho+O(\epsilon^7)\right).$$

Originating from a Hamiltonian system, this map necessarily preserves a measure defined by a smooth density. Moreover, it is a $O(\epsilon^7)$ perturbation of an integrable twist map whose twist is of size ϵ^6. This is a perfect ground for applying the main results of the general theory of conservative twist maps, a particular case of the theory of Hamiltonian systems with two degrees of freedom (see section 6):

1) Applied to the iterates of the return map, the *Birkhoff fixed point theorem* yelds an infinite number of periodic orbits of higher and higher periods to which correspond periodic orbits of long period of the Moon around the Earth in the rotating frame;

2) The *Moser invariant curve theorem* implies the existence of a positive measure Cantor set of invariant curves on which the map is conjugated to a diophantine irrational rotation and to which correspond quasi periodic orbits of the Moon;

3) To the Liouville rotation numbers, the *Aubry-Mather theory* associates invariant Cantor sets to which correspond orbits of the Moon with a Cantor caustic

5 Quasi-collision orbits

The Jacobi constant being fixed, the regularized flow takes place in a 3-sphere. For the original (i.e. non-regularized) dynamics, a KAM invariant torus $\mathcal{T}$ of the regularized flow corresponds either to an invariant torus if it does not intersect the collision curve $z=0$, or to a *punctured invariant torus* if it does intersect it. We first show that the collision curve does not lie in an invariant torus:

Theorem 1 ([CL]) *If ϵ is small enough (that is if the Jacobi constant $-1/\epsilon^2$ is large enough), the intersection of the collision circle $z = 0$ with its image under the Poincaré return map P_ϵ on the annulus of section $\mathcal{A}_\epsilon$ consists of exactly eight transversal points*[2].

Proof. Working with the expression of the return map we have just obtained is not convenient because in the coordinates we have used, the expression of the collision curve $z = 0$ turns out to be of the form $\rho = O(\epsilon^3)$, which is not precise enough. Instead, we directly blow up the circle $z = 0$ to a torus boundary by introducing polar coordinates in a transversal section and slowing down the time:

$$z = \epsilon r e^{i\theta}, \quad w = \epsilon(v + iu)e^{i\theta}, \quad dt' = r dt''.$$

In the McGehee-like coordinates coordinates (r, θ, u, v), the equations become

$$\begin{cases} dr/dt'' = rv, \\ d\theta/dt'' = u - 2\epsilon^3 r^3, \\ du/dt'' = -uv + \mu \mathrm{Im}\{\bar{z}(\partial g/\partial \bar{z})\}, \\ dv/dt'' = u^2 - r^2 + 4u\epsilon^3 r^3 + \mu \mathrm{Re}\{\bar{z}(\partial g/\partial \bar{z})\}. \end{cases}$$

The compactified energy level (in fact its two-fold covering)

$$(1/\epsilon^2)K = u^2 + v^2 - \nu + r^2 - 4\epsilon^3 r^3 u - \mu g(z) = 0$$

is diffeomorphic to the solid torus $u^2 + v^2 \leq \nu$, its boundary $u^2 + v^2 = \nu$ (that is $r = 0$) being the *collision manifold* whose flow is depicted on figure 5: the submanifolds W_- and W_+ are made of respectively of the trajectories negatively asymptotic to C_- and the trajectories positively asymptotic to C_+, where the two circles C_- $(u = 0, v = \sqrt{\nu})$ and C_+ $(u = 0, v = -\sqrt{\nu})$ are made of fixed points. They coincide if the perturbation term $\mu\epsilon^2 g(z)$ is absent, as being simply the avatar of the zero angular momentum invariant torus of the rotating Kepler problem.

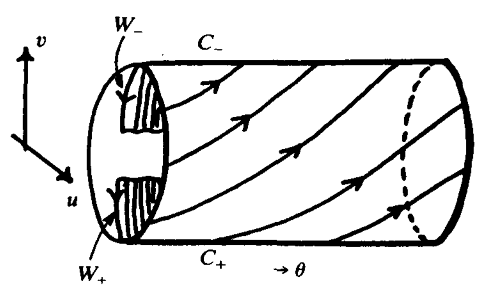

Figure 4: The flow on the collision manifold

[2]which become four if we consider the return map in the annulus of section obtained in $SO(3)$ after blowing down the 2-fold covering.

The proof of the theorem ([CL] page 67) consists in a standard perturbation computation from the even simpler case $\epsilon = 0$, which shows that the intersection of W_- and W_+ consists in exactly eight heteroclinic trajectories, negatively asymptotic to C_- and positively asymptotic to C_+, along which W_- and W_+ intersect transversally (figure 6).

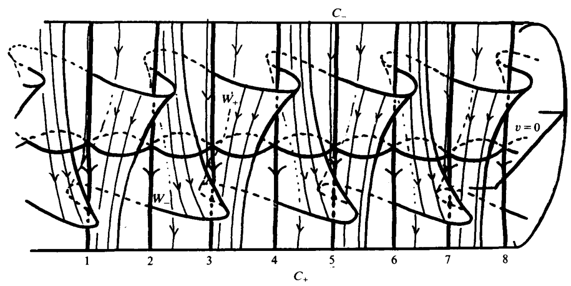

Figure 5: Transversal intersection

Corollary 2 (Existence of invariant punctured tori) *In any neighborhood of $\epsilon = 0$, there are intervals of values of ϵ such that, in the regularized energy level, the collision circle intersects an uncountable number of KAM invariant tori, each in a finite number of points. In the non regularized Kepler energy surface, these correspond to "invariant punctured tori".*

Proof. Theorem 1 implies that the collision circle is not contained in an invariant torus (that is, it is not an invariant curve of the Poincaré return map). Because it is analytic and KAM tori are also analytic, the intersections consist at most in a finite number of points. Varying the value of ϵ moves the invariant curve of a given rotation number across the annulus, which forces intersection with the collision curve. This proves the assertion.

Definition 1 *An orbit without collision which comes arbitrarily close to a collision is called a "quasi-collision orbit".*

Such orbits correspond to motions of the Moon which persistently change their direction of rotation around the Earth in the rotating frame without ever colliding with the Moon (they collide asymptotically when the time goes to infinity).

Lemma 3 *Most of the orbits in an invariant punctured torus are quasi-collision orbits.*

Proof. This comes from the fact that in an invariant punctured torus, the set of initial conditions leading to a collision (that is to one of the punctures) ia finite union of orbits, hence of 2-dimensional measure 0. Figure 6 corresponds to the case where $z = 0$ would intersect transversally in two points an invariant torus of the system in an energy hypersurface of the regularized system. An ejection trajectory is represented.

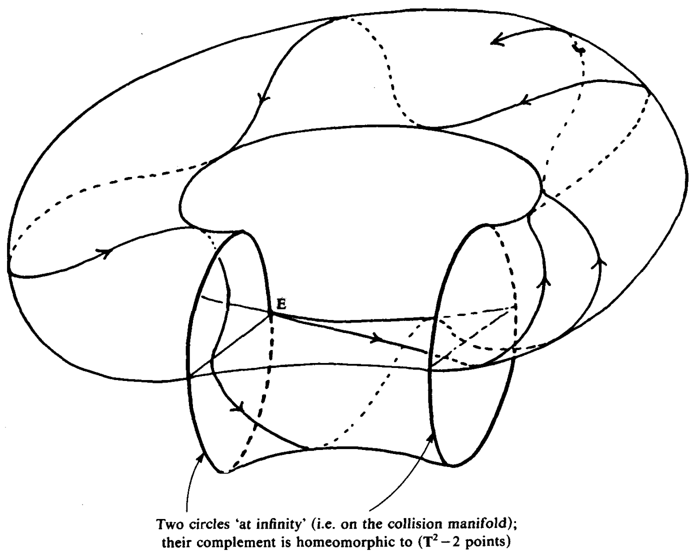

Figure 6: An invariant punctured torus

Remark. The detour through the Levi-Civita regularization was used for coherence with the construction of the annulus of section but it could have been bypassed. Indeed, coming back to the original x, y variables, we get

$$x = 2\epsilon^2 r^2 e^{2i\theta}, \quad y = (v + iu)\frac{e^{2i\theta}}{\epsilon r}, \quad dt = 4\epsilon^3 r^3 dt''.$$

Replacing 2θ by θ, this is essentially the McGehee regularization [McG].

6 Classical results about the dynamics of monotone twist maps of the annulus: a quick sketch

Recall the twist mapping of the annulus that we have obtained :

$$\mathcal{P}_\epsilon(\tilde{\varphi}, \rho) = \left(\tilde{\varphi} - \nu\epsilon^3 - 3\nu^2(1 - \frac{\mu}{4})\epsilon^6 \rho + 0(\epsilon^7),\ \rho + O(\epsilon^7)\right).$$

Because of its origin as the Poincaré return map on a surface of section in some energy hypersurface of a Hamiltonian system, it is *conservative*, that is, it preserves a finite measure equivalent to the Lebesgue measure. In particular, it charges open sets, which implies the *intersection property*: each curve homothetic to the boundaries of the annulus must intersect its image. To such *conservative twist mappings of the annulus*, three fondamental theorems apply, asserting respectively the existence of periodic orbits, Cantor-like invariant sets and regular invariant curves. For the original system, they mean the existence respectively (in the rotating frame) of periodic solutions (in general of long periods) of the Moon around the Earth, of solutions with a Cantorian caustic and of quasi-periodic solutions with a regular caustic, as illustrated on figure 7.

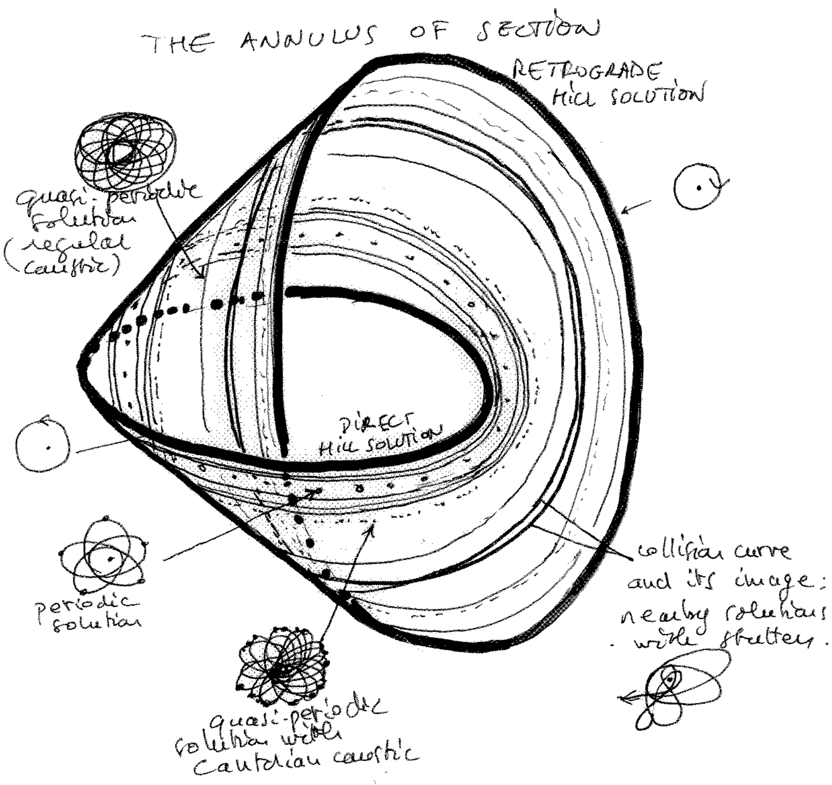

Figure 7: The annulus of section and the return map

6.1 Birkhoff's periodic orbits

Their existence is a consequence of the following theorem, conjectured by Poincaré in the last year of his life and proved by Birkhoff one year after Poincaré's death (see [C3], section 12.2). See also [C5] and the references therein.

A diffeomorphism F of the closed annulus $\mathbb{T}^1 \times [0,1]$ which preserves the boundaries "turns them in opposite directions" if there exists a lift $\tilde{F}$ of F to a diffeomorphism of the universal covering $\mathbb{R} \times [0,1]$ of the annulus which sends the points of the two boundaries in opposite directions.

Theorem 4 (Birkhoff's fixed point theorem) *A conservative distortion of a closed annulus possesses at least two fixed points*

Our diffeomorphism F of the annulus is a *monotone distorsion*, which means that the image of a segment $\tilde{\varphi} =$ constant is a graph over some sector of the circle. In such a case we have much more precise results (see the next paragraph) but let us show that one can readily apply Birkhoff's theorem: because of the monotonicity, the *rotation numbers* ρ_0 and ρ_1 of F on the two boundaries are distinct and it follows that if p/q is a rational number in between these two, the qth iterate F^q of F turns the boundaries in opposite directions. One deduces that there exists periodic points of any rational rotation number between ρ_0 and ρ_1 (recall that, as $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, the rotation numbers are defined modulo 1).

6.2 Aubry-Mather invariant Cantor sets

This is the 2 degrees of freedom case of what Albert Fathi has developped under the name *weak KAM theory*, a theory forrunned by Pierre-Louis Lions, which complements fundamental works by Ricardo Mañé and John Mather. See [C5] and the references therein. For an elementary introduction to the weak KAM theory, see [C6].

Theorem 5 (Existence of Aubry-Mather invariant sets) *For each irrational number ω in between the rotation numbers of the boundaries, a monotone distortion of the annulus possesses invariant sets (which can be of Cantor type but also regular curves as described in the next paragraph) on which the orbits of the restriction of F are circularly ordered as the orbits of the rotation ω.*

A short proof was given by A. Katok: these "well ordered" invariant sets are abtained as limits of "well ordered" periodic orbits; the crux of the argument is the uniform Lipschitz estimates verified by such periodic orbits (see [C5]).

6.3 Moser's invariant curves

Here is the statement of Moser's invariant curve theorem used in [CL].

Theorem 6 *Let $0 < \gamma \leq 1, C > 0, \beta \geq 0$ be three real numbers, ω a real number satisfying $\forall p/q, |\omega - p/q| \geq \gamma C/|q|^{2+\beta}$, and F a real analytic embedding of $(\mathbb{R}/\mathbb{Z}) \times [-\frac{1}{4}, \frac{1}{4}]$ into $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$,*

$$F(\varphi, \sigma) = \{\varphi + \omega + \gamma\sigma + \gamma\Phi_1(\varphi, \sigma),\ \sigma + \gamma\Phi_2(\varphi, \sigma)\}.$$

Suppose that F has the intersection property and consider a neighborhood $\mathcal{A} = \{(\varphi, \sigma), |Im\Phi| \leq a, \sigma \in \mathcal{A}'\}$ of $(\mathbb{R}/\mathbb{Z}) \times [-\frac{1}{4}, \frac{1}{4}]$ in $(\mathbb{C}/\mathbb{Z}) \times \mathbb{C}$ on which the complex extension of F is defined. For each $\eta > 0$ there is a $\delta > 0$ depending on $C, \beta, \mathcal{A}$ but not on γ, such that, if the C^0 norms on $\mathcal{A}$ of Φ_1 and Φ_2 satisfy $||\Phi_1||_{\mathcal{A}} + ||\Phi_2||_{\mathcal{A}} < \delta$, there exists a unique real analytic function $\psi : \mathbb{R}/\mathbb{Z}) \to [-\frac{1}{4}, \frac{1}{4}]$ whose graph is an invariant curve of F on which F is analytically conjugate to the rotation $\varphi \mapsto \varphi + \omega$, and such that $||\psi||_0 < \eta$.

7 Questions

1) When the collision curve does not lie in the closure of a Birkhoff region of instability, i.e. a subannulus not containing any invariant curve, the closure of the union of its iterates is of positive measure, as it contains a Cantor set of positive measure of quasi-periodic invariant curves. What if the collision curve is contained in a Birkhoff region of instability ? If it intersects "genreric" islands around elliptic fixed points, the same will be true but, is it always true ?

2) Is it even possible that, for some value of the Jacobi constant, the collision curve, in addition to lying in a domain of instability avoids also all the Aubry-Mather invariant Cantor sets and all the periodic orbits of the return map ?

3) When the Hill region opens, so that the 0-mass body may visit both primaries, Are there orbits which have a quasi-collision with both primaries. For the 2-fixed centers problem, there is an open subset of such quasi-collision orbits. By perturbation, this shows that the same is true for the restricted problem in case the primaries are far enough from each other, so that the angular velocity of the rotating frame is a small parameter.

8 Comments on the references

The primary sources are [Co, C1, C2, CL]. See also Kummer [K], who choosed to stick to symplectic changes of coordinates. The existence of quasi-collision orbits in the non-restricted three-body problem is proved in [F] in the planar case and in [Z] in the spatial case.

While our small parameter was the ratio of the distance Moon-Earth to the distance Moon-Sun, in the so-called *planetary problem* studied by Poincaré, the small parameter was the mass of one of the primaries (see [C3] section 9).

In chapter 4 of [C2], a dynamics similar to the one studied in this course is described for the equal mass three body problem in the neighborhood of the Lagrange relative equilbrium when restricted to a center manifold.

In [B], Birkhoff regularizes simultaneously the collisions of the zero mass body with the two primaries. Of course this is interesting only in case the Jacobi constant is small enough in absolute value so that one Hill region contains both primaries. The references [LMS, GMS, FGKR, KLMR, M] are some examples of the rich behaviour of the restricted problem for smaller (absolute) values of the Jacobi constant.
To-day, computers allow to get a quite good idea of the global structure of the phase space for 2 degrees of freedom systems. See for example [SiSt] on Hill's problem.
The Scholarpedia article [C4] gives a general view of the Three body problem with some basic references.

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