to Don

Action minimizing periodic orbits in the Newtonian n-body problem

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1. Central configurations, homographic motions, Sundman's inequality

I shall start with a crash course in Celestial Mechanics, to set up notation and to remind you, Don, of your first and everlasting (mathematical) love. For this, I shall use freely [2],[8],[9],[10], where more details and appropriate references will be found.

Equations and first integrals. After using Galilean invariance to fix the center of mass, the *phase space* of the n body problem in $\mathbb{R}^k$ may be identified with the tangent bundle of the *configuration space* $\hat{\mathcal{X}} = \mathcal{X} \setminus \{\text{collisions}\}$,

$$\hat{\mathcal{X}} = \Big\{x = (\vec{r}_1, \ldots, \vec{r}_n) \in (\mathbb{R}^k)^n, \sum_{i=1}^n m_i \vec{r}_i = 0\Big\} \setminus \big\{x,\ \exists\, i \neq j,\ \vec{r}_i = \vec{r}_j\big\}.$$

On $\mathcal{X}$, the *mass scalar product* is defined by

$$x \cdot x' = \sum_{i=1}^n m_i \left\langle \vec{r}_i \cdot \vec{r}_i{}' \right\rangle_{\mathbb{R}^k} \quad \text{if} \quad x = (\vec{r}_1, \ldots, \vec{r}_n),\ x' = (\vec{r}_1{}', \ldots, \vec{r}_n{}').$$

We shall identify the phase space with the cartesian product $\hat{\mathcal{X}} \times \mathcal{X}$ and we shall denote by (x, y) its elements. The equations of the n-body problem, written by Lagrange in 1777, take the form

$$\dot{x} = y, \quad \dot{y} = \nabla U(x),$$

where the dot stands for time derivative and ∇U is the gradient of the *potential function*, or *force function*, U – opposite to the potential energy – defined by

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

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(we chose the gravitational constant equal to one). Considered as a function $U(x,y)$ on $\hat{\mathcal{X}}\times\mathcal{X}$, it is invariant under the diagonal action of the isometry group of the euclidean space $\mathbb{R}^k$:

$$R\cdot(x,y)=(Rx,Ry)\quad\text{and}\quad R\cdot(\vec{r}_1,\ldots,\vec{r}_n)=(R\vec{r}_1,\ldots,R\vec{r}_n).$$

The same is true of the following functions on the phase space:

$$I=x\cdot x,\ J=x\cdot y,\ K=y\cdot y,\ H=\frac{1}{2}K-U,\ L=\frac{1}{2}K+U.$$

These are respectively the *moment of inertia with respect to the center of mass*, half of its time derivative, twice the *kinetic energy* in a frame which fixes the center of mass, the *total energy* and the *Lagrangian*. By a formula of Leibniz, the fact that the center of mass is at the origin implies

$$I=\frac{1}{\sum_{i=1}^n m_i}\sum_{i<j}m_im_j|\vec{r}_i-\vec{r}_j|^2.$$

The *size* $r=I^{\frac{1}{2}}$ of the configurations is a norm on $\mathcal{X}$ (in a more dramatic way, it is called "size of the universe" in [28], at least if we forget a most infortunate $\sqrt{2}$ factor: *please, Don, for the next sixty years, do not use any more 2I for what should be called I. Sundman's inequality is so much nicer without a factor 4 !*). Computing the derivatives $\dot{H}$ and $\ddot{I}$, one proves immediately the *conservation of energy* and the *Lagrange-Jacobi relation*

$$\dot{H}=0,\quad \frac{1}{2}\ddot{I}=K-U.$$

Finally, invariance under rotation implies the constancy of the *angular momentum bivector*

$$\mathcal{C}=\sum_{i=1}^n m_i\vec{r}_i\wedge\dot{\vec{r}}_i,$$

which, using the orientation, can be thought of as a real number (resp. a vector $\vec{\mathcal{C}}$) when $k=2$ (resp. $k=3$). We shall denote by $|\mathcal{C}|$ the norm of the bivector $\mathcal{C}$. Let us recall that the support of $\mathcal{C}$ is an even dimensional subspace of $\mathbb{R}^k$. If $\mathcal{C}\neq 0$, it coincides with the ambient plane in case $k=2$ and with the plane orthogonal to the vector representing $\mathcal{C}$ in case $k=3$. For any k, if $\mathcal{C}\neq 0$, the formula (where we identify $\mathbb{R}^k$ with its dual via the euclidean structure and $\mathcal{C}$ with an antisymmetric operator from $\mathbb{R}^k=(\mathbb{R}^k)^*$ to $\mathbb{R}^k$)

$$\mathcal{I}_{\mathcal{C}}\big(\vec{r}_1,\ldots,\vec{r}_n\big)=\frac{1}{|\mathcal{C}|}\big(\mathcal{C}(\vec{r}_1),\ldots,\mathcal{C}(\vec{r}_n)\big),\quad\text{or}\quad\frac{1}{|\vec{\mathcal{C}}|}(\vec{\mathcal{C}}\wedge\vec{r}_1,\ldots,\vec{\mathcal{C}}\wedge\vec{r}_n)\quad\text{if}\quad k=3,$$

defines a complex structure, that is an operator $\mathcal{I}_{\mathcal{C}}$ whose square is $-$Identity, on the subspace of $\mathcal{X}$ formed by n-tuples $(\vec{r}_1,\ldots,\vec{r}_n)$ with each component in the support of $\mathcal{C}$ (in particular, it is defined on the whole of $\mathcal{X}$ if $k=2$). The inequality $||\mathcal{I}_{\mathcal{C}}(x)||\leq||x||$ always holds. We shall freely speak of $\mathcal{I}_{\mathcal{C}}$ as a true complex structure and we shall call the set of elements of the space $\mathcal{X}$ of the form $\lambda x + \mu \mathcal{I}_{\mathcal{C}}(x)$ with real λ and μ, the *complex line generated by* x.

Two bodies. After fixing the center of mass, the motion of a two-body problem takes place on a fixed line if $\mathcal{C} = 0$, in a fixed plane otherwise. In the first case x and y are proportional and Cauchy-Schwarz inequality $IK - J^2 \geq 0$ becomes an equality. In the second case, y is always a complex multiple of x for the complex structure defined by the angular momentum. One easily deduces from this the identity $IK - J^2 = |\mathcal{C}|^2$ as a complex Schwarz equality.

More than two bodies. The previous equality becomes *Sundman's inequality*

$$IK - J^2 \geq |\mathcal{C}|^2,$$

which one obtains by replacing the norm $\|y\|$ of the velocities by the norm of its orthogonal projection on the complex line (real plane) generated by x.

Once written $K \geq J^2/I + |\mathcal{C}|^2/I$, Sundman's inequality is nicely interpreted in terms of *Saari's decomposition of the velocities* [28]: some linear algebra shows that for each (x, y), the "velocity configuration" y is the orthogonal sum of a component y_h, proportional to x, which induces a homothetic variation of the configuration, a purely rotational component y_r, i.e. such that there exists an antisymmetric operator Ω on the euclidean space $\mathbb{R}^k$ satisfying for each i, $\dot{\vec{r}}_i = \Omega \vec{r}_i$, and a component y_d which corresponds to a deformation of the normalized configuration $r^{-1}x = I^{-\frac{1}{2}}x$. From the orthogonality of the three components one deduces that $K = ||y||^2 = ||y_h||^2 + ||y_r||^2 + ||y_d||^2$. Computing $x \cdot y = x \cdot y_h$, one sees that $y_h = I^{-1}Jx$, that is $||y_h||^2 = I^{-1}J^2 = (\dot{r})^2$. Finally, one checks that y_d is orthogonal to the complex line generated by x. So Sundman's inequality amounts to bounding from below the rotation component $||y_r||^2$, in reality the squared norm of its projection onto this complex line, by $I^{-1}|\mathcal{C}|^2$, and ignoring the deformation component $||y_d||^2$.

The equality $IK - J^2 = |\mathcal{C}|^2$ holds at a given moment if and only if ([2] lemma 3.1)
– on the one hand $y_d = 0$ and y_r belongs to the complex line generated by x,
– on the other hand $||\mathcal{I}_{\mathcal{C}}(x)|| = ||x||$, that is if the motion takes place in a plane.
The equality holds for every t if and only if y is at each moment a complex multiple of x (a real multiple if $\mathcal{C} = 0$). This implies (see [2]) that

$$x(t) = \zeta(t)x_0, \quad \text{where} \quad \zeta(t) \in C, \quad \ddot{\zeta}(t) = -U\left(\frac{x_0}{|x_0|}\right)\frac{\zeta}{|\zeta|^3},$$

which means that, in such motions, the bodies describe around their center of mass similar keplerian motions.

Central configurations. The cases of equality of Sundman's inequality described above can exist only with very special configurations, the *central configurations*. These are precisely the critical points of the homogeneous function of degree zero, $\tilde{U} = \sqrt{I}U = rU$ (called "configuration measure" in [28]) or, equivalently, the critical points of the restrictions of U to the spheres $I =$ constant. The corresponding motions are called *homographic motions with central configuration*. The particular cases of the circular motions, where the configuration remains constant up to isometry, are called *relative equilibria* (for the specialists, see [2] to discover that if $k \geq 4$, a quasi-periodic relative equilibrium motion can arise with a non-central configuration). Conversely, every central configuration admits *homothetic motions* and, provided the ambient space dimension k is even, periodic *homographic* motions for which the configuration remains constant up to similarity (that is composition of homothety and isometry) [2].

The determination of the similarity classes of central configurations is a major problem which pertains to algebraic geometry. The only cases completely understood are the following:

(i) the collinear case ($k = 1$): for each set of n (positive) masses, Moulton's theorem asserts the existence of exactly one central configuration up to homothety for a given order of the bodies. This gives $n!/2$ different configurations up to similarity;

(ii) the case of three bodies: to the Moulton's collinear configurations, due to Euler in this case, one must add the Lagrange equilateral configuration;

(iii) the case of four bodies of equal masses: Albouy's theorem says that, excepting the Moulton's collinear configurations and the regular tetrahedron, one has only three more planar configurations, namely the square, the equilateral triangle with a mass at the center of mass and a particular isosceles triangle with a mass on the axis of symmetry a little above the center of mass.

For general n and k, not even the finiteness of the number of similarity classes of central configurations is known.

Homographic motions with central configuration are the simplest periodic motions of the n-body problem and the only explicit ones (if one forgets that the possible configurations are unknown for n bigger than 3). They can be defined by mimizing $IK - J^2 - |\mathcal{C}|^2$. Among them, only those whose configurations minimize $\tilde{U}$ will be met again in the sequel.

2. Minimizing the action

Let T be a positive real number and $\Lambda_T = H^1(I\!R/T\mathbb{Z}, \mathcal{X})$ be the Sobolev space of those mappings ("loops") which are, together with their first derivative in the sense of distributions, square integrable. We call

$$\mathcal{A}_T : \Lambda_T \to R \cup \{\infty\}, \quad \mathcal{A}_T(x) = \int_0^T L\big(x(t), \dot{x}(t)\big)\, dt,$$

the *action functional* or simply the *action*. It is well known that, as for any hamiltonian system, T-periodic orbits of the n-body problem are critical points of the action functional. For such a positive functional, the critical points easiest to find are certainly the absolute minimizers – which are the subject of this paper – but three obstacles lay on the way:

A) non-coercivity. Minimizing $\mathcal{A}_T$ over the full functional space $H^1(I\!R/T\mathbb{Z}, \mathcal{X})$ is not very rewarding as one readily checks that the minimum is attained "at infinity" by configurations with all bodies infinitely separated and moving infinitely slowly on infinitely small closed curves. The minimum value is of course zero.

B) collisions. Due to Sundman's estimates

$$U = O(|t - t_0|^{-\frac{2}{3}}) \quad \text{and} \quad K = O(|t - t_0|^{-\frac{2}{3}})$$

in the neighborhood of a collision time t_0 (see [10]), the action of a segment of solution leading to a collision remains finite. So, a critical point of $\mathcal{A}_T$ may include such segments and even an infinite number of them, though the set of collision times must have measure zero.

C) triviality. One has to find conditions under which an absolute minimum cannot be a homographic periodic solution (we shall consider that these are "known" even if their actual determination is a very hard open problem).

A) Coercivity. The action functional is called *coercive* if it goes to infinity as I goes to infinity. Coercivity prevents an absolute minimizer from being a "critical point at infinity".

There are two natural ways of restricting the functional space so as to make the action functional coercive:

i) Topological constraints. These can be *homological* or *homotopical*, the two notions being the same in the case of two bodies.

ii) symmetry constraints. We shall see that, up to now, these appear to be much more tractable than the the topological ones as far as proofs are concerned.

In both cases, the trick is to impose conditions on the loops such that if at any given time the configuration is big, then the loop itself, and consequently the action, must be big.

A-i1) Topological constraints: homology. The best known example and the one which, in the Newtonian case, is the mother of all topological ones, is Gordon's theorem [19] on the planar Kepler problem, equivalent to the case $n = 2$, $k = 2$. It asserts that among all planar loops which encircle the attracting center (i.e. which are homologically non trivial), the ones which minimize the action are exactly the Keplerian motions with the given period, including the "ejection-collision" ones. Moreover, if the index of the loop is fixed to a value different from $-1, 0, 1$, the minimizers are exclusively the ejection-collision Kepler motions. The main observation is the convexity of the action functional for the Kepler problem (it is proportional to $T^{\frac{1}{3}}$):

$$\mathcal{A}_T < \mathcal{A}_{T_1} + \mathcal{A}_{T-T_1}.$$

But as early as November 30th 1896, Henri Poincaré [27] had already foreseen part of the story described hereafter, even if he didn't prove himself anything concerning Newton's potential because of the problem of collisions (Poincaré didn't know of course Sundman's work, which is posterior, but he certainly could compute the action in the two-body case and discover it stayed finite along a collision solution; we shall see in the paragraph on collisions how he eliminated this problem by changing the potential). What Poincaré proposed for the three-body problem in the plane, was to minimize the action on a space of loops representing a fixed 1-dimensional homology class in the configuration space. For three bodies in the plane, the configuration space $\hat{\mathcal{X}}$ is diffeomorphic to $\mathbb{R}^4 \setminus 3$ collision planes and its first homology group is isomorphic to $\mathbb{Z}^3$, the three components being the algebraic number of turns that each side of the triangle defined by the bodies undergoes along the loop (see [23]). In reality, Poincaré was interested only in periodic orbits modulo rotation, that is in a rotating frame, but this is immaterial for us, it only changes $\mathbb{Z}^3$ to $\mathbb{Z}^3/(1,1,1)\mathbb{Z} \cong \mathbb{Z}^2$. Indeed, the quotient of the configuration space by the $SO(2)$ symmetry is realized by the Hopf map $\mathbb{R}^4 \to \mathbb{R}^3$

and the reduced configuration space is diffeomorphic to $\mathbb{R}^3 \setminus 3$ half-lines and hence homotopic to a 2-sphere minus 3 points (see [13] and references therein). One checks readily, for example, that coercivity is insured as soon as the homology class $(k_1, k_2, k_3) \in \mathbb{Z}^3$ is such that at least two of the k_i are different from 0 (in his note of 3 pages, Poincaré does not explicitely address the problem of coercivity but he was certainly aware of it). The first results in the Newtonian case using the Poincaré strategy were obtained in July 2000 by Andrea Venturelli [35]. We describe them below (Theorem 1).

A-i2) Topological constraints: homotopy. It is strange that Poincaré did not think to replace the first homology group $H_1(\hat{\mathcal{X}})$ by the first homotopy group $\pi_1(\hat{\mathcal{X}})$ – the fundamental group – that he had just defined the year before in his famous paper on *Analysis situs*. The homotopy class of a loop is a much richer invariant. For example, in the case of three bodies in the plane, it is coded by the braid type of the (colored) braid defined by the bodies in space time $\mathbb{R}^2 \times \mathbb{R}$ (see [23] where one finds more generally a complete study of the coercive classes in the general case of the n-body problem in the plane). The analogous problem for periodic geodesics on negatively curved surfaces was studied quite early but the first study of minimization of the action over a given homotopy class for potentials including Newton's, was done – and only numerically – in 1993 by Cris Moore [25]. We shall come back after stating Theorem 5 to his very interesting paper which, by the way, was completely ignored by specialists in Celestial Mechanics till June 2000 when Phil Holmes made me aware of it after I showed him reference [13]. The same ideas were independantly developed theoretically in 1998 by Richard Montgomery [23] (similar results by Luca Sbano for $n = 3$ appeared at the same time in [29]), but only for "strong force" potentials which avoid collision problems (see below).

In these proceedings, Richard Montgomery [24] shows well the difficulties of minimizing on a given homotopy class: for example, with three bodies, the Lagrange (= equilateral) ejection-collision orbit of the given period is adherent to any homotopy class (put the topology in a tiny ball), so that a collision-free minimizer in any homotopy class must have strictly smaller action than that of Lagrange.

A-ii) Symmetry constraints. The first papers to make use of symmetry constraints to force coercivity seem to have been [18] in the case of 2 bodies (=Kepler problem) and [15] in the general case. In both cases, one restricts the action functional to the subspace Λ_T^a of Λ_T consisting of loops x which satisfy for all t the condition

$$x(t + T/2) = -x(t).$$

This condition is called, for obvious reasons, antisymmetry by analysts and symmetry by geometers. For the sake of œcumenism, we shall call it *(anti)symmetry*. It clearly implies coercivity. Note that among Keplerian motions, only the circular ones are (anti)symmetric.

Let us rephrase the definition of Λ_T^a in the following way: it is the set of invariant elements under the action a of the group $\mathbb{Z}/2\mathbb{Z}$ on Λ_T defined by

$$\big(a(1) \cdot x\big)(t) = -x(t - T/2).$$

More generally, let us consider an orthogonal (i.e. by isometries) representation ρ of a finite (or compact) group G in the real Hilbert space Λ_T (with the H^1 scalar product), such that, for any g in G,

$$\mathcal{A}_T(\rho(g)\cdot x) = \mathcal{A}_T(x).$$

This is the case here because $\mathbb{Z}/2\mathbb{Z}$ acts on R^k by isometry, the symmetry with respect to the origin, and on the circle $R/T\mathbb{Z}$ also by isometry, the rotation by angle π.

Notation. *We call Λ_T^ρ the linear subspace of Λ_T formed by the elements which are invariant under the representation ρ. We call $\mathcal{A}_T^\rho$ the restriction of $\mathcal{A}_T$ to Λ_T^ρ.*

In the sequel we shall consider only representations of the form

$$\rho(g)\cdot x(t) = \alpha(g)\cdot x(\beta(g)^{-1}\cdot t),$$

where α and β are isometric actions of G respectively on $\mathcal{X}$ and on the circle: a loop $x(t)$ is invariant under ρ if and only if it is equivariant under α and β.
There are of course trivial cases in which $\mathcal{A}_T^\rho$ is not coercive, for example if α or β is trivial. But in all the cases we shall consider, the coercivity will be easy to prove. Moreover, the following lemma is a particular instance of Palais' principle of symmetric criticality [26]:

Lemma. *Any critical point of $\mathcal{A}_T^\rho$ is a critical point of $\mathcal{A}_T$.*

Proof. After identification of Λ_T with its tangent space at any point by translation, we deduce from the ρ-invariance of $\mathcal{A}_T$ that, for any $g\in G$,

$$d\mathcal{A}_T(\rho(g)\cdot x)(X) = d\mathcal{A}(x)(\rho(g)^{-1}\cdot X).$$

If x belongs to the vector subspace Λ_T^ρ of elements which are fixed by the action,

$$d\mathcal{A}(x)(\rho(g)^{-1}\cdot X) = d\mathcal{A}(x)(X).$$

As $\rho(g)$ preserves the H^1-scalar product, the H^1-gradient $\nabla\mathcal{A}(x)$ at any point $x\in\Lambda_T^\rho$ satisfies

$$(\rho(g)\cdot\nabla\mathcal{A}(x))\cdot X = \nabla\mathcal{A}(x)\cdot(\rho(g)^{-1}\cdot X) = \nabla\mathcal{A}(x)\cdot X.$$

This means that $\nabla\mathcal{A}(x)$ belongs (=is tangent) to Λ_T^ρ, which implies that a critical point of the restriction $\mathcal{A}_T^\rho$ of $\mathcal{A}$ to Λ_T^ρ is indeed a critical point of $\mathcal{A}$.
As we said, we shall be mainly interested in global minimizers of Λ_T^ρ. Note that, once we have coercivity, the weak lower semi-continuity of the functional implies in a standard way the existence of a minimizer in Λ_T^ρ.
Example: relative equilibrium motions. We suppose $k=2$. Let us consider the "standard" representation ρ_{st} of the circle $G=S^1$ in Λ_T defined by

$$\rho_{st}(\theta)\cdot x(t) = (R_\theta\cdot x)(t-\frac{\theta}{2\pi}T),$$

where $R_\theta\cdot x$ is the rotation of the whole configuration x by the angle θ . A critical point of $\mathcal{A}_T^{\rho_{st}}$ is a relative equilibrium motion $x(t)=R_{\frac{2\pi t}{T}}x(0)$. Note that $\Lambda_T^{\rho_{st}}$ is a subspace of Λ_T^a. We shall come back to this in our first theorem. Related "standard" actions are obtained when replacing the circle by the subgroup $\mathbb{Z}/m\mathbb{Z}$ of m-th roots of unity. Such actions were used in [5].

B) Collisions. More than coercivity, this is certainly the main difficulty of the variational approach to the problem of finding periodic orbits for the Newtonian n-body problem. Proving that a given critical point, say a minimum of $\mathcal{A}_T$ in a well chosen functional space, is a "true" solution without collision, requires in general much work. This explains the popularity of the so called "strong force" problem where the Newtonian exponent -1 in the potential is replaced by an exponent $-a \leq -2$ for which any path leading to a collision has infinite action (see [12]). Few people – including the author before he was told so by Robert McKay in July 2000 at the Rio conference – know that it is Henri Poincaré (and not Gordon, for example) who introduced this notion in the note [27] and proved there that any segment of solution leading to a collision has infinite action.

Proving directly that a minimizer is collision-free appears often to be easier under symmetry constraints than under topological constraints. Here is an example

A simple example: exclusion of total collisions in the (anti)symmetric case. Before giving an overview of known cases where the problem of collisions could be overcome, we show the simplest reasoning by which collisions can be excluded (the first instances of this kind of reasoning are again [18] and [15]). Namely, we shall show that, if $k = 2$, any $x \in \Lambda_T^a$ minimizing $\mathcal{A}_T^a$ has no total collision. If not, we can suppose after translating the time, that a total collision occurs at $t = 0$. The (anti)symmetry assumption implies that a total collision occurs again at $t = T/2$.

Let us use Sundman's inequality to compare the kinetic energy of the ejection-collision path $x|_{[0,T/2]}$ to the one of a Kepler problem on the line: $K \geq (\dot{r})^2$. We cannot do better as Sundman's theorem implies that the angular momentum $\mathcal{C}$ must be zero (see [10]).

Let $U_0 = U_0(m_1, \ldots, m_n; k)$ be the minimal value of $\tilde{U}$. For three bodies in the plane $U_0(m_1, m_2, m_3; 2) = 3\sqrt{3}/\sqrt{m_1 + m_2 + m_3}$ is the value of U on an equilateral triangle of unit size. For four bodies in space, $U_0(m_1, m_2, m_3, m_4; 3) = 6\sqrt{6}/\sqrt{m_1 + m_2 + m_3 + m_4}$ is the value of U on a regular tetrahedron of unit size. For four bodies of unit mass in the plane, $U_0(1, 1, 1, 1; 2) = 4\sqrt{2} + 2$ is the value of U on a square of unit size. One has by definition $U \geq U_0/r$.

Finally, the Lagrangian L may be bounded from below by the Lagrangian of a Kepler problem on the line:

$$L \geq \frac{1}{2}\dot{r}^2 + \frac{U_0}{r}\,.$$

Then Gordon's theorem implies that the action of $x|_{[0,T/2]}$ is greater or equal to the Kepler action of an ejection-collision solution in time $T/2$ of the Kepler problem on the line with potential U_0. Moving to the plane and applying the convexity argument, we see that the action of the (anti)symmetric loop x is strictly greater than the Kepler action of any T-periodic solution of the Kepler problem, in particular of a T-periodic circular solution. But this last action is the same as the one of a T-periodic relative equilibrium solution of the n-body problem in $\mathbb{R}^2$ with a central configuration such that $\tilde{U} = U_0(m_1, \ldots, m_n; 2)$. As such a solution is (anti)symmetric, we have found an element x_0 of Λ_T^a without collision, whose action is strictly smaller than the action of any loop x in Λ_T^a with at least one total collision. This ends the proof that a minimizer in Λ_T^a cannot have any total collision.

Unfortunately, this argument cannot work for partial collisions because if such a collision occurs at time $t=0$, $x(T/2)$ will only be symmetric to $x(0)$ with respect to the origin, not equal to it. It cannot work either in 3-space for 4 bodies or more because a relative equilibrium motion with $\tilde{U}=U_0$ simply does not exist in 3-space (see [2]).

We describe now a few known cases where the absolute minimizers of the action functional $\mathcal{A}_T^\rho$ can be proved to be collision-free solutions of the Newtonian n-body problem.

3. Some collision-free minimizers

i) I shall start with the only result I know – excepting Gordon's one – which uses homological constraints with Newton's potential. This result was proved by Andrea Venturelli in July 2000. It is an exact generalization of Gordon's theorem to the case of three bodies in the plane, and a partial answer to the expectations of Poincaré.

Theorem 1 ([35]). *Let us fix an element $(k_1,k_2,k_3)\in H_1(\hat{\mathcal{X}})=\mathbb{Z}^3$ in the first homology group of the configuration space of the three-body problem in $\mathbb{R}^2$. If $(k_1,k_2,k_3)=(1,1,1)$ or $(k_1,k_2,k_3)=(-1,-1,-1)$, the minimizers of the action $\mathcal{A}_T$ among the loops in this homology class are exactly the Lagrange (i.e. equilateral) homographic solutions. If $(k_1,k_2,k_3)\neq(1,1,1),(-1,-1,-1)$ but each k_i is different from 0, the minimizers of the action $\mathcal{A}_T$ among the loops in this homology class are exactly the Lagrange homothetic ejection-collision solutions.*

The proof, which works only for three bodies, is based on the possibility of writing the action as a sum of three "two-body" actions:

$$\mathcal{A}_T(x)=\sum_{(i,j)=(1,2),(2,3),(3,1)}\frac{m_im_j}{\sum m_i}\int_0^T\left(\frac{|\dot{\vec{r}}_i-\dot{\vec{r}}_j|^2}{2}+\frac{\sum m_i}{|\vec{r}_i-\vec{r}_j|}\right)dt.$$

It is interesting to note that proving directly that a Lagrangian homographic motion with non-zero excentricity is a local minimum of the action seems by no means easy.

ii) All the other results that I shall describe use symmetry constraints. In the (anti)symmetric setting, the first results asserted

1) the existence of a minimizer,
2) the fact that this minimizer is collision-free.

They did not address the problem of triviality because they considered more general potentials than just the Newtonian one, in particular non homogeneous and/or non $SO(2)$-symmetric potentials. They appeared in [5] for the n-body problem in the plane and in [30] [31] for the case of three bodies in space. In the first paper, collisions are shown to be absent from a minimizer by a global estimate. On the contrary, in the last ones, local perturbation arguments are used to show that the presence of a collision, double or triple, prevents a loop from being a local minimizer. These results made the following theorem not too unexpected, even if, as we shall see, it contained some surprise.

Theorem 2 ([11]). *If k is even, or if $n \leq k$, a relative equilibrium motion whose configuration minimizes $\tilde{U}$ is a minimizer of $\mathcal{A}_T^a$. If moreover the similarity classes of central configurations which minimize $\tilde{U}$ are isolated, there are no other minimizers.*

In [11], the second part of the theorem was mistakenly stated without the hypothesis that similarity classes of central configurations be isolated (I am grateful to Vittorio Coti Zelati who did not trust the "proof" given there of equality in Sundman's inequality). I shall discuss this point at the end of the paragraph. The conditions on n, k are necessary to insure the existence of such a relative equilibrium motion (see the comment at the very end of paragraph 2). They had also been forgotten at first and it is a question of Vittorio Coti Zelati (the same!) to the author of the present paper at the Aussois meeting in June 1998 which made this apparent. The question was about the first case where the conditions are not satisfied, namely $n = 4$ and $k = 3$, and it was instrumental in the discovery of the less expected

Theorem 3 [14]. *For four bodies of equal masses in 3-space, any minimizer of $\mathcal{A}_T^a$ is free from total collision but it cannot be a relative equilibrium motion.*

The absence of total collision is proved by almost the same argument we used in the plane: any loop with a total collision has greater action than the ejection-collision loop of the regular tetrahedron, which in turn has greater action than the relative equilibrium motion of the square.
The next candidate is thus a relative motion $x(t)$ of the the square, but a well chosen vertical deformation

$$x(t) \mapsto x(t) + \epsilon \cos \frac{2\pi t}{T}\big((0,0,1),(0,0,-1),(0,0,1),(0,0,-1)\big)$$

shows that such a motion is not even a local minimum. The above deformation is invariant under the action of $\mathbb{Z}/4\mathbb{Z}$ on R^3 (coordinates a, b, c) generated by the isometry

$$S(a,b,c) = (-b, a, -c).$$

This strongly suggests restricting the action $\mathcal{A}_T$ to the subspace Λ_T^{hh} (hh means Hip-Hop) defined by the loops which are invariant under the following orthogonal representation hh of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ in Λ_T:

$$\begin{cases} hh(1,0)\cdot x(t) = -x(t - T/2) \\ hh(0,1)\cdot x(t) = \big(S\vec{r}_4(t), S\vec{r}_1(t), S\vec{r}_2(t), S\vec{r}_3(t)\big), \end{cases}$$

where $x = (\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)$. The action $\mathcal{A}_T$ is invariant under hh because all four masses are equal. One can then prove the

Theorem 4 [14]. *For four bodies of equal masses in $\mathbb{R}^3$, an absolute minimizer of $\mathcal{A}_T^{hh}$ is a collision-free solution of Lagrange equations. Such a solution, called* Hip-Hop, *hesitates periodically between the shape of a tetrahedron and the shape of a square.*

When proving the existence of the Hip-Hop, we were sure that this orbit was new, but I was made aware by Ian Stewart that the family of spatial equal-mass symmetric periodic orbits to which the Hip-Hop belongs was first discovered numerically in 1983 by Ian Davies, Aubrey Truman and David Williams [17]. The symmetry principle is very clearly stated in this nice paper which inspired the abstract developments of Ian Stewart in [34] for non-singular potentials. These solutions were also rediscovered – still numerically – under the suggestive name of "pelotes" by Georges Hoynant [20]. A related paper was written by Ken Meyer and Dieter Schmidt in a perturbative situation [21]. Nevertheless, none of these papers uses the variational method and our existence proof seems to be the first one.

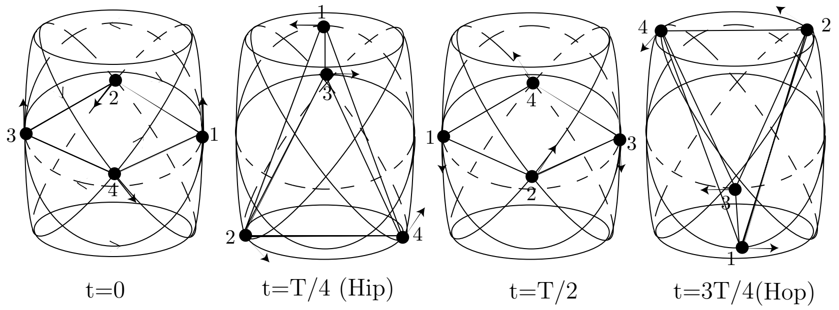

Figure 1

I already said that in the topological setting, a thorough study of the topological hypotheses implying the coercivity property for the n-body problem in the plane had been made by Richard Montgomery in [23]. The related collision problem is discussed by Richard in these proceedings [24]. His study was an important step towards the surprising theorem which follows. Note that in spite of its origin, it is nevertheless based on (very strong) symmetry constraints and not on topological ones. *To describe this result, it will be convenient to label the bodies* $(0, 1, \ldots, n-1)$ *instead of* $(1, 2, \ldots, n)$.

We take $n = 3$ and $k = 2$ and all three masses equal to 1 (planar three body problem with equal masses). We identify $\mathbb{R}^k = \mathbb{R}^2$ with the complex plane $\mathbb{C}$ and denote as usual complex conjugation by $z \mapsto \overline{z}$. For equal masses, Lagrange's equilateral relative equilibrium solution has the peculiarity that all three bodies follow one and the same circle. If we replace the circle by a closed curve symmetric with respect to both coordinate axes, this property is a consequence of the invariance of the corresponding loop under a symmetry of the Lagrangian which we proceed to describe.
Let us recall that the dihedral group D_p, of order $2p$, is the symmetry group of a regular p-gon. It admits the following presentation by generators and relations:

$$D_p = \langle s, \sigma;\ s^p = 1,\ \sigma^2 = 1,\ s\sigma = \sigma s^{-1} \rangle.$$

We shall call "natural" the representation β_0 of D_p by isometries of the circle $S^1_T = \mathbb{R}/T\mathbb{Z}$ defined by

$$\beta_0(s) \cdot t = t + T/p, \quad \beta_0(\sigma) \cdot t = -t.$$

We define a representation $\mathcal{L}$ (for $\mathcal{L}$agrange) in Λ_T of the dihedral group D_6 which leaves invariant Lagrange relative equilibrium. As already said in paragraph A-ii), we define $\mathcal{L}$ by giving representations α and β of D_6 as isometries respectively of $\mathcal{X}$ and S^1_T. We shall take $\beta = \beta_0$ and define α by

$$\alpha(s) \cdot (x_0, x_1, x_2) = (-x_2, -x_0, -x_1), \quad \alpha(\sigma) \cdot (x_0, x_1, x_2) = (\overline{x}_0, \overline{x}_2, \overline{x}_1).$$

The generator s^2 of the normal subroup $\mathbb{Z}/3\mathbb{Z}$ acts by circular permutation:

$$\alpha(s^2) \cdot (x_0, x_1, x_2) = (x_1, x_2, x_0).$$

Note that the action defined by restricting α and β to $\mathbb{Z}/3\mathbb{Z}$, leaves $\mathcal{A}_T$ invariant only because all three masses are equal. It is indeed a remarkable action: the invariants in Λ_T are loops of the form:

$$x(t) = x_q(t) = \big(q(t), q(t + T/3), q(t + 2T/3)\big).$$

This means exactly that the three bodies chase each other around a planar loop $q \in \Lambda_{3,T}$, where

$$\Lambda_{3,T} = \left\{q \in H^1(\mathbb{R}/T\mathbb{Z}), \mathbb{R}^2), \quad q(t) + q(t + T/3) + q(t + 2T/3) = 0\right\}.$$

Moreover, it follows from Theorem 2 that Lagrange's relative equilibrium is the only absolute minimum of $\mathcal{A}^{\mathcal{L}}_T$.

The surprise is that we can define another very natural representation $\mathcal{E}$ (for $\mathcal{E}$ight) in Λ_T of the group D_6. We still take $\beta = \beta_0$ but define α by

$$\alpha(s) \cdot (x_0, x_1, x_2) = (-\overline{x}_2, -\overline{x}_0, -\overline{x}_1), \quad \alpha(\sigma) \cdot (x_0, x_1, x_2) = (-x_0, -x_2, -x_1).$$

The action of the $\mathbb{Z}/3\mathbb{Z}$ part is the same and invariance under it has the same interpretation as above. But if moreover $x(t)$ is invariant under the full representation $\mathcal{E}$ of D_6, the curve q is symmetric with respect to both axes and must satisfy $q(0) = q(T/2) = 0$. In particular, *it cannot be a circle.*

Theorem 5 [13]. *An absolute minimizer of $\mathcal{A}^{\mathcal{E}}_T$ has no collision: it is a zero angular momentum T-periodic solution of the planar three-body problem with equal masses of the form*

$$x(t) = \big(q(t), q(t + T/3), q(t + 2T/3)\big),$$

where q is an "eight-shaped" curve in $\in \Lambda_{3,T}$, symmetric with respect to both axes.

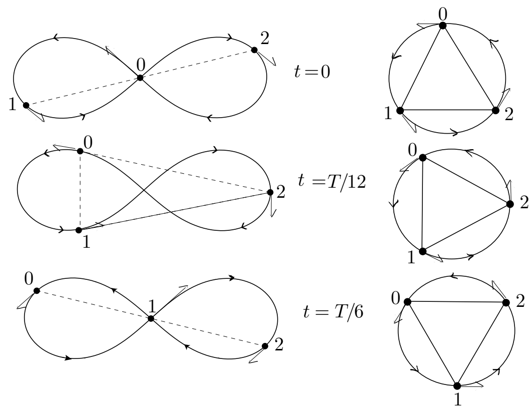

Figure 2

During a period, every Euler configuration is realized twice, once along each of two fixed lines (figure 2). "Eight" orbits are in some sense comparable to the Lagrange relative equilibrium solutions, but in the latter case the angular momentum, instead of being equal to zero, is maximal.

Of course, here too we (and not only we) were sure that this orbit was new. But it had indeed been already discovered numerically by Cris Moore in 1993, in the paper [25] that I already quoted in section A-i2). This is quite remarkable, and even astonishing, because it follows from the theorem of Richard Montgomery [24] alluded to at the end of section A-i2 that the eight cannot be an absolute minimizer in its homotopy (= braid) class (thank you, Richard, for having pointed out this fact). More precisely, the Lagrange ejection-collision solution of the same period, which is adherent to any homotopy class, has smaller action than the eight: it follows from the values of the action given in ([13] Appendix 1) that the action of the eight with period $T = 2\pi$ is equal to $12 \times 2.0309938 = 24.3719256\ldots$ while the action of the Lagrange ejection-collision orbit with the same period is only equal to $2 \times 5.39433 \times 6^{1/3} = 19.60429\ldots$ This situation should be compared to the one described at the beginning of section (Ai1) for the Kepler problem: in the space of loops with an index different from -1, 0 or 1, the minimizer is the ejection-collision Kepler orbit.

Nevertheless, if one imposes the $\mathbb{Z}/3\mathbb{Z}$ symmetry, then, in order that it belongs to the space of invariant loops, the Lagrange ejection collision orbit has to be repeated 3 times, each time with period $T/3$ and with exchange of the bodies. The corresponding action is $3 \times 2 \times 5.39433 \times 2^{1/3} = 40.778579\ldots$ which is now greater than the action of the eight. But up to now, we have no proof of the existence of the eight as a minimum of the action in its homotopy class under the $\mathbb{Z}/3\mathbb{Z}$ symmetry constraint only, even if the numerical computations of Carles Simó ([12],[33]) indicate that it should be true.

More precisely, Simó's computations indicate that"the" eight orbit is probably unique and that "its" shape in the plane is very close to the one of a quartic. Simó [32] has also shown numerically that "this" orbit is stable, that is completely elliptic with (indefinite) torsion. It can be continued for different masses (the three bodies then move along slightly different eight-shaped curves) and the domain of "stability" in the normalized mass triangle is a tiny triangular neighborhood of the equal mass point. This is also in sharp contrast with the Lagrange relative equilibrium solutions, which are "stable" only when one of the masses is much bigger than the two others. That a minimizing periodic orbit may be "stable" is not as surprising as it appears at first sight (see [4]).

Remark. The dihedral group D_6 is isomorphic to the direct product $D_3 \times \mathbb{Z}_2$, that is to the group formed by the isometries of the reduced configuration space $\hat{\mathcal{X}}/SO(2)$ which leave the potential invariant in the case of equal masses (see the end of A-i1)); hence, in contrast with Lagrange relative equilibrium, the eight possesses the full symmetry of the reduced configuration space (there is not a direct eight and a retrograde one because changing the orientation amounts to rotating of π). The group D_6 can also be written as a semi-direct product of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Its quotient by $\mathbb{Z}_3$ is the symmetry group of the planar curve q.

Theorems 4 and 5 have in common that they rely heavily on symmetry asumptions on the masses. Also, in both theorems, the minimizers are non trivial. Indeed the natural candidates to be absolute minimizers – the relative equilibrium motions – are ruled out: in theorem 4 because the regular tetrahedron would need one dimension more to be allowed such a motion, in theorem 5 because the circle does not possess the right D_6-symmetry. Nevertheless, the actual minimizers have much to do with relative equilibrium motions and central configurations. In a sense, they do their best to achieve this impossible goal: *keep I (and in consequence U) constant*. Numerical computations – by Jacques Laskar for theorem 4 and Carles Simó for theorem 5 – show indeed that I and U are not far from being constant in the corresponding solutions. This is pertinent in view of the following

Saari's conjecture: if I is constant along a solution of the n-body problem, the motion is *rigid*, i.e. all r_{ij} remain constant (it is then a relative equilibrium, see [2] prop. 2.5).

Remark on theorem 2 and Saari's conjecture. What is really proved in [11] is that a minimizer $x(t)$ of $\mathcal{A}_T^a$ is a uniform motion along a round circle in the metric space $\mathcal{X}$ of configurations. This implies immediately that, at each time, $x(t)$ is a central configuration because $\ddot{x}(t)$ and $\nabla U(x(t))$ are both proportional to $x(t)$ (this also follows from the fact that $\tilde{U}(x(t))$ is minimal). Hence, if similarity classes of central configurations minimizing $\tilde{U}$ are isolated, the motion is homographic and even a relative equilibrium because of the (anti)symmetry. If this is not the case, a counter-example would exist as soon as an affine straight line (D) of non similar central configurations minimizing $\tilde{U}$ would exist in $\mathcal{X}$: it would be provided by a circle of well chosen radius in the (vector) plane generated by (D). The corresponding solution would look quite strange, each body running around an ellipse centered on the center of mass. And, as $I = cste$, it would be at the same time a counter-example to Saari's conjecture.

Other results of interest. Other cases exist where the minimum has no collision:

1) In [16], V. Coti Zelati shows the existence of (anti)symmetric periodic solutions for the problem of n small masses in $\mathbb{R}^k$, $k \geq 2$, revolving in near-circles around a massive sun. No critical point has collisions because of the perturbative setting.

2) In [3], G. Arioli, F. Gazzola and S. Terracini show that the retrograde Hill's orbit minimizes the action functional in an appropriate functional space.

3) Following the method of [13], Kuo Chang Chen found in [7] new periodic solutions of the 4-body problem in the plane in the case of equal masses. At each instant, the bodies form a parallelogram; two of them rotate in one direction on a curve which looks like an ellipse with center at the center of mass of the system, while the two others rotate in the opposite direction on a similar curve, with the same center, orthogonal to the first one. Here the group is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the normal subgroup $\mathbb{Z}/2\mathbb{Z} \times \{1\}$ expressing that the bodies lie on two curves, the quotient $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ defining the two symmetries which exchange these two (oriented) curves.

Remark. I should stress the fact that I was interested exclusively here in the Newtonian problem and in action minimizing periodic orbits. It was pointed out to me by Vittorio Coti Zelati that in most of the papers on the subject which followed Gordon's paper, the point was not so much to find new results on the n-

body problem as to get results which could survive a perturbation of the potential. These are not discussed here.

4. About the proofs of Theorems 2,4 and 5.

Theorems 2,4 and 5 all rely on symmetry constraints. We try only here to give the main ideas and to put them in perspective, referring to the papers [11],[13],[14] for complete proofs.
In theorem 2, the (anti)symmetry of $x(t)$ implies that its mean is zero, which implies in turn the Poincaré inequality

$$\int_0^T K dt \geq \frac{4\pi^2}{T^2}\int_0^T I dt,$$

with equality if and only if the Fourier expansion of $x(t)$ contains no harmonic but the first one. This implies

$$\mathcal{A}(x) \geq \int_0^T \left(\frac{2\pi^2}{T^2} I + \frac{U_0}{\sqrt{I}}\right) dt,$$

where $U_0 = U_0(m_1, \ldots, m_n; k)$. Taking $I = I_0$, the minimum of the function under the integral sign, leads to a candidate for a minimizer, namely the relative equilibrium motion with size $\sqrt{I_0}$ and configuration corresponding to U_0. The conditions on n, k insure that such a motions does exist. The only subtlety is in the converse assertion that any minimizer is of this type: one shows that equality must be satisfied in Sundman's inequality, which implies that the motion is homographic. Being (anti)symmetric, it is a relative equilibrium.

In theorem 4, the $\mathbb{Z}/4\mathbb{Z}$ symmetry and the invariance under rotation around the vertical axis reduce the system to a 3 degree of freedom system with $SO(2)$ symmetry (the position of one body determines the position of the other three). To exclude the collisions we use a very simple deformation: given a loop with at least one collision, one can assert the existence of another loop with the same action which is contained in a vertical plane containing the z axis. A small rotation of this plane around its intersection with the horizontal plane is then shown to decrease the action thanks to Sundman's estimates in the neighborhood of a collision. Similar studies can be pursued with $\mathbb{Z}/3\mathbb{Z}$ replacing $\mathbb{Z}/4\mathbb{Z}$ and more generally with $\mathbb{Z}/n\mathbb{Z}$ and n or more bodies. The reduced problems one obtains are natural generalizations of the spatial isosceles problem of three bodies.

In theorem 5, a minimizer cannot be a relative equilibrium motion, simply because this would imply that the curve q is a circle and the circle does not possess the required D_6 symmetry. On the other hand, critical points with collisions do exist in $\Lambda_T^{\mathcal{E}}$. One has for example the ejection-collision homothetic motion of an equilateral triangle repeated symmetrically after time $T/2$ ("symmetric" homothetic motion), or the collinear "Schubart's orbit" modified in such a way that the bodies be exchanged after a binary collision (they traverse each other instead of bouncing). To show that an absolute minimizer of $\mathcal{A}_T^{\mathcal{E}}$ has no collision, we introduce a "test curve" $q_0 \in \Lambda_{3,T}$ and show that the action of the corresponding loop $x_{q_0}(t) \in \Lambda_T^{\mathcal{E}}$ is smaller than the action of any loop with at least one double or triple collision. The test curve, which has the shape of an eight, is uniquely defined up to isometry by the following conditions: along the corresponding loop $x_{q_0}(t)$, the angular momentum is zero, the size I_0, the kinetic energy $K_0/2$ and the potential U_0 are constant. Moreover, I_0 is chosen so that the action be minimal among similar loops. This test curve has such a low action that in the evaluation of the action of a loop with collision, one can even forget one mass (the one not participating in the collision in case of a binary collision) and reduce the computation to the case of two bodies. In the original proof, a numerical estimate of the length of some implicitely defined curve was needed (and gracefully provided by Carles Simó and Jacques Laskar who agreed up to an impressive number of decimals). Thanks to refined estimates on the action of loops with collision obtained by Kuo Chang Chen [6], no numerical estimate is necessary any more.

5. Many bodies: choreographies. At the Chicago meeting, I ventured saying that theorems 4 and 5 were probably the beginning of a story, and indeed they were, even if, as we saw, the story really began in 1896. A whole new world of periodic solutions of the n-body problem with equal masses in the plane or in space is being discovered, at the moment only numerically (see [12] and [33]). All these solutions, named *choreographies*, share the property that the n bodies chase each other around a fixed curve. After the "eight", a solution where 4 bodies sit on a "supereight" with one more lobe was found by Joseph Gerver: the configuration stays symmetric with respect to the origin (parallelograms) and the angular momentum is different from zero. Then Carles Simó found and is still finding ad libitum such solutions with more and more general supporting planar curves (in particular curves with no symmetry) when the number of bodies increases. He finds them as *local* minimizers of $\mathcal{A}_T^{\mathcal{R}_n}$ where $\mathcal{R}_n$ is the representation in Λ_T of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ which generalizes the representation of $\mathbb{Z}/3\mathbb{Z}$ used for the "eight":

$$\mathcal{R}_n(1)\cdot\big(x_0(t),x_1(t),\ldots,x_{n-1}(t)\big)=\big(x_1(t-T/n),x_2(t-T/n),\ldots,x_0(t-T/n)\big).$$

As for $n=3$, $\Lambda_T^{\mathcal{R}_n}$ is isomorphic to the space $\Lambda_{n,T}$ of planar curves defined by

$$\Lambda_{n,T}=\big\{q\in H^1(\mathbb{R}/T\mathbb{Z}),\mathbb{R}^2),\quad q(t)+q(t+T/n)+\cdots+q(t+(n-1)T/n)=0\big\}.$$

Trading this action against an action of another cyclic group, one could find solutions where the bodies stay on two curves (see [7], already described at the end of paragraph 3), three curves… Symmetric solutions such as supereights (chains) with up to $n-1$ lobes, flowers, etc, can be obtained as *local* minimizers of $\mathcal{A}_T^{\rho}$ where ρ is a representation of a cyclic or dihedral extension ($\mathbb{Z}/nm\mathbb{Z}$ or D_{nm}) of $\mathbb{Z}/n\mathbb{Z}$ or a product of such an extension by a subgroup of $O(2)$ acting trivially on the circle.

For example, the (direct) flower with a three-fold symmetry and four bodies depicted in Fig. 3c of [12] is a fixed loop of the representation of D_{12} defined (compare to paragraph 3) by $\beta=\beta_0$ and

$$\alpha(s)\cdot(x_0,x_1,x_2,x_3)=(e^{-2\pi i/3}x_3,e^{-2\pi i/3}x_0,e^{-2\pi i/3}x_1,e^{-2\pi i/3}x_2),$$
$$\alpha(\sigma)\cdot(x_0,x_1,x_2,x_3)=(\overline{x}_0,\overline{x}_3,\overline{x}_2,\overline{x}_1).$$

Gerver's (direct) supereight with four bodies depicted in Fig. 3b of [12] is a fixed loop of the representation of $D_4 \times \mathbb{Z}_2$ (different from D_8 !) defined by

$$\beta = \beta_0, \quad \alpha(s)(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3, x_0),$$
$$\alpha(\sigma)(x_0, x_1, x_2, x_3) = (\overline{x}_0, \overline{x}_3, \overline{x}_2, \overline{x}_1),$$

for the D_4 factor,

$$\beta \text{ trivial}, \quad \alpha(1)(x_0, x_1, x_2, x_3) = (-x_2, -x_3, -x_0, -x_1)$$

for the $\mathbb{Z}_2$ factor.

But only the solutions where an odd number n of bodies chase each other around a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric "eight" seem to be obtainable as global minimizers: as in the case $n = 3$, they are fixed loops of the representation of D_{2n} defined by $\beta = \beta_0$ and

$$\alpha(s) \cdot (x_0, x_1, \ldots, x_{n-1}) = (-\overline{x}_{(n+1)/2}, -\overline{x}_{(n+3)/2}, \ldots, -\overline{x}_0, -\overline{x}_1, \ldots, -\overline{x}_{(n-1)/2}),$$
$$\alpha(\sigma) \cdot (x_0, x_1, \ldots, x_{n-1}) = (-x_0, -x_{n-1}, -x_{n-2}, \ldots, -x_1)$$

which is not a symmetry of the relative equilibrium of the regular n-gon.

In contrast, the $D_4 \times \mathbb{Z}_2$-symmetry of Gerver's supereight (direct, resp. retrograde) and the D_{12}-symmetry of the (retrograde, resp. direct) three-lobed flower with four bodies are also shared by the (direct, resp. retrograde) relative equilibrium motion of the square.
Hence such choreographies can be only a relative minimizer of the action in their symmetry class. To prove their existence, one must introduce homotopical constraints.

For spatial solutions also, one already finds choreographies in [17] (implicitly) and in [20] (explicitly): tuning appropriately the period of a reduced periodic solution and the period of vertical oscillations of the diagonals produces indeed solutions where the four bodies stay on a fixed spatial curve. This was later independently noticed by several people, and in particular by Carles Simó and Susanna Terracini. In this way, a potentially infinite number of *spatial choreographies* for four bodies was found numerically to exist.

6. Conclusion: symmetry versus topology.

Let us start with the Newtonian problem of three bodies in the plane. Homological constraints do detect Lagrange's solution whose class is $\pm(1,1,1)$ (Theorem 1) but fail to detect the "figure eight" solution whose class is $(0,0,0)$. Numerically, it was possible to detect this solution as a *local* minimizer using a homotopical constraint [25], or as a *global* one using a homotopical constraint combined with $\mathbb{Z}/3\mathbb{Z}$ symmetry ([12],[33]), but to my knowledge, no existence proof is available in these contexts. Indeed, the only existence proof is the one of Theorem 5, which relies solely on the D_6 symmetry constraint. It is certainly an interesting question to decide if the equality of the three masses implies that there is a collision-free minimum in the homotopy class of the "eight" when imposing only $\mathbb{Z}/3\mathbb{Z}$ symmetry, that this minimum is unique up to isometry and that it possesses the

D_6-symmetry imposed in Theorem 5. For more general choreographies, symmetry constraints are not enough to define the class and must be mixed with homotopy constraints, which makes possible proofs *a priori* more difficult.

As another example, in Theorem 4, one may conjecture that, in the reduced space, a minimizer is a "brake" orbit but we can only prove that such a brake orbit exists by imposing the corresponding symmetry on the space of paths we consider. Also the $\mathbb{Z}/4\mathbb{Z}$ symmetry could be automatic for a minimizer in the space of (anti)symmetric loops.

A beautiful theorem of Albouy [1] states that a planar central configuration of four equal masses must have some symmetry. Numerical experiments by Rick Moeckel [22] show that such a property is no longer true for eight bodies or more. But it could be true for central configurations which minimize $\tilde{U}$. One could then ask more generally if periodic solutions of the n-body problem with equal masses which minimize the action in a "reasonable" subspace of Λ_T must always have some extra-symmetry.

This is certainly not true in general: examples of choreographies found numerically by Carles Simó ([12], figure 4f) show that one cannot expect in general that symmetry in the minimizers be an automatic consequence of symmetry in the masses, but maybe one could expect that under topological constraints with "some" symmetry, the minimizers be "more symmetric". The story goes on...

Many thanks to

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and .... Happy birthday, Don.

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