SYMMETRIES AND "SIMPLE" SOLUTIONS OF THE CLASSICAL N-BODY PROBLEM

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The Lagrangian of the classical n-body problem has well known symmetries: isometries of the ambient euclidean space (translations, rotations, reflexions) and changes of scale coming from the homogeneity of the potential. To these symmetries are associated "simple" solutions of the problem, the so-called *homographic motions*, which play a basic role in the global understanding of the dynamics. The classical subproblems (planar, isosceles) are also consequences of the existence of symmetries: invariance under reflexion through a plane in the first case, invariance under exchange of two equal masses in the second. In these two cases, the symmetry acts at the level of the "shape space" (the oriented one in the first case) whose existence is the main difference between the 2-body problem and the ($n \geq 3$)-body problem. These symmetries of the Lagrangian imply symmetries of the action functional, which is defined on the space of regular enough loops of a given period in the configuration space of the problem. Minimization of the action under well-chosen symmetry constraints leads to remarkable solutions of the n-body problem which may also be called simple and could play after the homographic ones the role of organizing centers in the global dynamics. In [13] and [16], I have given a survey of the new classes of solutions which had been obtained in this way, mainly *choreographies* of n equal masses in a plane or in space and *generalized Hip-Hops* of at least 4 arbitrary masses in space. I give here an updated overview of the results and a quick glance at the methods of proofs.

1. The Newtonian n-body problem and its symmetries

Given a configuration $x = (\vec{r}_1, \dots, \vec{r}_n)$ of n point masses $m_1, \dots, m_n$ in a euclidean space $(E, \cdot)$, the newtonian potential function (opposite to potential energy) of x is

$$U(x) = \sum_{i<j} \frac{m_i m_j}{||\vec{r}_i - \vec{r}_j||}.$$

The classical Newton equations may be written

$$\ddot{x} = \nabla U(x),$$

provided we introduce on the configuration space E^n the *kinetic energy metric*, also called *mass metric*, associated with the scalar product

$$x \cdot y = \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{s}_i \quad \text{if} \quad x = (\vec{r}_1, \dots, \vec{r}_n),\ y = (\vec{s}_1, \dots, \vec{s}_n).$$

Isometries of E acting diagonally on E^n leave the equations invariant. A reduction of the translation symmetry may be obtained by fixing the center of mass at the origin or, more conceptually by considering a configuration as an element of $\mathcal{D} \otimes E \cong E^{n-1}$, where $\mathcal{D} = \mathbb{R}^n/(1, \dots, 1)\mathbb{R}$ is the *disposition* space introduced in [2] in the spirit of Jacobi. A reduction of the symmetry under all isometries may be found in [29] in the case of 3 bodies and [2] in the general case: the idea is that the natural reduced variables are the (squared) mutual distances, which are independant only if the dimension of E is at least $2(n-1)$. Hence for $dim(E) \leq 3$, the dimension of E appears as a constraint within the space of mutual distances. Joined to the invariants of the angular momentum, it defines the symplectic leaves of the Poisson structure on the quotient phase space. At last, the homogeneity of the Newtonian potential implies the existence of a symmetry under change of scale: if $x(t)$ is a solution and λ is a positive real number, $\lambda^{-2/3}x(\lambda t)$ is also a solution. This fact is at the basis of the analysis of the behaviour of solutions near a partial or total collision (section 3). The above symmetries imply the complete integrability of the 2-body problem. After reduction of the translation symmetry, this problem is equivalent to the *Kepler problem* of a particle attracted by a fixed center and central force problems (not even invariant under rotation) are known since Newton to be integrable. In the next section, we review those solutions of the n-body problem which are Keplerian.

2. The "simplest" solutions: homographic motions

The main difference between the $(n \geq 3)$-body problem and the 2-body problem is the existence of a *shape*. Indeed, all segments are similar but the set of triangles in $\mathbb{R}^2$ up to oriented similarity has naturally the geometry of a round 2-sphere (abandoning the orientation or going up to $\mathbb{R}^3$, one replaces the sphere by a disk; for n bodies up to oriented similarity in $\mathbb{R}^2$, one gets $\mathbb{CP}_{n-2}$; in $\mathbb{R}^3$ the corresponding set is no more a manifold).

Hence it is not astonishing that the "simplest" solutions of the newtonian n-body problem, and in fact the only ones whose dynamics is "explicit", are the so-called *homographic* motions where the shape does not change up to similarity and the motion of each body is Keplerian. When the energy is negative, one gets a family of periodic motions characterized by an eccentricity e between 0 (*relative equilibria* where the configuration rotates as a single rigid body around its center of mass) and 1 (*homothetic* motion ending in total collapse on the center of mass). For intermediate values of e, one gets "complex homothetic" solutions for a complex structure defined on the support of the angular momentum bivector. The existence of these motions is clearly related to the symmetries of the problem under isometries and changes of scale.

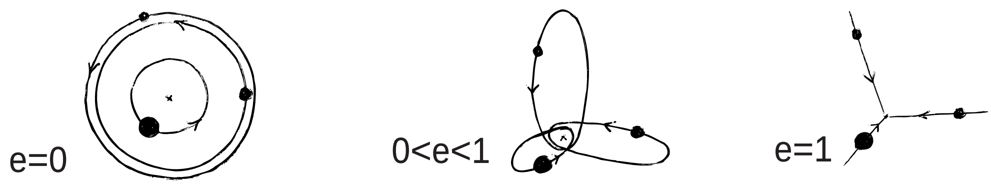

Figure 1. The equilateral homographic motions discovered by Lagrange

2.1. *Homothetic motions and central configurations*

The study of Keplerian motions of n-body systems goes back to Euler and Lagrange at the end of eighteenth century in the case of 3 bodies. If one excepts relative equilibria in dimension at least 4, such motions are possible only for the so-called *central configurations* which can be defined as the ones which when released without initial velocities undergo a homothetic collapse onto their center of mass $\vec{r}_G$. They are characterized by the collinearity of the forces $\nabla U(x)$ and the relative configuration $x - x_G = x - (\vec{r}_G, \dots, \vec{r}_G)$, that is the collinearity of the two gradients $\nabla U(x)$ and $\nabla I(x)$, where

$$I(x) = ||x - x_G||^2 = \sum_{i=1}^{n} m_i ||\vec{r}_i - \vec{r}_G||^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j ||\vec{r}_i - \vec{r}_j||^2$$

is the moment of inertia of the configuration with respect to its center of mass. This collinearity is equivalent to saying that x is a critical point of the restriction of U to the sphere $I = I(x)$, or a critical point of the scaled potential $\tilde{U} = \sqrt{I}U$, homogeneous of degree 0 (and constant in case $n = 2$). In the case of 3 non collinear bodies, the squared mutual distances may be taken as independant coordinates and the collinearity for the gradients of U and I yields immediately the equality of all three mutual distances (the celebrated *Lagrange equilateral configuration*). The actual determination of central configurations for more than 3 bodies is a very difficult task. Even the finiteness of the number of solutions up to similarity is known only in the case of 3 bodies (Lagrange [29]), or 4 bodies of the same mass (Albouy [1]) and of collinear configurations (Moulton). It is known to be false if one allows negative masses, which is an indication of the difficulty of the problem. Central configurations are the object of active research which in most cases must be computer-assisted.

2.2. *Relative equilibria and balanced configurations*

As René Thom kept saying, to understand a mathematical object, one must start understanding its singularities. For the n-body problem, these are the relative equilibrium solutions, which become equilibria after one has "reduced" by the rotational symmetry (see [2] for a general discussion of the reduction).

Relative equilibria in dimension at least 4 are shown in [2] to exist for more general configurations, the *balanced* configurations, which may be defined as critical points of the restriction of U to the set of configurations whose inertia form is the same up to rotation (this set is also an orbit of the so-called *democracy group* which, in the langage of [2], is simply the orthogonal group of the disposition space $\mathcal{D}$ endowed with the mass metric). For 3 bodies, one must fix not only I, which is the trace of the endomorphism associated to the inertia form via the euclidean structure of the ambient space E, but also the area of the triangle, which is proportional to the determinant). The only 2-dimensional central configuration is the equilateral triangle. However *any* isosceles triangle with equal masses is balanced and, as so, admits periodic or quasi-periodic relative equilibrium motions in $\mathbb{R}^4$.

2.3. *Complex homothetic motions and Sundman's inequality*

Homographic motions with a central configuration (this is automatic if $e \neq 0$) are in fact complex homothetic for a complex structure defined on the support of the angular momentum bivector [2]:

$$x(t) = \zeta(t) x_0, \quad \text{where the complex-valued function } \zeta(t) \text{ satisfies} \quad \ddot{\zeta} = -U_0 |\zeta|^{-3} \zeta.$$

They may be characterized in the following way: the quadratic isometry invariants of an element (x, y) of the phase space are the moment of inertia $I(x, y) = ||x - x_G||^2$ with respect to the center of mass, (half) its derivative $J(x, y) = (x - x_G) \cdot (y - y_G)$ along the flow of Newton equations and (twice) the kinetic energy $K(x, y) = ||y - y_G||^2$ in a galilean frame fixing the center of mass: The angular momentum bivector

$$\mathcal{C} = \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_G) \wedge (\vec{v}_i - \vec{v}_G) \quad \text{if} \quad x = (\vec{r}_1, \ldots, \vec{r}_n),\ y = (\vec{v}_1, \ldots, \vec{v}_n),$$

defines a complex structure on its support (see [2] for precise definitions). A slightly refined version of the complex Cauchy-Schwartz inequality for this structure gives the classical Sundman's inequality $IK - J^2 \geq c^2$, where c denotes the norm of the bivector $\mathcal{C}$. Equality characterizes the homographic motions with central configuration. For relative equilibrium motions in $\mathbb{R}^p$ with (balanced) non central configuration (this implies $p \geq 4$), $IK - J^2$ is a constant strictly greater than c^2 (see [2]).

Another fundamental consequence of the homogeneity is the analysis, due to Sundman in the case of 3 bodies, of the solutions which undergo a collision. We review it in the next section as it will be basic in the application of the variational method to the n-body problem.

3. Solutions ending in collision

3.1. *The scaling symmetry*

The scaling symmetry due to the homogeneity of the potential function admits a less effective reduction than the former ones. This symmetry is expressed in the fact that if $x(t)$ is a solution of Newton's equations and λ is a positive real number, $x_\lambda(t) = \lambda^{-\frac{2}{3}} x(\lambda t)$ is also a solution. Identifying these solutions leads to a "reduced" system living in a "reduced" phase space which may be identified with the subspace of the phase space $\hat{\mathcal{X}} \times \mathcal{X}$ defined by $I = 1$. The total energy H_λ and angular momentum $\mathcal{C}_\lambda$ of $x_\lambda(t)$ are obtained from those of $x(\lambda t)$ by multiplication respectively by $\lambda^{\frac{2}{3}}$ and $\lambda^{-\frac{1}{3}}$. Hence, this reduction leads to a gain in dimension only in the case $H = 0$ and $\mathcal{C} = 0$. Otherwise, the reduced system admits only the first integral $\sqrt{|H|}\mathcal{C}$, better known after taking the quotient by the rotations under the form $H|\mathcal{C}|^2$. The dynamics in the open set defined by $H > 0$ (resp. $H < 0$) reflects, up to change of time, the dynamics of any energy level of the same sign. They are glued together along a common "boundary", the quotient of the energy level $H = 0$ by the scaling symmetry. In this "boundary", sits the quotient of $H = 0$, $\mathcal{C} = 0$, which can be identified with the so-called "collision manifold". Introduced by McGehee, this manifold plays a fundamental rôle in the study of solutions which get close to total collision(see [32]) and allows to recover more geometrically the classical results of Sundman. Sundman had proved that a total collision can occur only if the angular momentum is equal to 0. Solutions ending or starting in total collision correspond in the reduced space to solutions which are asymptotic to singularities of the collision manifold. As these singularities are the solutions invariant under the scaling transformation, one sees readily that they are exactly the parabolic homothetic motions: $x(t) = t^{\frac{2}{3}}\bar{x}$, where $\bar{x} = x(1)$ is a central configuration. In particular, in a total collision solution, the configuration of the bodies tends, after normalization of its size, to the set of central configurations. All this can be presented via the introduction of the "symmetry vector-field" $Y : \dot{x} = x, \dot{y} = -\frac{1}{2}y$, which becomes a symmetry of Newton's equations after the change of time defined by $d\tau/dt = I^{\frac{3}{4}}$ (see [12]). The case of partial collisions is more delicate [40, 45, 25] but the final estimates are the same: in a cluster undergoing a (partial) collision, the normalised configuration of the cluster tends to the set of central configurations and the behaviour of its size is the same as the one in a homothetic solution, that is the same as for a 2-body collision: if in the solution $(\vec{r}_1(t), \dots, \vec{r}_n(t))$ the bodies i and j collide at time t_0, their mutual distance satisfies

$$||\vec{r}_i(t) - \vec{r}_j(t)|| = O(|t - t_0|^{\frac{2}{3}}), \quad ||\dot{\vec{r}}_i(t) - \dot{\vec{r}}_j(t)|| = O(|t - t_0|^{-\frac{1}{3}}).$$

Two problems occur when one tries to understand the behaviour of the rescaled configuration of the bodies in a cluster undergoing a collision: (i) the possible existence of a continuum of non equivalent central configurations, which implies that there is not necessarily a limit shape; (ii) the infinite spin problem, that is the possibility that if a limit shape exists, its orientation does not stabilize asymptotically. The blow-up technique, proposed by S. Terracini and developed in [45] (see also [25]) helps dealing with these problems.

3.2. *Blow-up*

Let $x : [a, b] \to \mathcal{X}$ be a solution of Newton's equations with an isolated total collision at $t = t_0$. Let us call x_λ the restriction to $[a, b]$ of $x_\lambda(t) = \lambda^{-\frac{2}{3}} x(t_0 + \lambda(t - t_0))$. The x_λ's form a bounded, and hence weakly compact, subset of $H^1([a, b], \mathcal{X})$. Hence there exists a sequence λ_n tending to 0 such that x_{λ_n} converges weakly to a limit $\bar{x}$; one can show [45, 25] that $\bar{x}$ is a parabolic homothetic collision-ejection solution, that is a solution of the form $\bar{x}(t) = (t_0 - t)^{\frac{2}{3}} \bar{x}_0$ if $a \le t \le t_0$ and $\bar{x}(t) = (t - t_0)^{\frac{2}{3}} \bar{x}_1$ if $t_0 \le t \le b$, where $\bar{x}_0$ and $\bar{x}_1$ are central configurations. If the collision is not total, one gets the same conclusion when restricting the attention to one colliding cluster (the others " go to infinity"). The existence of such limits will be fundamental in the next section.

4. Action minimization and the collision problem

Newton equations coincide with the Euler-Lagrange equations of the action functional S, which to a path $x : [t_0, t_1] \to \mathcal{X}$ in the configuration space associates

$$\mathcal{A}(x(t)) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt,$$

where the Lagrangian L is defined by

$$L(x, \dot{x}) = \frac{1}{2} ||\dot{x}||^2 + U(x).$$

Hence critical points $t \mapsto x(t)$ with values in the subset $\hat{\mathcal{X}}$ of non-collision configurations are exactly the solutions of Newton's equations. If one replaces the interval $[t_0, t_1]$ by the circle $S^1_T = \mathbb{R}/T\mathbb{Z}$, the critical points with value in $\hat{\mathcal{X}}$ become the periodic solutions of period T.

Recall that, by Weierstrass theorem, the fact that classical Lagrangians have positive-definite Hessian with respect to the velocities implies that small extremals do minimize $\mathcal{A}$.

4.1. *The necessity of constraints*

The first attempt to find (relative) periodic solutions of the n-body problem by way of minimizing the action is the short note [38] of Poincaré in 1896, the year after he defined the fundamental group. In modern langage, what he proposed was to fix a non trivial homology class of loops in the (relative) configuration space of the planar 3-body problem and to minimize the action among such loops of a given period. Not waiting for Sundman's theory of collisions (1913) or for Tonelli's proof of the existence of minimizers (1930), he was well aware of the fact that the action of a solution ending in a collision is finite (one integrates a function of t of the order of $|t-t_0|^{-\frac{2}{3}}$) and that a minimizer could well be the concatenation of pieces of solution separated by a closed set (of zero measure) of collision times. This prevented him from concluding existence in the Newtonian case and led him to introduce the so-called "strong force" potential ($1/r^2$ instead of $1/r$) for which this problem disappears because $|t-t_0|^{-\frac{2}{3}}$ is replaced by $|t-t_0|^{-1}$.

Constraints of some kind are certainly necessary in order to force coercivity of the problem, that is to prevent that the minimum of the action functional be attained only at infinity. Otherwise, minimizers are obviously stationary loops of infinitely separated bodies.

And here comes the main problem: to find constraints for which the minimizers are collision-free. A general remark comes first: it is very unlikely that one can obtain interesting results by constraining only the homology or homotopy class of the loops among which one minimizes. This is because, as noted by Montgomery [34], minimizers are most likely to undergo collisions. The only exception I know is Venturelli's generalization [44] to the 3-body case of Gordon's theorem [26], which is a partial answer to Poincarés quest. We shall see in what follows that symmetry constraints behave much better.

4.2. *Symmetry constraints*

One defines naturally an action $x \mapsto g \cdot x$ of a finite group G on the loop space $\Lambda = H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{X})$ by setting

$$(g \cdot x)(t) = (\rho(g)\vec{r}_{\sigma(g^{-1})1}(\tau(g^{-1})t), \dots, \rho(g)\vec{r}_{\sigma(g^{-1})n}(\tau(g^{-1})t),$$

where $\tau : G \to O(2)$, $\sigma : G \to \mathcal{S}_n$ and $\rho : G \to O(E)$ are respectively actions of G on the time circle $\mathbb{R}/T\mathbb{Z}$ (by isometries), the index set $\{1, 2, \dots, n\}$ (by permutations) and the ambient euclidean space E (by orthogonal transformations). Such a group action leaves $\mathcal{A}$ invariant provided only permutations of equal masses occur. Invariant loops $x \in \Lambda^G$ are those which are equivariant under the combined actions τ of G on $\mathbb{R}/T\mathbb{Z}$ and $g \cdot (\vec{r}_1, \dots, \vec{r}_n) = (\rho(g)\vec{r}_{\sigma(g^{-1})1}, \dots, \rho(g)\vec{r}_{\sigma(g^{-1})n})$ of G on $\mathcal{X}$. It follows from the Palais *symmetric criticality principle* ([37, 13]) that critical points of the restriction $\mathcal{A}^G$ of the action to Λ^G are also critical points of the action $\mathcal{A}$ on the full loop space, and hence solutions provided that they are collision-free. We shall use the following notations: $I = [t_0, t_1]$ will be a fundamental domain of the action τ of G on the time circle. The restriction to I of a minimizer x of $\mathcal{A}^G$ is a path $x|_I : I \to \mathcal{X}^{Ker(\tau)}$ which is a minimizer among similar paths with the same endpoints (if not, symmetrize a path with lower action to get a contradiction). There are two cases, one case in which the image of τ is cyclic and the other case in which this image does contain reflections. In the second case, the endpoints t_0, t_1 of I will be fixed by some reflection and their isotropies G_{t_0} and G_{t_1} will be larger than $Ker(\tau)$. The two cases are best illustrated by the $(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z})$-symmetric Hip-Hop [20] and the D_6-symmetric Eight [19] (see figure 2).

5. Beating the collisions: from local deformations to solving the fixed ends problem

Various tools are used to get a contradiction from the hypothesis that an action minimizer has a collision. In their discussion, we allude to the solutions described in the last section.

5.1. *Making local deformations*

This works only for isolated collision times. Action decreasing deformations must be dictated by the geometry of the limit central configuration provided such a limit exists (see par. 3). As this geometry is not thoroughly known for $n \geq 4$ bodies, this implies that only few bodies collisions can be analyzed in this way. Explicit local deformations are just impossible to design in general ! Examples are the early works of the italian school [4, 39, 24]

5.2. *Evaluating the action of collisions by comparison to a 2-body problem*

Comparing an n-body system to a 2-body is a fundamental tool. It amounts to forgetting the shape: this is exactly what Sundman's inequality $IK - J^2 \geq c^2$ accomplishes by forgetting the deformation velocity (write it as $K = K_{scale} + K_{rot} + K_{def} \geq J^2/I + c^2/I$). It is the main tool available to get global results on the behaviour in the large – in particular on the size – of an n-body system [11].

In our context, it is the knowledge of the action of solutions of the 2-body problem which may be used to estimate a lower bound of the action of loops (or paths)

1) Gordon's theorem [26] was the first attempt since Poincaré to apply variational methods to the n-body problem: it asserts that minimizers of the action $\int_0^T(\frac{1}{2}|\dot{\vec{r}}|^2 + \frac{g}{|\vec{r}|})dt$ of the planar Kepler problem (Newtonian fixed center at the origin) are exactly the Keplerian solutions of the given period T provided one minimizes among loops whose index with respect to the attracting center is different from zero. It has the virtue of showing the simplest example where all minimizers of a reasonable problem have collisions because if one insists in minimizing the action among loops with any given index different from $(-1, 0, +1)$, one gets only ejection-collision loops as minimizers.

2) K.C. Chen [7] has greatly increased the applicability of Gordon's theorem by proving that the homology (index) constraint could be replaced by the much less demanding condition that every line through the center meets the loop. Among the minimizers are also half Keplerian orbits way and back but the action is the same. He also noticed [8, 9] that only arcs of Keplerian circle minimize the action among paths joining two half-lines making an angle $\theta < \pi$ in given time. Both extensions are used in the quoted papers to get good lower bounds of the action of collision in problems constrained by symmetry conditions. This is done by writing the action of the n-body problem as a sum of actions of $n(n-1)$ virtual 2-body problems (with centers of mass not at rest): each mass m_i is decomposed into $n-1$ masses m_{ij} bound together at $\vec{r}_i$ and m_{ij} is made to act only on m_{ji}. For $n = 3$, a natural decomposition into three 2-body problems had been used by Venturelli in his proof of the generalization to the planar 3-body problem of Gordon's theorem [44]

3) C. Marchal noticed that, given two points $\vec{r}'$ and $\vec{r}''$ in the punctured plane, both the direct and indirect Keplerian arc joining these two points in given time have lesser action than the collision-ejection path between them. This is a key ingredient in [43] where Terracini and Venturelli prove the existence of a family of solutions of the $2n$-body problem in 3-space with equal masses, which contains (when $n = 2$) the original Hip-Hop and a spatial choreography already observed numerically by G. Hoynant [27].

5.3. *Constructing test paths*

One constructs by hand a path satisfying the constraints with low enough (and computable) action. This is more global and allows beating even a local minimizer provided one has an estimate from below of the action of a path with collision satisfying the constraints. Examples are the proof of the existence of the Eight [19], where a horizontal lift of the equipotential containing the Euler points in the shape sphere is used, of the P_{12} family of Marchal [15], where one uses a Lagrange horizontal family, and of Chen's retrograde orbits of Hill type [9], where the model is explicit with circles.

5.4. *Averaging deformations: Marchal's theorem and its invariant form*

The following theorem, mainly due to Marchal (see [30, 16]), is the most powerful tool available to prove that minimizers under well chosen symmetry constraints have no collision.

Theorem 5.1. *Let $x' = (\vec{r}'_1, \vec{r}'_2, \cdots, \vec{r}'_n)$ and $x'' = (\vec{r}''_1, \vec{r}''_2, \cdots, \vec{r}''_n)$ be two arbitrary configurations, possibly with collisions, of n material points with positive masses $m_1, m_2, \cdots, m_n$ in the plane or in space. For any $T > 0$, any local minimizer of the action among paths $x(t) = (\vec{r}_1(t), \vec{r}_2(t), \cdots, \vec{r}_n(t))$ in the configuration space which start at $x(0) = x'$ and end at $x(T) = x''$ is a true (i.e. collision-free) solution of Newton's equations (in the plane or in space) in the open interval $]0, T[$.*

Marchal's key idea to show that local minimizers have no collision was to compute the average of the action after small deformations in all possible directions of one of the bodies involved in a collision. For such perturbations the change in the integrated potential dominates the increase in the integrated energy. To estimate the change in potential, Marchal used the fact that in three dimensions the Newtonian potential is the Green function for the Laplacian. He made the striking observation that the effect of his averaged perturbations of the potential term is equivalent to replacing the colliding particle by a sphere centered at that particle, of uniform density, and of time-varying radius. The potential due to such a massive sphere is *constant* inside that sphere, whereas the original point-particle potential increases without bound, and as a result the averaged perturbation decreases the action. In particular there must be at least one direction of perturbation among those averaged over which decreases the action. Marchal also gave a modification of the averaging process, which took care of the planar problem. This idea was a breakthrough which permitted to completely bypass the necessity of any precise knowledge on central configurations. Marchal's proof worked only in the case of isolated collisions with limit configuration. Using the work of several people (R. Montgomery, S. Terracini and A. Venturelli), I was able to complete the proof and I exposed it in [16]. The two main steps in this proof are 1) the existence of an *isolated* collision if there is a collision in a minimizer $x(t)$ and 2) the reduction, via blow-up, of the case of an arbitrary isolated collision to the case of a parabolic homothetic collision-ejection solution. For 1), one uses the minimizing property of $x(t)$ with respect to reparametrizations (internal variations) to show that the energy of a colliding cluster $\mathbf{k}$ remains continuous on a neighborhood of the collision time t_0 provided that no collisions occur between bodies in $\mathbf{k}$ and bodies not belonging to $\mathbf{k}$; one then uses this continuity and the so-called Lagrange-Jacobi relation to prove that an accumulation at t_0 of collision times would lead to a contradiction provided no subcluster of $\mathbf{k}$ undergoes a collision near t_0 (this is the case if the collision at t_0 concerns the smallest possible number of bodies). For 2), one proves that if a minimizer $x(t)$ has an isolated collision at t_0, the blow-up $\bar{x}$, defined as a limit of $\lambda^{-\frac{2}{3}}x(t_0+\lambda(t-t_0))$, is also a minimizer of the fixed ends problem. In both cases, the technical elaboration came from Venturelli's thesis [45]

This theorem applies immediately to local minimizers $x \in \Lambda^G$ of $\mathcal{A}^G$, that is to local minimizers under symmetry constraints, in case τ is a faithfull action on the time circle (i.e. $Ker(\tau)=0$). Let $[t_0,t_1] \subset [0,T]$ be a fundamental domain of this action: the restriction of x to $[t_0,t_1]$ must be an unrestricted local minimizer of the action $\mathcal{A}$ among paths with the same endpoints, and as such collision-free in the open interval $]t_0,t_1[$. But if τ acts only through non trivial rotations, the starting point t_0 may be chosen arbitrarily and x cannot have a collision. Applications of this remark are given in [16]. When this is not the case, the methods described in 5.1,5.2 and 5.3 can sometimes be applied to prove that no collision occurs at the boundary of a fundamental domain [16, 7] but a much more systematic (not universal, however) way of dealing with this problem results from the generalization of Marchal's theorem to invariant paths given by Ferrario and Terracini [25]. Their key observation is that averaging perturbations along *circles* is sufficient to get a decrease of the averaged modified action. This results directly from Marchal's treatment (see [16]) for the planar Newtonian problem but works with much greater generality for any $1/r^\alpha$ potential with $0<\alpha<2$ in $\mathbb{R}^d, d=2,3,\ldots$, due to the the following inequality: for $\vec{r},\vec{s} \in \mathbb{R}^3 \setminus \{0\}$, let

$$S(\vec{r},\vec{s}) = \int_0^\infty \left[\frac{1}{|t^{(2/2+\alpha)}\vec{r}+\vec{s}|^\alpha} - \frac{1}{|t^{(2/2+\alpha)}\vec{r}|^\alpha}\right] dt.$$

Then, for every $\vec{r} \in \mathbb{R}^3 \setminus \{0\}$ and for every circle $\mathbb{S} \subset \mathbb{R}^3$ with center at 0,

$$\tilde{S}(\vec{r},\mathbb{S}) = \frac{1}{|\mathbb{S}|}\int_{\mathbb{S}} S(\vec{r},\vec{s})d\vec{s} < 0.$$

One first notices that $S(\vec{r}_\gamma,\mathbb{S})$ is a monotonically decreasing function of the angle $\gamma \in [0,\frac{\pi}{2}]$ between $\vec{r}_\gamma$ and the plane generated by $\mathbb{S}$; then, one computes the integrals when $\vec{r}$ belongs to the plane generated by $\mathbb{S}$ with the help of series expansions of the inverse-distance function. On the other hand, if a parabolic homothetic solution $x(t)=t^{(2/2+\alpha)}(\vec{r}_1,\ldots,\vec{r}_n)$ of the n-body problem is perturbed by the addition of a variation $v^\delta(t)=R(t)(\vec{\delta}_1,\ldots,\vec{\delta}_n)$, where $R(t)$ is equal to 1 if $t\in[0,T-\sum|\vec{\delta}_i|^2]$, to 0 for $t\geq T$ and decreases linearly in between, the variation in action is

$$\mathcal{A}(x+v^\delta)-\mathcal{A}(x) = |\delta|^{1-\alpha/2}\sum_{i<j} S\left(\vec{r}_i-\vec{r}_j, \frac{1}{|\delta|}(\vec{\delta}_i-\vec{\delta}_j)\right) + O(|\delta|),$$

and hence the average of the variation is negative *provided this average is taken over a set of δ's such that each difference vector $\vec{\delta}_i - \vec{\delta}_j$ covers a circle.* Taking a single non-zero $\vec{\delta}_i$ varying along any small circle $\mathbb{S} \subset \mathbb{R}^3$ will suffice in the absence of group action. But let us suppose now that we restrict to the paths $x : [0,T] \to \mathcal{X}^K$, where $\mathcal{X}^K$ is the subset of configurations which are invariant under the action on $\mathcal{X}$ of a finite group K. In the application to minimizers of $\mathcal{A}^G$, K will be the kernel of the action τ. Its action on $\mathcal{X}$ is defined by $g \cdot (\vec{r}_1, \ldots, \vec{r}_n) = (\rho(g)\vec{r}_{\sigma(g^{-1})1}, \ldots, \rho(g)\vec{r}_{\sigma(g^{-1})n})$. The proof of the existence of an isolated collision time (if any) goes on without modification when one replaces $\mathcal{X}$ by $\mathcal{X}^K$. Now, if a cluster $\mathbf{k}$ of bodies collides at some time, its images under the action of K will also collide and every deformation of the chosen cluster must be repeated via K to preserve invariance. But, the clusters being at distances from each other which are bounded away from zero, the contributions to the variation of the action of their mutual interactions and of their interactions with the non colliding bodies won't play any significant role in the estimations. Hence, one can focus on a single colliding cluster or, what is equivalent, suppose that the subgroup H which leaves it invariant is K itself. Finally, one can use blow-up to replace this collision by a parabolic homothetic one. Let us then choose an index $i \in \mathbf{k}$ and start again with a family of deformations $\vec{\delta}_i$ of body i which covers a circle $\mathbb{S}$. To preserve the invariance property, this circle must be contained in the subspace E^{H_i} fixed by the isotropy subroup of the index i for the action σ; moreover, one must define $\vec{\delta}_j = \rho(g)\vec{\delta}_i$ whenever $g \in H$ and $j = \sigma(g)i$. For the $\vec{\delta}_i - \vec{\delta}_j$ to all vary on circles, the $\vec{\delta}_j$ must all lie on $\mathbb{S}$ and move rigidly with $\vec{\delta}_i$. This forces to chose $\mathbb{S}$ invariant under the action ρ of H and moreover to be such that this action consists only of rotations (the presence of a reflexion would irremediably destroy the rigidity !). Such a circle is called in [25] *rotating* for i under H. As at least two bodies are present in a collision, one must ask for the existence of such a circle for at least $n-1$ bodies in order to be able to choose at least one such index i in any case. Each time this existence is granted, one has a Marchal theorem for invariant paths. Moreover, the same construction allows to settle in many cases the problem of collisions at the boundaries of a fundamental domain, in which cases $Ker(\tau)$ must be replaced by the isotropy group G_t of the corresponding time t. Here is the precise statement found in [25]:

Definition 5.1. The group G acts on the loop space Λ with the rotating circle property if for every isotropy subgroup $G_t \subset G$ of the τ action and for at least $n-1$ indices i, there exists a rotating circle $\mathbb{S}$ under G_t, that is a circle on which G_t acts by rotations and which is contained in the subspace $E^{G_t^i}$ fixed by the isotropy subgroup G_t^i of i for the action σ of G_t on $\{1, 2, \ldots, n\}$.

Theorem 5.2. *Let G be a finite group which acts on Λ in such a way that every maximal isotropy subgroup of the action τ on $\mathbb{R}/T\mathbb{Z}$ either has the rotating circle property or has trivial action σ on the set of indices. Then any local minimizer of $\mathcal{A}^G$ is collision-free.*

The gain is two-fold: (i) one can deal with group actions for which some instants of the time t have non trivial stabilizer G_t, typically, when some element acts by reversing the time (this doe not work all the time, for example it does not work for the D_6-symmetric eight but it does for the eights with less symmetry [16]), (ii) one can deal with constraints which at any time restrict the shape of the configurations in play (the simplest examples are the isosceles subproblems or the original Hip-Hop [20]).

Note that the requirements of the rotating circle property are easier to meet in space than in the plane. The following figure, which is made up of two transparencies from the oral lecture, illustrates the theorem in the first two cases studied by the author respectively with A. Venturelli [20] and R. Montgomery [19].

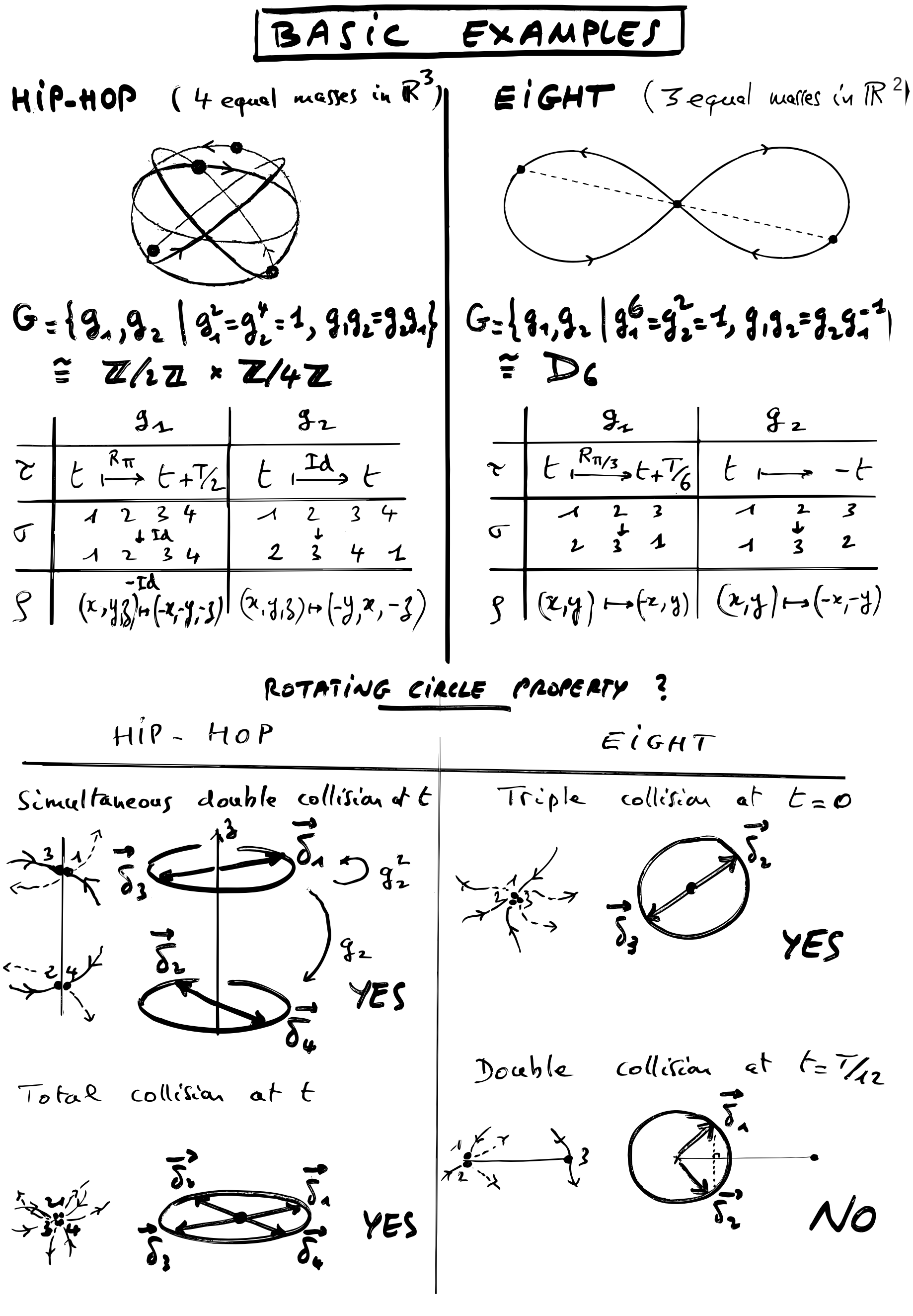

Figure 2. The basic examples : two transparencies from the lecture

6. The next "simplest" solutions: constrained action minimizers

6.1. *Action minimizing properties of planar homographic solutions*

1) The existence of the planar subproblem may be seen, via the unicity of solutions of differential equations, as a consequence of the invariance of the equations under reflexion through a plane. This implies in particular that solutions of the Kepler problem are necessarily planar. The same is true of any homographic and non homothetic solution of the n-body problem in $\mathbb{R}^3$: for example, a regular tetrahedron can collapse homothetically onto its center of mass but it cannot rotate or have a homographic motion of eccentricity $e < 1$. That it cannot rotate is easy to verify (it would have to become more and more flat) but the result for $0 < e < 1$ is deep and Lagrange, who proved the result for 3 bodies [29], was well aware of this depth. More generally, homographic but non homothetic solutions of the n-body problem can occur only in even dimension [2].

2) The planar relative equilibria whose configuration minimizes $\tilde{U} = \sqrt{I}U$ are minimizers of the action among loops satisfying the *italian symmetry* $x(t + T/2) = -x(t)$. Other minimizers could possibly exist if there were continua of similarity classes of central configurations which minimize $\tilde{U}$ [18, 13]. As this is not the case for $n = 3$, the Lagrange relative equilibrium (with equilateral configuration whatever be the masses) of the given period is the only minimum of the action among loops with the italian symmetry. A characterization of the equilateral homographic solutions as minimizers in their homology class was given by Venturelli in [44] (the homology class is determined by the three integers k_{ij} which give the effective number of turns accomplished by each side of the triangle during one period). As in Gordon's theorem, which Venturelli's theorem generalizes, the minimizers in any other homology class for which each k_{ij} is different from 0 are the homothetic collision-ejection solutions. In particular, such minimizers have necessarily a collision.

To get new solutions as minimizers, one has to exclude the homographic solutions by an appropriate choice of the constraints. This is done in various ways in the following sections: imposing pure symmetry constraints in $\mathbb{R}^3$ or a mixture of symmetry and topological constraints in $\mathbb{R}^2$ but I do not know of a single example where topological constraints alone lead to interesting non-collision minimizers (see [34]). There are two main classes of examples, according to whether or not equality of masses occurs, that is whether or not permutations of bodies are allowed.

6.2. *New "simple" solutions: arbitrary masses*

I discuss two classes: in the first one, purely spatial, the homographic solutions – necessarily planar – are excluded because their action has a non positive Hessian, which forbids them to be even a local minimizer; in the second one, planar, they are excluded by the combination of symmetry constraints and a homotopy constraint.

6.2.1. *Generalized HipHops: the simplest non planar solutions*

It follows immediately from Marchal's theorem that a minimizer of the action among loops of configurations with the italian symmetry are collision-free. For the spatial problem, such minimizers are never planar as soon as the number of bodies is greater than three [16, 14]. This comes from the fact that if, as we said, a planar relative equilibrium whose configuration minimizes $\tilde{U}$ is a minimizer for the planar problem with italian symmetry, it is *never* a minimizer for the spatial problem: the Hessian of the action has a negative direction for exactly the reason which makes the spatial configurations which minimize $\tilde{U}$ non planar [31]. The original Hip-Hop has all its masses equal and an extra $\mathbb{Z}/4\mathbb{Z}$-symmetry (figure 3). The first proof of its existence is in [20] but it had already been found numerically in [23]. The $\mathbb{Z}/4\mathbb{Z}$-symmetry is probably a consequence of minimization under the italian $\mathbb{Z}/2\mathbb{Z}$-symmetry alone. If, inspired by the next example, we replace the italian symmetry by the conditions $x(t+\tau) = e^{i\phi}x(t)$ and $z(t+\tau) = -z(t)$ (we identified $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$), we obtain periodic or quasi-periodic motions which, for $\pi - \phi$ small enough, are non planar.

6.2.2. *Retrograde Hill's orbits*

Here, one comes back to the planar 3-body problem and one uses the rich topology of the shape space. Relative equilibria with lowest action are excluded by the choice of an extra symmetry constraint (reflexion) which forces the shape to be collinear at t=0 and $t = \tau/2$. More precisely, the constraints are that $x(t+\tau) = e^{i\phi}x(t),\ 0 < \phi \leq \pi$, and $x(-t) = \bar{x}(t)$ (one identifies $\mathbb{R}^2$ with $\mathbb{C}$ and hence $\mathcal{X}$ with $\mathbb{C}^{n-1}$). Collinear relative equilibria, which may minimize locally, are excluded by choosing the homotopy class of *retrograde Hill-type* loops. The absence of collision when $\pi - \phi$ is small is proved in [9] for "good" masses by comparing the action of an explicit test path to the action of collisions estimated as in 5.2.2. Analogous results for n bodies are announced in [10]. See also [3] for the classical Hill case

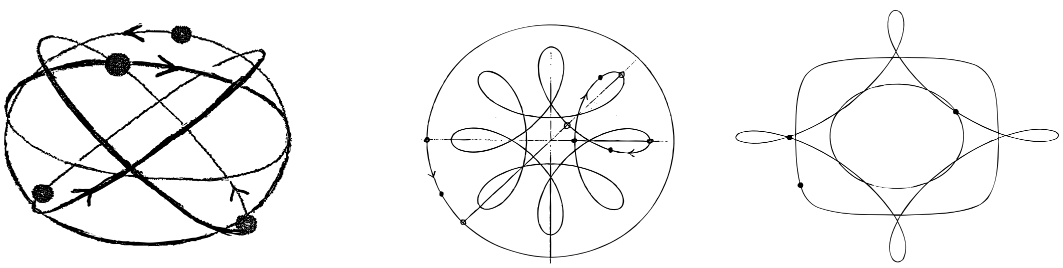

Figure 3. A Hip-Hop solution (4 equal masses) and two retrograde Hill solutions (equal and unequal masses)

6.3. *New "simple" solutions: equal masses*

Invariance under permutation of equal masses accounts for the existence of the generalized isosceles subproblems, where the configuration is assumed to be at any time invariant under a group of symmetries of $\mathcal{X}$ which may permute equal masses. Subtler are the symmetries of the action which act simultaneously on the time circle S^1_T and the configuration space $\mathcal{X}$. The quotient of the configuration space of the planar n-body problem by the natural action of $SO(2)$ can be identified with the complex projective space $\mathbb{CP}_{n-2}$, quotient of $(\mathbb{R}^2)^{n-1} \equiv \mathbb{C}^{n-1}$ by the diagonal action of $\mathbb{C}\setminus\{0\}$. For $n = 3$, this is the Riemann sphere and it is called in this context the *shape sphere*. In the spatial problem, the quotient of $\mathcal{X}$ by the rotation group leads to spaces which are singular in the neighborhood of collinearities. In the 3-body case, one gets the 2-disk of unoriented triangles of fixed size. This disk can be conveniently parametrized by the three squared mutual distances $r_{ij}^2 = |\vec{r}_i - \vec{r}_j|^2$ subject to the positivity of the polynomial which gives the squared area (Heron's formula) [2].

6.3.1. *The D_6-symmetry of the shape sphere and the Eight*

The dihedral group D_6 is the natural symmetry group of the shape sphere, that is of the set of similarity classes of oriented triangles. It can be presented as the group generated by two generators g_1, g_2 and the relations $g_1^6 = g_2^2 = 1, g_1 g_2 = g_2 g_1^{-1}$. The following figure shows the action of the generators and the decomposition of the sphere into a fundamental domain and its 11 images under the action of the group on $\mathcal{X}$ (compare to the formulas in figure 2). The boundaries are made from the equator, which represents the flat triangles, and the three meridians which represent the isosceles triangles. The two poles are the equilateral triangles with the two orientations. For each choice of masses, the shape sphere inherits a metric from the mass metric on $\mathcal{X}$. The metrics corresponding to different sets of masses are different but they share the same conformal structure. For a nice application of this, see [35]. The Eight [19] (already discovered numerically in [36]) is a minimizer among loops invariant under the action of D_6 given in figure 2, which lifts the action on the shape sphere. A fundamental domain of the action on the time circle is the interval $[0, T/12]$. The invariant version of Marchal's theorem applies everywhere, except at the boundary point $t = T/12$ where the configuration must be isosceles and the rotating circle property is not available due to the required invariance under reflexion through the horizontal axis (figure 2). It applies however for proving the existence of eights with a priori symmetry D_3 or $\mathbb{Z}/6\mathbb{Z}$ [16, 25]. The original proof in [19] (see also [15]) uses the fact that the zero angular momentum lift of the equipotential through the Euler points in the shape sphere is an eight with the D_6-symmetry and that its action is smaller than the one of any invariant loop with collision. The actual minimizer is not qualitatively different from this model; this makes it close to be a counterexample to Saari's conjecture which asserts that the only solutions whose moment of inertia with respect to the center of mass is constant are the relative equilibria. A proof of this conjecture for 3 arbitrary masses in the plane was given by Moeckel [33].

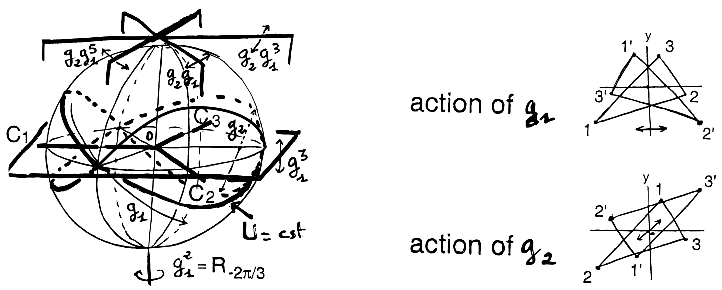

Figure 4. The shape sphere and the D_6-invariant lift of the equipotential curve

In 3-space, the Eight gains a new symmetry, the reflexion through its plane. It can be continued to families of periodic solutions in a frame rotating around any one of its symmetry axes. Each of these correspond to a breaking of the $D_6 \times \mathbb{Z}/2\mathbb{Z}$-symmetry into a symmetry under a subgroup isomorphic to D_6 (see [22])

6.3.2. *The choreographies*

These are solutions of the n-body problem in E of the form

$$x(t) = (q(t), q(t + T/n), \ldots, q(t + (n-1)T/n)),$$

where $q : S^1_T \to E$ is a closed curve. They are invariant under group actions which contain the action of $\mathbb{Z}/n\mathbb{Z}$ by $t \mapsto (t+T/n)$ on S^1_T and circular permutation on the set $\{1, 2, \ldots, n\}$. For this action of $\mathbb{Z}/n\mathbb{Z}$ alone, it is shown in [5] that the minimizers are the relative equilibria of the regular n-gon (this may be different in rotating frame). Many planar choreographies were computed numerically by C. Simó [21, 42] as local minimizers of the action under the $\mathbb{Z}/n\mathbb{Z}$-symmetry, some supported by a curve without any symmetry, but for most of them, no existence proof is available because topological constraints are mixed to the symmetry constraints. The same is true of the many 3-body choreographies found numerically by Simó [41] by a shooting method followed by symmetrization, in a way reminiscent of Birkhoff's method for finding Hill's solutions in the restricted 3-body problem. We already alluded in 5.2.3 to the first proof of existence of a non planar choreography (4 equal masses [43]); it is done by minimization under a mixture of symmetry and topological constraints which is made to work by a clever use of the blow-up and the property of described in 5.2.3. *Multiple choreographies* with several curves correspond to the action of a product of cyclic groups; the first one was proved to exist by Chen [6] (number 1 of figure 5).

Questions: (i) is equality of all the masses necessary to get this type of solutions ? (see [17]); (ii) do choreographies with non equal time intervals exist ?

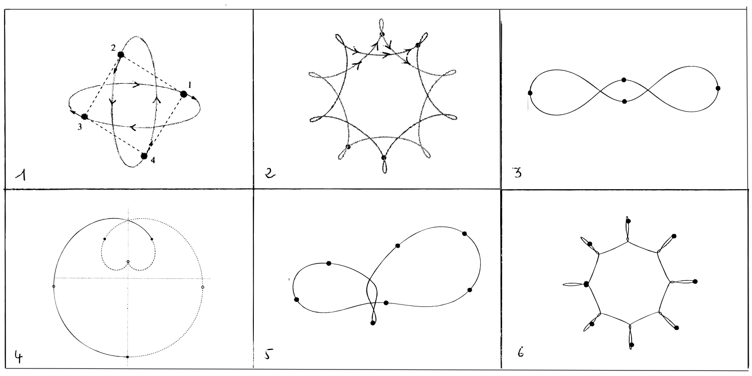

Figure 5. Some choreographies (1-2 by Chen, 3 by Gerver, 4-6 by Simó)

7. CONCLUSION: What next ?

Finding new solutions is not necessarily a goal in itself. It would be really interesting if these new solutions, when unstable, could serve as organizing centers of the global dynamics (compare Moeckel [32]). Up to now, only the Eight with 3 bodies has been found (numerically) stable [41]. The study of the invariant manifolds will certainly need to be computer assisted, as are already the proofs of local unicity of the Eight and existence of Gerver's orbit (number 3 in figure 5) [28]. Many other problems are pending: gains of symmetry (in many cases, the minimizers seem to possess more symmetry than was asked for), proof of the existence of minimax solutions (see [5] for a minimax interpretation of the Eight), understanding how to find solutions satisfying topological constraints, limits of choreographies for $n \to \infty$, etc...

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