Москва

30/11/2021

ELLIPTIC FIXED POINTS WITH AN INVARIANT FOLIATION: SOME FACTS AND MORE QUESTIONS

A. Chenciner, D. Sauzin, 孙善忠 (Shanzhong Sun), 魏巧玲 (Qiaoling Wei)

Памяти Алексея Борисова
для которого математика была
больше чем абстрактная игра

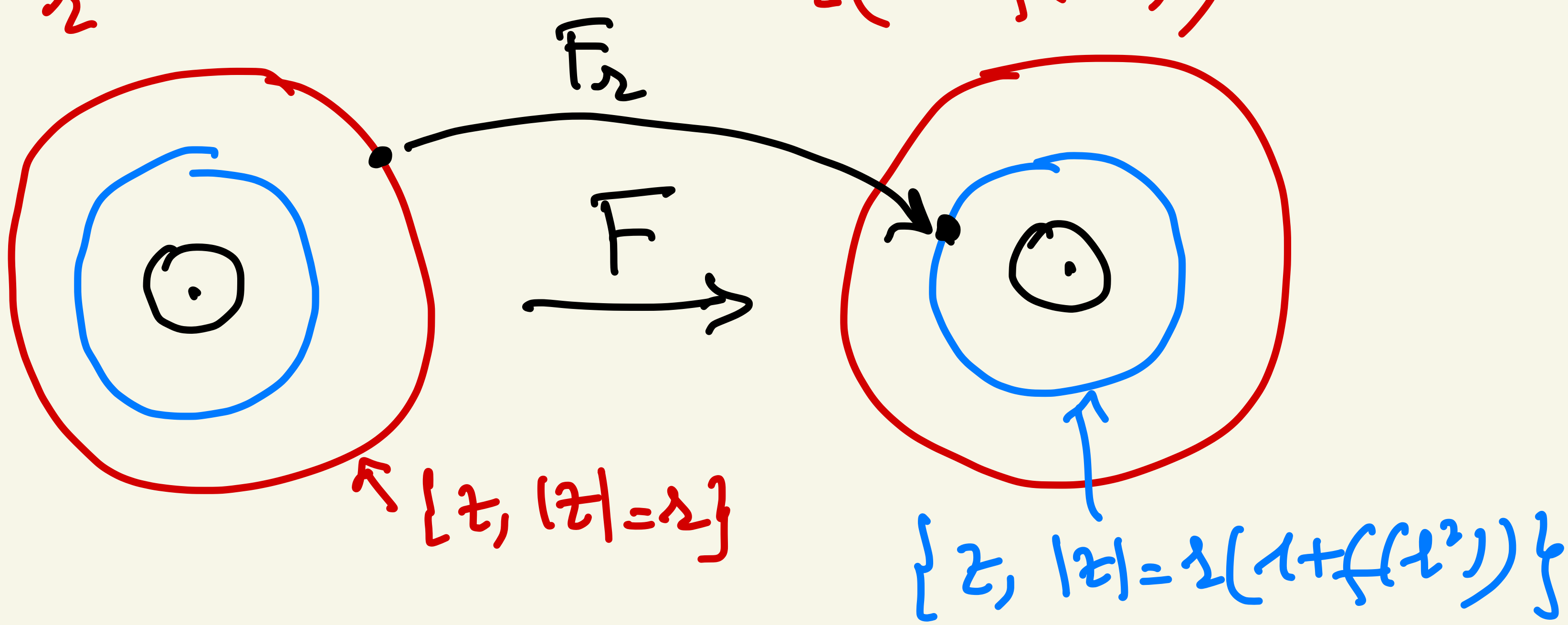
The two main characters:

$$A_{\lambda,a,d}(z) = \lambda z\left(1+a|z|^{2d}\right) e^{2\pi i \operatorname{Im} z}$$

$$B_{\lambda,a,d}(z) = \lambda z\left(1+a|z|^{2d}\right) e^{2\pi i |z|^2 (1+\operatorname{Im} z)}$$

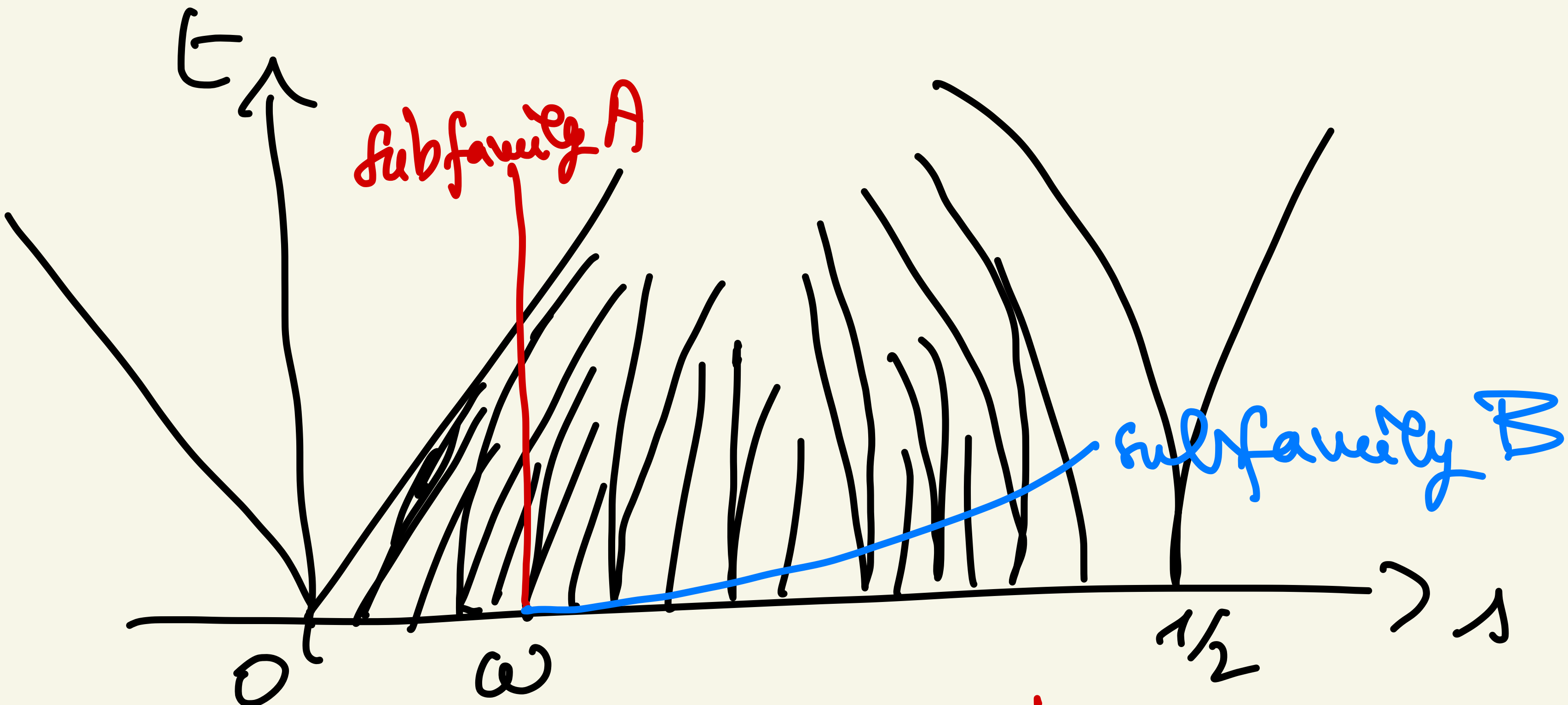
$$\lambda = e^{2\pi i \omega}, \quad \omega \notin \mathbb{Q}, \quad a \le 0$$

$$
\begin{array}{ccc}
\mathbb{R}^+ \times \mathbb{T}^1 & \cdots\longrightarrow & \mathbb{R}^+ \times \mathbb{T}^1 \\
(r, \theta) & & \big(r(1+f(r^2)),\ \theta+\omega+g_r(\theta)\big) \\
\downarrow & & \downarrow \\
\mathbb{R}^2, 0 & \cdots\xrightarrow{\ F\ } & \mathbb{R}^2, 0 \\
z = re^{2\pi i\theta} & \text{real analytic} & \lambda z(1+f(|z|^2))e^{2\pi i g(z,\bar{z})} \\
\downarrow & & \downarrow \\
\mathbb{R}^+ & \cdots\longrightarrow & \mathbb{R}^+ \\
r & & r(1+f(r^2))
\end{array}
$$



Arnold's family $\subset \mathrm{Diff}^\omega \mathbb{T}^1$

$$\theta \longmapsto \theta + s + t \sin 2\pi\theta$$



A: $\theta \longmapsto \theta + \omega + \lambda \sin 2\pi\theta$

B: $\theta \longmapsto \theta + \omega + \lambda^2 + \lambda^3 \sin 2\pi\theta$

The problem

Does there exist a convergent normalization i.e. a local analytic diffeo φ such that

$$\begin{array}{ccc} \mathbb{R}^2, 0 & \xrightarrow{F} & \mathbb{R}^2, 0 \\ \varphi \downarrow & & \downarrow \varphi(z) = z\left(1 + \sum_{p+q \geq 1} c_{pq} z^p \bar{z}^q\right) \\ \mathbb{R}^2, 0 & \xrightarrow{N} & \mathbb{R}^2, 0 \end{array}$$

write $N(z) = \lambda z \left(1 + \sum_{k \geq 1} \gamma_k |z|^{2k}\right)$

$$= \lambda z \left(1 + \alpha(|z|^2)\right) e^{2\pi i \beta(|z|^2)}$$

real

2 extreme contexts for $F(z) = \lambda z + O(|z|^2)$

(no hypothesis of preservation of foliation)

F strong contraction $(|\lambda| < 1)$: $\exists$ local analytic Ψ

(Poincaré) conjugating F to $dF(0)z = \lambda z$

F preserves area $(\lambda = e^{2\pi i \omega}$ and $\omega \notin \mathbb{Q})$

(Birkhoff)

$\exists$ formal conjugacy Ψ to a 1! normal form N

and Ψ, N are generically divergent.

(Siegel) (Krikorian)

$\Longrightarrow$ More precise question:

F weak contraction foliation preserving

$$F(z) = \lambda z\left(1+f(|z|^2)\right)e^{2\pi i g(z,\bar z)}$$

$$\lambda = e^{2\pi i\omega},\quad \omega \notin \mathbb{Q}$$

$$f(|z|^2) = a|z|^{2d} + \cdots,\quad a<0$$

Does the strength of the weak contraction influence the analytic nature of normalization(s) of F ?

Our result:

NO if ω is not a Bruno number

(i.e. if convergents $\frac{p_n}{q_n}$ do not satisfy $\sum \frac{\log q_{n+1}}{q_n} < +\infty$)

Non unicity of normal forms : (Formal theory)

Theorem 1 $N_1(z) = \lambda z\left(1+\sum_{k\geq 1} \gamma_k^1 |z|^{2k}\right)$

$N_2(z) = \lambda z\left(1+\sum_{k\geq 1} \gamma_k^2 |z|^{2k}\right)$

two normal forms of $F(z) = \lambda z + O(|z|^2)$
are formally conjugate by

$$H(z) = z\left(1+\sum_{k\geq 1} h_k |z|^{2k}\right)$$

Moreover, first nonzero coefficients $\gamma_{k_0}^1, \gamma_{\ell_0}^2$

coincide : $k_0 = \ell_0$ and $\gamma_{k_0}^1 = \gamma_{\ell_0}^2$

2 If F is a contraction (if $f \not\equiv 0$)
$\exists$ formal conjugacy to $N = \lambda z P(|z|^2)$

↑ polynomial

The foliation preserving case:

Proposition: $F(z) = \lambda z\left(1+f(|z|^2)\right)e^{2\pi i g(z,\bar z)}$

$\exists$ special conjugacy $\Phi(z) = z e^{2\pi i \varphi(z,\bar z)}$ to

$$N(z) = \lambda z\left(1+f(|z|^2)\right)e^{2\pi i n(|z|^2)}$$

$$\varphi(z,\bar z) = \sum_{p+q\geq 1} \varphi_{pq} z^p \bar z^q, \quad n(|z|^2) = \sum_{s\geq 1} n_s |z|^{2s}$$

The φ_{pp} can be chosen arbitrarily

$\Downarrow$

If $f(|z|^2) = \underset{\neq 0}{a}|z|^{2d} + \cdots$,

$$\begin{cases} n_1, \dots, n_d \text{ are } 1! \text{ determined} \\ n_{d+1}, \dots \text{ can be chosen arbitrarily (e.g. } = 0) \end{cases}$$

(in the "conservative case" $f \equiv 0$, all n_i are $1!$ determined)

Corollary : Any formal normalization Ψ of $F(z) = \lambda(1+f(|z|^2))e^{2\pi i g(z,\bar z)}$ "preserves the foliation by circles"

i.e.
$$\Psi(z) = z(1+a(|z|^2))e^{2\pi i\psi(z,\bar z)}$$

$\Longrightarrow$ Provides an obvious difference between the cases $f \equiv 0$ and $f \not\equiv 0$

(1) "Conservative" case: $f \equiv 0$

$$\theta \longmapsto \theta + \omega + g_r(\theta)$$

$$\begin{tikzcd} r \arrow[r, "F_r"] \arrow[d, "\Psi_r"'] & r \arrow[d, "\Psi_r"] \\ r' \arrow[r, "N_{r'}"] & r' \end{tikzcd}$$

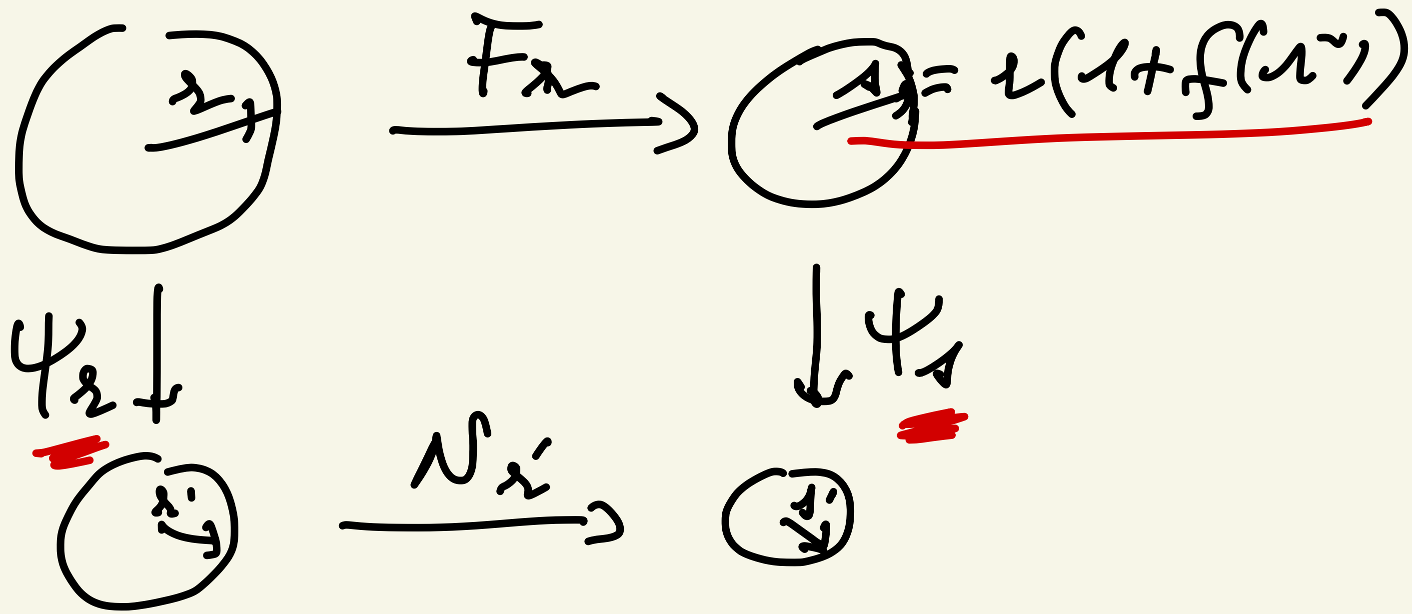
Ψ arbitrary formal conjugacy

$$\theta \longmapsto \theta + \omega + n(r^2)$$

divergent as soon as the family F_r contains elements not conjugated to rotation

(e.g. $F = A$ or $F = B$ when $f \equiv 0$)

(2) Non conservative case: $f \not\equiv 0$ (say < 0 near 0)



No more a conjugacy between F_1 and N_1,

$\Longrightarrow$ no obvious obstruction to convergence of Ψ

Theorem 4 (divergence implied by the holomorphic part $F^0(z) = F(z,0)$ of F)

If $F(z,\bar z) = \lambda z\left(1 + f(|z|^2)\right) e^{2\pi i g(z,\bar z)}$ is such that $F^0(z) = \lambda z e^{2\pi i g(z,0)}$ is not $\mathbb{C}$-analytically linearizable, any formal conjugacy Ψ of F to a normal form N is divergent

<u>Corollary</u> If ω is not a Bruno number, any formal conjugacy of $A_{e^{2\pi i\omega}, a, d}$ to a normal form diverges

$\left\{\begin{array}{l}\text{can be } 0 \\ \text{or } \neq 0\end{array}\right.$

Proof of theorem A 1) The homological equation

$$\begin{array}{ccc} z & \xrightarrow{F} & \lambda z(1+f(|z|^2))e^{2\pi i g(z,\bar z)} \\ \text{special } \Phi \downarrow & & \downarrow \Phi \\ u = z e^{2\pi i \varphi(z,\bar z)} & \xrightarrow[N]{} & \lambda u(1+f(|u|^2))e^{2\pi i n(|z|^2)} \end{array}$$

$$\Updownarrow$$

$$(H) \quad \boxed{\varphi \circ F(z,\bar z) - \varphi(z,\bar z) + g(z,\bar z) - n(|z|^2) = 0}$$

- **Proof of theorem A** ②

(case of special conjugacy $\underset{\text{easy}}{\Longrightarrow}$ general case)

$$\phi(z) = z\, e^{2\pi i \varphi(z,\bar z)} \qquad \psi = H \circ \phi \qquad H(z) = z(1 + h(|z|^2))$$

$$\phi \circ F = N \circ \phi \qquad \varphi(F(z,\bar z), \bar F(z,\bar z)) - \varphi(z,\bar z) + g(z,\bar z) - n(|z|^2) = 0$$

$$\Downarrow \; (\bar z = 0)$$

$$\phi^0 \circ F^0 = N^0 \circ \phi^0 \qquad \varphi(F^0(z),0) - \varphi(z,0) + g(z,0) = 0$$

- **Proof of corollary:**

$$A^0(z) = \lambda z e^{\pi z}$$

Geyer's theorem

$\Uparrow$

Yoccoz's theorem

($e^{2\pi i \omega} z(1-z)$ analytically conjugate to $e^{2\pi i \omega} z \Leftrightarrow \omega$ Brjuno)

Thm does not apply to B : $B^\circ(z) = \lambda z$

• **Theorem B** : if $F(z,\bar z) = \lambda z\left(1+f(|z|^2)\right)e^{2\pi i g(z,\bar z)}$

is such that $\rho = \frac{N}{M} = \sup\limits_{g_{pq}\neq 0} \frac{p-q}{p+q} < 1$,

let $z = r e^{2\pi i\theta}$, $Z = r^M e^{2\pi i N\theta}$

$$g^\circ(Z) = \sum_k g_{p_k q_k} z^{p_k}\bar z^{q_k} = \sum_k g_{p_k q_k} Z^k$$

$$\varphi^\circ(Z) = \sum_k \varphi_{p_k q_k} z^{p_k}\bar z^{q_k} = \sum_k \varphi_{p_k q_k} Z^k$$

k such that $\left\{\begin{array}{l} p_k+q_k = kM, \quad p_k - q_k = kN \\ \exists (p_k, q_k) \end{array}\right\}$

Then $\Phi^\circ(Z) = Z e^{2\pi i N\varphi^\circ(Z)}$ linearizes $F^\circ(Z) = \lambda^N Z e^{2\pi i N g^\circ(Z)}$

Thm B applies to B: $\rho = \frac{1}{3}$, $B^\circ(Z) = \lambda Z e^{\pi Z}$

QUESTIONS

- $F(z,\bar{z}) = \lambda z(1+f(|z|^2))e^{2\pi i g(z,\bar{z})}$

  rôle of the weak attraction f ?

  Pérez-Marco dichotomy (originating from an idea of Ilyashenko) applies:

  ω, f fixed $\Rightarrow$ (1) either normalization generically dv
  (2) or ——— always cv

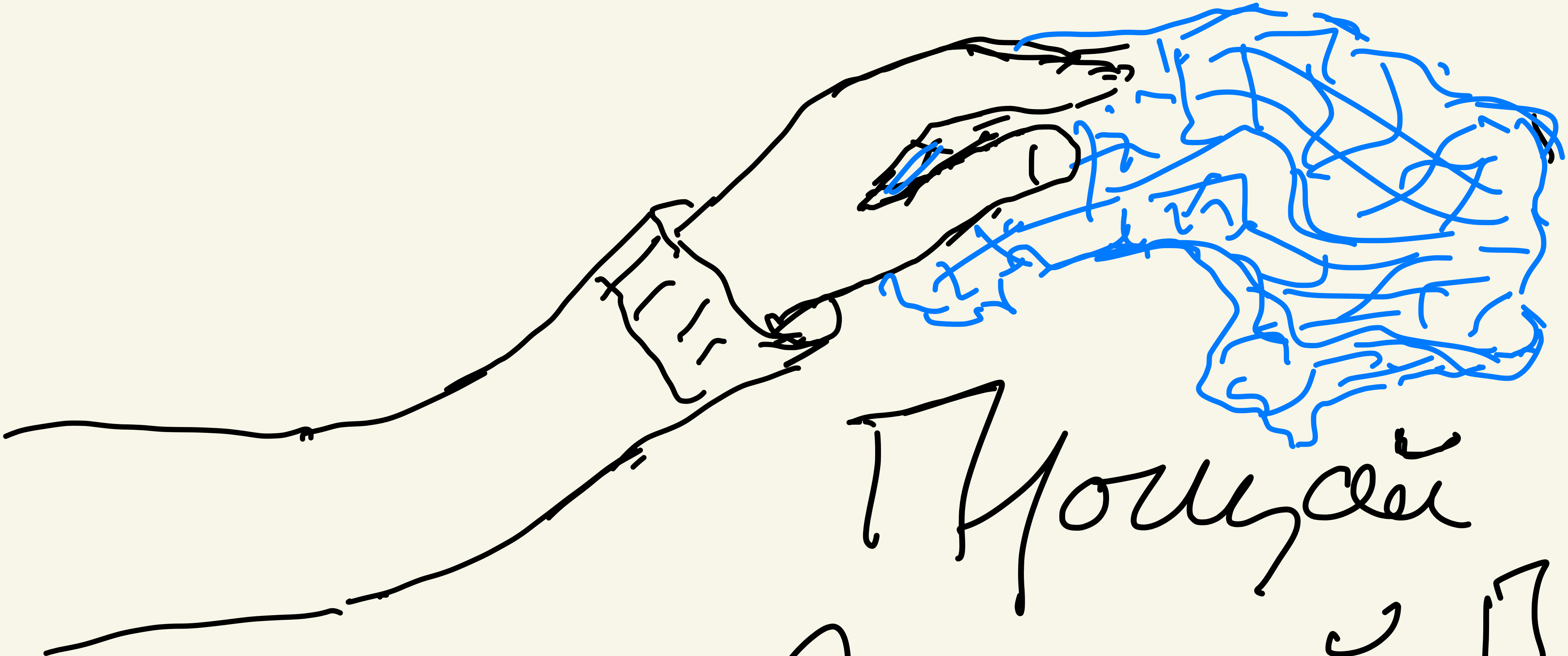
  if ω non Bruno $\Rightarrow$ (1)

  if ω Bruno ?

- Prop: $f \neq 0 \Rightarrow \exists\ \Phi(z) = \lambda z e^{2\pi i \varphi(z,\bar{z})}$ C^0
  such that φ continuous even at $(0,0)$
  and $\Phi \circ F = F \circ N$

  is such a Φ necessarily injective ?

- Role of translated objects (Rk: not invariant)



Прощай

Алексей !