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Elliptic fixed points with an invariant foliation: Some facts and more questions

Alain Chenciner^{*†}, David Sauzin^{*}, Shanzhong Sun^{‡§}, Qiaoling Wei[‡]

*dedicated to the memory of our friend
and colleague Alexey Borisov*

Abstract

We address the following question: let $F : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be an analytic local diffeomorphism defined in the neighborhood of the non resonant elliptic fixed point 0 and let Φ be a formal conjugacy to a normal form N . Supposing F leaves invariant the foliation by circles centered at 0, what is the analytic nature of Φ and N ?

1 Motivation: the two families $A_{\lambda,a,d}$, $B_{\lambda,a,d}$

Understanding the normalization of the following examples of local analytic diffeomorphisms of the plane with an elliptic fixed point was the motivation for raising the questions studied in this paper. Preserving the foliation by circles, these examples are radially trivial but angularly subtle; a normalization is a formal change of coordinates which makes the angular behavior trivial.

$A_{\lambda,a,d}$ and $B_{\lambda,a,d}$ are the local maps from $(\mathbb{R}^2, 0)$ to itself respectively defined by

$$\begin{cases} A_{\lambda,a,d}(z) = \lambda z(1 + a|z|^{2d})e^{\pi(z-\bar{z})} \\ B_{\lambda,a,d}(z) = \lambda z(1 + a|z|^{2d})e^{\pi|z|^2(2i+z-\bar{z})} \end{cases},$$

where $\lambda = \rho e^{2\pi i\omega}$, $0 < \rho \leq 1$ and $a \in \mathbb{R}$, $a < 0$. In polar coordinates $z = re^{2\pi i\theta}$:

$$\begin{cases} A_{\lambda,a,d}(r, \theta) = (\rho r(1 + ar^{2d}), \theta + \omega + r \sin 2\pi\theta), \\ B_{\lambda,a,d}(r, \theta) = (\rho r(1 + ar^{2d}), \theta + \omega + r^2 + r^3 \sin 2\pi\theta). \end{cases}$$

We shall use the notations

$$\begin{cases} A_{\lambda}(z) = A_{\lambda,0,d}(z) = \lambda z e^{\pi(z-\bar{z})}, \\ B_{\lambda}(z) = B_{\lambda,0,d}(z) = \lambda z e^{\pi|z|^2(2i+z-\bar{z})}. \end{cases}$$

^{*}IMCCE (Observatoire de Paris, PSL Research University, CNRS)

[†]University Paris 7

[‡]Department of Mathematics, Capital Normal University, Beijing 100048, China

[§]Academy for Multidisciplinary Studies, Capital Normal University, Beijing 100048, China

The families, parametrized by r , of analytic diffeomorphisms of the circle defined by the angular component of $A_{\lambda,a,d}$ and $B_{\lambda,a,d}$ are subfamilies of *Arnold's family*

$$\theta \mapsto \theta + s + t \sin 2\pi\theta,$$

whose resonant zones (parameter values for which the rotation number is rational, the so-called *Arnold's tongues*) are depicted on figure 1. In particular, each rational rotation number corresponds to an interval of values of r ($[A, H]$).

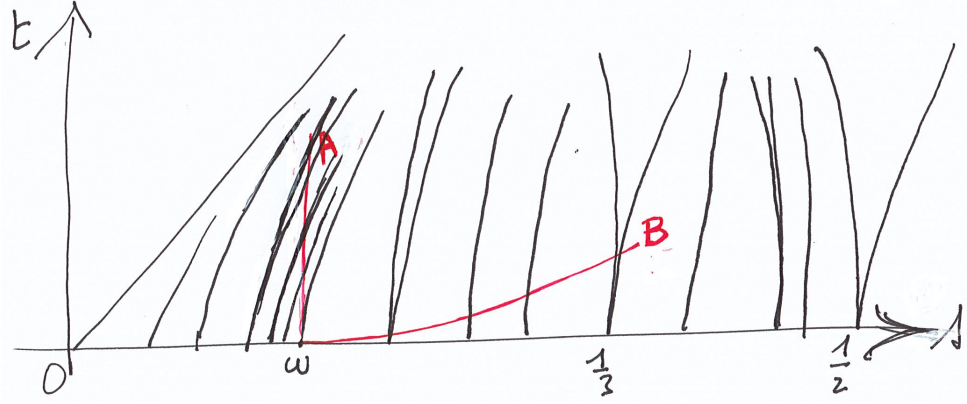


Figure 1 : Families A and B .

2 Formal theory

2.1 Special normal forms

Definition 1 Let \mathcal{F}_0 be the foliation of \mathbb{R}^2 by circles centered at 0. A formal diffeomorphism $F : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is said to preserve \mathcal{F}_0 if $|F(z)|^2$ depends only on $|z|^2$. Identifying \mathbb{R}^2 with \mathbb{C} , this means that it is of the form

$$\begin{cases} F(z) = \lambda z (1 + f(|z|^2)) e^{2\pi i g(z)}, & \text{where } \lambda \neq 0 \in \mathbb{C}, \\ f(u) = \sum_{n \geq 1} f_n u^n, \quad f_n \in \mathbb{R}, & g(z) = \sum_{j+k \geq 1} g_{jk} z^j \bar{z}^k, \quad g_{kj} = \bar{g}_{jk} \in \mathbb{C}. \end{cases}$$

Blowing up the fixed point, that is using polar coordinates $z = r e^{2\pi i \theta}$, turns the (formal) diffeomorphism F into a skew-product over the half-line \mathbb{R}^+ (which we shall still call F):

$$F : \mathbb{R}^+ \times \mathbb{T}^1 \rightarrow \mathbb{R}^+ \times \mathbb{T}^1, \quad F(r, \theta) = (r(1 + f(r^2)), \theta + \omega + g(r, \theta)).$$

Hence, iterating F amounts to composing sequences of (formal) circle diffeomorphisms.

The eigenvalues of the linear part $dF(0)$ of F are λ and $\bar{\lambda}$. The case $|\lambda| < 1$ is well understood since Poincaré: F is then locally formally conjugate to $dF(0)$

(and analytically if F is analytic). From now on, we shall suppose that $|\lambda| = 1$ and even that

$$\lambda = e^{2\pi i\omega}, \quad \omega \in \mathbb{R} \setminus \mathbb{Q}$$

In that case, the only “resonant monomials”, i.e. monomials $z^p \bar{z}^q$ such that $\lambda^p \bar{\lambda}^q = 1$, are those of the form $|z|^{2p}$, since

$$p \neq q \quad \Rightarrow \quad \lambda^p \bar{\lambda}^q - 1 \neq 0. \quad (1)$$

Lemma 2 (Special normal forms) *Let F be a formal diffeomorphism of \mathbb{R}^2 defined in the neighborhood of the elliptic fixed point 0. Suppose that F preserves the foliation \mathcal{F}_0 . If the derivative $dF(0)$ is a non-periodic rotation $z \mapsto \lambda z$, there exists a formal conjugacy Φ of F to a normal form N such that Φ preserves formally each circle centered at 0, hence N sends formally each circle centered at 0 on the same circle as F does, that is $\Phi \circ F = N \circ \Phi$ with*

$$\begin{cases} \Phi(z) = ze^{2\pi i\varphi(z)}, & \varphi(z) = \sum_{p+q \geq 1} \varphi_{pq} z^p \bar{z}^q, \quad \varphi_{qp} = \bar{\varphi}_{pq}, \\ N(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n(|z|^2)}, & n(|z|^2) = \sum_{s \geq 1} n_s |z|^{2s}. \end{cases}$$

Such conjugacies and the corresponding normal forms will be called “special”. The coefficients φ_{pp} can be chosen arbitrarily in \mathbb{R} .

Proof. Starting with

$$F(z) = \lambda z(1 + f(|z|^2))e^{2\pi i g(z)}, \quad \lambda = e^{2\pi i\omega}, \quad \omega \notin \mathbb{Q},$$

let us look for a formal change of coordinates

$$\Phi(z) = ze^{2\pi i\varphi(z)}, \quad \varphi(z) = \sum_{p+q \geq 1} \varphi_{pq} z^p \bar{z}^q, \quad \varphi_{qp} = \bar{\varphi}_{pq},$$

which transforms F into a normal form

$$N(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n(|z|^2)}.$$

The equation $\Phi \circ F = N \circ \Phi$ is equivalent to the homological equation

$$g(z) - n(|z|^2) + \varphi \circ F(z) - \varphi(z) = 0. \quad (\text{H})$$

Writing

$$\begin{aligned} (1 + f(|z|^2))e^{2\pi i g(z)} &= (1 + \sum_r f_r |z|^{2r}) \left(1 + \sum_s \frac{1}{s!} (2\pi i \sum_{t,u} g_{tu} z^t \bar{z}^u)^s \right) \\ &= 1 + \sum_{\alpha, \beta} c_{\alpha, \beta} z^\alpha \bar{z}^\beta, \end{aligned}$$

the homological equation becomes

$$\begin{aligned} & - \sum_{j+k \geq 1} g_{jk} z^j \bar{z}^k + \sum_{s \geq 1} n_s |z|^{2s} \\ & = \sum_{p+q \geq 1} \varphi_{pq} \left[\lambda^p \bar{\lambda}^q \left(1 + \sum_{\alpha, \beta} c_{\alpha, \beta} z^\alpha \bar{z}^\beta \right)^p \left(1 + \sum_{\alpha, \beta} \bar{c}_{\alpha, \beta} \bar{z}^\alpha z^\beta \right)^q - 1 \right] z^p \bar{z}^q. \end{aligned}$$

Once the coefficients $\varphi_{p'q'}$, $p' + q' < p + q$ and n_s , $2s < p + q$, are determined, identification of the terms in $z^p \bar{z}^q$ determines $\varphi_{pq}(\lambda^p \bar{\lambda}^q - 1)$ if $p \neq q$ (resp. determines n_p if $p = q$). In view of (1), this determines by induction the coefficients φ_{pq} such that $p \neq q$. For $p = q$ the coefficient φ_{pp} can be chosen arbitrarily provided it is real, hence the non unicity.

Finally, the first member of the homological equation is real; replacing the equation by its conjugate and exchanging q and p amounts to the original equation apart from transforming φ_{pq} into $\bar{\varphi}_{qp}$. This proves the lemma.

Definition 3 We shall call “basic” and denote by $\Phi^*(z)$ the unique special formal conjugacy without resonant terms in its angular component, i.e.

$$\Phi^*(z) = z e^{2\pi i \varphi^*(z)}, \quad \varphi^*(z) = \sum_{p+q \geq 1, p \neq q} \varphi_{pq} z^p \bar{z}^q.$$

The corresponding normal form $N^* = \Phi^* \circ F \circ (\Phi^*)^{(-1)}$ is called the basic normal form.

All the other special formal conjugacies $\Phi(z)$ of F to a normal form can be written

$$\Phi(z) = z e^{2\pi i (\varphi^*(z) + b(|z|^2))},$$

where b is an arbitrary real formal series in one variable without constant term. A natural choice is given by the following lemma:

Lemma 4 1) If the valuation of f is d , that is if $f(|z|^2)$ starts with a term in $|z|^{2d}$, then the coefficients n_s , $1 \leq s \leq d$, of a special normal form are uniquely determined; moreover, the φ_{pp} can be chosen so that n_{d+p} takes any given value, in particular 0, for $p \geq 1$. If $f \equiv 0$ (a case which we call conservative), then the special normal form is uniquely determined.

2) The initial form \tilde{g}_k of the formal function

$$\tilde{g}(z) = g(z) - \sum_{s \geq 1} n_s |z|^{2s}$$

does not contain any resonant term. In other words,

$$\tilde{g}(re^{2\pi i \theta}) = \tilde{g}_k(re^{2\pi i \theta}) + O(r^{k+1}), \quad \text{with} \quad \tilde{g}_k(re^{2\pi i \theta}) = r^k P_k(\theta),$$

where $P_k(\theta)$ is a real trigonometric polynomial of degree at most k with mean value zero.

Hence, the stronger the contraction (i.e. the smaller is d), the less constrained is the torsion of a formal normal form (compare to section 5.3 where we recall that in case of a linear contraction (i.e. $|\lambda| < 1$), the normal form can be chosen linear in the angle).

Proof. 1) The homological equation (H) expresses \tilde{g} as

$$\begin{aligned}\tilde{g}(z) &= \varphi(z) - \varphi \circ F(z) \\ &= \sum_{p+q \geq 1} \varphi_{pq} \left[1 - \lambda^{p-q} (1 + a|z|^{2d} + O(|z|^{2(d+1)}))^{p+q} e^{2\pi i(p-q)g(z)} \right] z^p \bar{z}^q.\end{aligned}$$

The coefficient of φ_{pp} is $|z|^{2p} \left[(1 + f(|z|^2))^{2p} - 1 \right]$ which starts with a term in $|z|^{2(p+d)}$: the choice of φ_{11} allows choosing n_{d+1} , then the choice of φ_{22} allows choosing n_{d+2} , and so on.

2) Let k be the smallest degree of monomials in \tilde{g} , then

$$\tilde{g}_k(z) = \sum_{p+q=k} \varphi_{pq} [1 - \lambda^{p-q}] z^p \bar{z}^q$$

does not contain any resonant term.

2.1.1 The formal conjugacy equations for $F = A_{e^{2\pi i\omega}, a, d}$

We look for a conjugacy $\Phi(z) = ze^{2\pi i\varphi(z)}$ to a special normal form

$$N(z) = \lambda z(1 + a|z|^{2d})e^{2\pi i n(|z|^2)}, \quad \lambda = e^{2\pi i\omega}, \quad \omega \text{ irrational.}$$

The conjugacy equation $\Phi \circ F = N \circ \Phi$ becomes the homological equation

$$\frac{z - \bar{z}}{2i} + \varphi(\lambda z(1 + a|z|^{2d})e^{\pi(z - \bar{z})}) - \varphi(z) = n(|z|^2),$$

that is

$$\frac{z - \bar{z}}{2i} + \sum_{j+k \geq 1} \varphi_{jk} \left(\lambda^{j-k} (1 + a|z|^{2d})^{j+k} e^{(j-k)\pi(z - \bar{z})} - 1 \right) z^j \bar{z}^k = \sum_{s \geq 1} n_s |z|^{2s}.$$

Expanding the exponential we get

$$\begin{aligned}\sum_{j+k \geq 1} \varphi_{jk} \left(\lambda^{j-k} (1 + a|z|^{2d})^{j+k} \left(1 + \sum_{n \geq 1} \frac{\pi^n}{n!} (j-k)^n (z - \bar{z})^n \right) - 1 \right) z^j \bar{z}^k \\ = -\frac{z - \bar{z}}{2i} + \sum_{s \geq 1} n_s |z|^{2s}.\end{aligned}$$

Separating terms of degree 1 and 2 in z, \bar{z} and the ones containing a leads to

$$\varphi_{10} = \frac{1}{2i(1 - \lambda)} = \bar{\varphi}_{01}, \quad \varphi_{20} = \frac{\pi\lambda}{2i(1 - \lambda)(1 - \lambda^2)} = \bar{\varphi}_{02}, \quad n_1 = -\operatorname{Im} \frac{\pi\lambda}{1 - \lambda},$$

and

$$\begin{aligned}
& \sum_{j+k=N \geq 3} \varphi_{jk} (\lambda^{j-k} - 1) z^j \bar{z}^k + \sum_{n \geq 1} \sum_{j+k=N-n} \varphi_{jk} \lambda^{j-k} \frac{\pi^n}{n!} (j-k)^n (z - \bar{z})^n z^j \bar{z}^k \\
& + \sum_{(j+k, l) \in I_N} \varphi_{jk} \lambda^{j-k} a^l \binom{j+k}{l} |z|^{2dl} z^j \bar{z}^k \\
& + \sum_{n \geq 1} \sum_{(j+k, l) \in I_{N-n}} \varphi_{jk} \lambda^{j-k} a^l \binom{j+k}{l} |z|^{2dl} \frac{\pi^n}{n!} (j-k)^n (z - \bar{z})^n z^j \bar{z}^k = \begin{cases} n_{\frac{N}{2}} |z|^N & \text{if } N \text{ even} \\ 0 & \text{if } N \text{ odd} \end{cases}
\end{aligned}$$

where the condition $(j+k, l) \in I_N$ means $j+k = N - 2dl \geq 1$ and $1 \leq l \leq \frac{N}{2d+1}$.

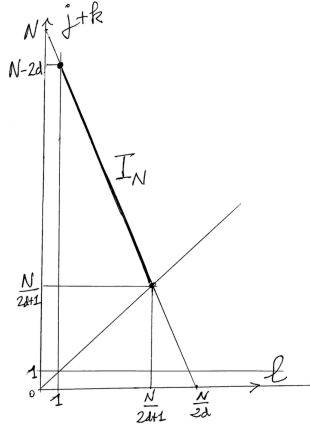


Figure 2: The condition $(j+k, l) \in I_N$.

According to what we previously noticed, the coefficients φ_{pp} can be chosen so that $n(|z|^2) = n_1 |z|^2 + \dots + n_d |z|^{2d}$. Each n_s , $1 \leq s \leq d$, is a well-defined function of (λ, a, d) , polynomial in a and rational in λ .

Thus, if $F = A_{e^{2\pi i \omega}, a, d}$,

$$\tilde{g}(z) = \operatorname{Im} z - \frac{\pi}{2i} \frac{\lambda + 1}{\lambda - 1} |z|^2 - n_2 |z|^4 - \dots - n_d |z|^{2d}.$$

2.1.2 The formal conjugacy equations for $F = B_{e^{2\pi i \omega}, a, d}$

The corresponding homological equation is

$$|z|^2 \left(1 + \frac{z - \bar{z}}{2i} \right) + \varphi(\lambda z (1 + a |z|^{2d}) e^{\pi |z|^2 (2i + z - \bar{z})}) - \varphi(z) = n(|z|^2),$$

that is

$$\begin{aligned} & \sum_{j+k \geq 1} \varphi_{jk} \left(\lambda^{j-k} (1 + a|z|^{2d})^{j+k} e^{(j-k)\pi|z|^2(2i+z-\bar{z})} - 1 \right) z^j \bar{z}^k \\ &= -|z|^2 \left(1 + \frac{z - \bar{z}}{2i} \right) + O(|z|^4). \end{aligned}$$

Expanding the exponential we get

$$\begin{aligned} & \sum_{j+k \geq 1} \varphi_{jk} \left[\lambda^{j-k} (1 + a|z|^{2d})^{j+k} \left(1 + \sum_{n \geq 1} \frac{\pi^n}{n!} (j-k)^n |z|^{2n} (2i + z - \bar{z})^n \right) - 1 \right] z^j \bar{z}^k \\ &= -|z|^2 \left(1 + \frac{z - \bar{z}}{2i} \right) + \sum_{s \geq 1} n_s |z|^{2s}. \end{aligned}$$

Equating terms of degree up to 4 in z, \bar{z} leads to

$$\begin{cases} \varphi_{10} = \bar{\varphi}_{01} = 0, & \varphi_{20} = \bar{\varphi}_{02} = 0, & n_1 = 1, & \varphi_{30} = \bar{\varphi}_{03} = 0, & \varphi_{21} = \bar{\varphi}_{12} = \frac{-1}{2i(\lambda - 1)}, \\ \varphi_{40} = \bar{\varphi}_{04} = 0, & \varphi_{31} = \bar{\varphi}_{13} = 0, & n_2 = 0 \text{ if } d \geq 2; & n_3 = \text{Im} \frac{\lambda}{\lambda - 1} \text{ if } d \geq 3. \end{cases}$$

From these computations, we deduce that, if $F = B_{e^{2\pi i \omega}, a, d}$,

$$\tilde{g}(z) = |z|^2 \text{Im } z + O(|z|^6).$$

2.2 Non unicity of formal normal forms

As we have already noticed in the special case, normal forms are not unique but, even in the general case, this non unicity is mild, more precisely:

Lemma 5 (Non unicity of normal form) *If $\lambda = e^{2\pi i \omega}$ with ω irrational, then any two formal normal forms of the same formal diffeomorphism*

$$N_1(z) = \lambda z \left(1 + \sum_{k \geq 1} \alpha_k |z|^{2k} \right) \quad \text{and} \quad N_2(z) = \lambda z \left(1 + \sum_{k \geq 1} \beta_k |z|^{2k} \right)$$

are formally conjugated by a formal diffeomorphism of the form

$$H(z) = z \left(1 + h(|z|^2) \right) = z + \sum_{l \geq 1} h_l |z|^{2l} z, \quad h_l \in \mathbb{C}.$$

Moreover, the first non vanishing coefficient α_{k_0} of N_1 and the first non vanishing coefficient β_{l_0} of N_2 coincide:

$$l_0 = k_0 \quad \text{and} \quad \beta_{k_0} = \alpha_{k_0}.$$

Proof. Let $H(z) = z + \sum_{p+q \geq 2} h_{pq} z^p \bar{z}^q$ be such that $H \circ N_1 = N_2 \circ H$. Identifying degree 2 terms in this identity implies that $(\lambda^p \bar{\lambda}^q - \lambda) h_{pq} = 0$ for all p and q such that $p + q = 2$. Moreover, degree 3 terms satisfy

$$\lambda \alpha_1 |z|^2 z + \sum_{p+q=3} \lambda^p \bar{\lambda}^q h_{pq} = \lambda \beta_1 |z|^2 z + \lambda \sum_{p+q=3} h_{pq},$$

from which it follows that $\alpha_1 = \beta_1$ and $h_{pq} = 0$ for all $(p, q) \neq (2, 1)$ such that $p + q = 3$. Let us suppose by induction that

$$H(z) = z \left(1 + \sum_{l=1}^m h_l |z|^{2l} \right) + \sum_{p+q \geq 2m+2} h_{pq} z^p \bar{z}^q. \quad (H_m)$$

As all the terms of $N_1(z) \left(1 + \sum_{l=1}^m h_l |N_1(z)|^{2l} \right)$ are of odd degree, the only terms of degree $2m + 2$ in $H \circ N_1(z)$ are $\sum_{p+q=2m+2} \lambda^p \bar{\lambda}^q h_{pq} z^p \bar{z}^q$.

Similarly, all the terms of $N_2 \left(z \left(1 + \sum_{l=1}^m h_l |z|^{2l} \right) \right)$ being of odd degree, the only terms of degree $2m + 2$ in $N_2 \circ H(z)$ are $\lambda \sum_{p+q=m+2} h_{pq} z^p \bar{z}^q$.

One deduces that $h_{pq} = 0$ for all p and q such that $p + q = 2m + 2$. Hence

$$H(z) = z \left(1 + \sum_{l=1}^m h_l |z|^{2l} \right) + \sum_{p+q \geq 2m+3} h_{pq} z^p \bar{z}^q.$$

Terms of degree $2m + 3$ in $H \circ N_1(z)$ are the ones of

$$N_1(z) \left(1 + \sum_{l=1}^m h_l |N_1(z)|^{2l} \right) + \sum_{p+q=2m+3} \lambda^p \bar{\lambda}^q h_{pq} z^p \bar{z}^q$$

and those of $N_2 \circ H(z)$ are the ones of

$$N_2 \left(z \left(1 + \sum_{l=1}^m h_l |z|^{2l} \right) \right) + \lambda \sum_{p+q=2m+3} h_{pq} z^p \bar{z}^q.$$

One deduces that

$$\begin{aligned} & \lambda (\alpha_{m+1} + \varphi(\alpha_1, \dots, \alpha_m)) |z|^{2m+2} z + \sum_{p+q=2m+3} \lambda^p \bar{\lambda}^q h_{pq} z^p \bar{z}^q \\ &= \lambda (\beta_{m+1} + \psi(\beta_1, \dots, \beta_m)) |z|^{2m+2} z + \lambda \sum_{p+q=2m+3} h_{pq} z^p \bar{z}^q, \end{aligned}$$

where φ and ψ are polynomials without constant term. It follows that (H_{m+1}) is verified and that $\alpha_{m+1} = \beta_{m+1}$ if all the α_k and the β_k vanish for $k \leq m$, which concludes the proof.

Corollary 6 *Under the hypotheses of lemma 2, any formal conjugacy Ψ of F to a normal form preserves the foliation \mathcal{F}_0 .*

Indeed, writing

$$H(z) = z(1 + h(|z|^2)) = z(1 + a(|z|^2))e^{2\pi ib(|z|^2)},$$

where a and b are real series, it follows from lemmas 2 and 5 that, once the basic normal form

$$N^*(z) = \Phi^* \circ F \circ \Phi^{*-1}(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n^*(|z|^2)}, \quad \Phi^*(z) = ze^{2\pi i \varphi^*(z)},$$

is known, the most general conjugacy of F to a normal form is a composition

$$\Psi(z) = H \circ \Phi^*(z) = z(1 + a(|z|^2))e^{2\pi i(\varphi^*(z) + b(|z|^2))}$$

where $a(X)$ and $b(X)$ are arbitrary real formal series in one variable without constant term.

A direct computation leads to

Lemma 7 *Corresponding to a general formal conjugacy $\Psi = H \circ \Phi^*$ as above, the most general normal form for F is*

$$N(z) = \Psi \circ F \circ \Psi^{-1}(z) = \lambda z(1 + \alpha(|z|^2))e^{2\pi i \beta(|z|^2)},$$

where α and β are given by the following formulas

$$\begin{aligned} 1 + \alpha(|z|^2) &= (1 + f(|H^{-1}(z)|^2)) \frac{1 + a(|F \circ H^{-1}(z)|^2)}{1 + a(|H^{-1}(z)|^2)}, \\ \beta(|z|^2) &= n^*(|H^{-1}(z)|^2) + b(|F \circ H^{-1}(z)|^2) - b(|H^{-1}(z)|^2), \text{ with} \\ |H(u)|^2 &= |u|^2(1 + a(|u|^2))^2, \text{ hence } |H^{-1}(z)|^2 = |z|^2(1 + \rho(|z|^2)). \end{aligned}$$

Remarks.

1) If the conjugacies Φ_1 and Φ_2 are special, the composition $H = \Phi_2 \circ \Phi_1^{-1}$ must preserve individually each circle: $H(z) = ze^{2\pi ib(|z|^2)}$. Hence the corresponding special normal forms $N_k(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n_k(|z|^2)}$, $k = 1, 2$, satisfy

$$n_2(|z|^2) - n_1(|z|^2) = b(|F(z)|^2) - b(|z|^2).$$

2) If $f \equiv 0$, i.e. $|F(z)| = |z|$, a case which we shall call *conservative*, then $\alpha \equiv 0$ and $\beta(|z|^2) = n^*(|H^{-1}(z)|^2)$. This implies that β can be chosen to be a polynomial and even a monomial: indeed, if $n^*(X) = n_p X^p + O(|X|^{p+1})$ with $n_p \neq 0$, we can write $\frac{n^*(r^2)}{n_p r^{2p}} = (1 + a^*(r^2))^{2p}$ with a suitable real series a^* and, by choosing $a = a^*$ in H (with any b), we get $n^*(r^2) = n_p(r(1 + a(r^2)))^{2p}$, whence $\beta(r^2) = n_p r^{2p}$.

Hence, if F is conservative, there always exist a non conservative formal transformation to a convergent normal form $N(z) = \lambda ze^{2\pi i |z|^{2p}}$.

3) In general, even if f is not identically 0, we can always achieve $\beta(r^2) = n_p r^{2p}$, i.e. an “angularly polynomial normal form”, by choosing $a = a^*$ as above and $b = 0$. However, the resulting normal form $N(z) = \lambda z(1 + \alpha(|z|^2))e^{2\pi i |z|^{2p}}$ is not convergent if n^* is not convergent.

Definition 8 A formal conjugacy Ψ (resp. a normal form $\Psi \circ F \circ \Psi^{-1}$) such that H (or what is equivalent, the series a and b) converge will be called an RC (resonant part convergent) formal conjugacy (resp. RC normal form).

Lemma 9 The three properties: F admits a convergent normalization, F admits a convergent RC normalization, every RC-normalization of F is convergent, are equivalent.

In particular, if its basic normalization Φ^* is divergent, then all normalizations of F are divergent.

Proof. It follows from the observation that the terms $z|z|^{2s}$ in $\Psi = H \circ \Phi^*$ originate only from H , the Cauchy-Hadamard formula for the radius of convergence of a formal series in several variables implies that the convergence of Ψ implies the ones of H and Φ^* .

Hence, one can restrict the discussion of convergence to RC-normal forms and even to the basic special one $N^*(z)$. Notice that it is not yet known whether a polynomial normal form is an RC-normal form but this seems unlikely.

Definition 10 Let $G : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a formal diffeomorphism; we shall say that the formal diffeomorphism F preserves the “formal foliation” $\mathcal{F} = G^{-1}(\mathcal{F}_0)$ if $G \circ F \circ G^{-1}$ preserves \mathcal{F}_0 .

Corollary 11 A formal diffeomorphism F whose derivative $dF(0)$ is an irrational rotation cannot preserve more than one formal foliation.

Proof. If F preserves $\mathcal{F}_1 = G_1^{-1}(\mathcal{F}_0)$ and $\mathcal{F}_2 = G_2^{-1}(\mathcal{F}_0)$, and Ψ is a formal conjugacy of F to a normal form N , corollary 6 implies that the formal diffeomorphisms $\Psi \circ G_1^{-1}$ and $\Psi \circ G_2^{-1}$ must both preserve \mathcal{F}_0 . This means that Ψ sends both \mathcal{F}_1 and \mathcal{F}_2 on \mathcal{F}_0 , hence that $\mathcal{F}_1 = \mathcal{F}_2 = \Psi^{-1}(\mathcal{F}_0)$.

The case F a pure homothety is the simplest counter-example to Corollary 11 when the hypothesis on $dF(0)$ is not satisfied.

3 Polynomial normal forms

Proposition 12 As soon as F is a weak contraction, there exists a formal conjugacy to a polynomial normal form $N_1 = \lambda z P(|z|^2)$.

Recall remark 2 above: if F is conservative, there exists a non conservative conjugacy to a normal form $N(z) = \lambda z e^{2\pi i |z|^{2p}}$; on the other hand, if $|\lambda| < 1$, Poincaré has proved that there exists an analytic conjugacy of F to $N = dF(0)$,

that is $N(z) = \lambda z$. Note that here, as in Poincaré's case, no preservation of a foliation is required.

Proof. We look for a formal conjugacy of a normal form

$$N_2(z) = \lambda z \nu_2(|z|^2), \quad \text{such that} \quad |\nu_2(z)|^2 = 1 - b|z|^{2r} + O(|z|^{2(r+1)}), \quad b > 0,$$

to a polynomial normal form $N_1(z) = \lambda z \nu_1(|z|^2)$. The main variable being $|z|^2$, the use of symplectic polar coordinates $z = t^{\frac{1}{2}} e^{2\pi i \theta}$ is mandatory. The two normal forms $N_i, i = 1, 2$, become

$$(t, \theta) \mapsto (F_i(t) = t f_i(t), \theta + \omega + g_i(t)), \quad f_i(t) = |\nu_i(t)|^2.$$

According to section 2.2, a conjugacy is necessarily of the form

$$\Phi(z) = z (1 + h(|z|^2)),$$

that is

$$(t, \theta) \mapsto (\phi(t) = t \varphi(t), \theta + \gamma(t)), \quad \varphi(t) = |1 + h(t)|^2, \quad \gamma(t) = \arg(1 + h(t)).$$

The following diagram summarizes the situation.

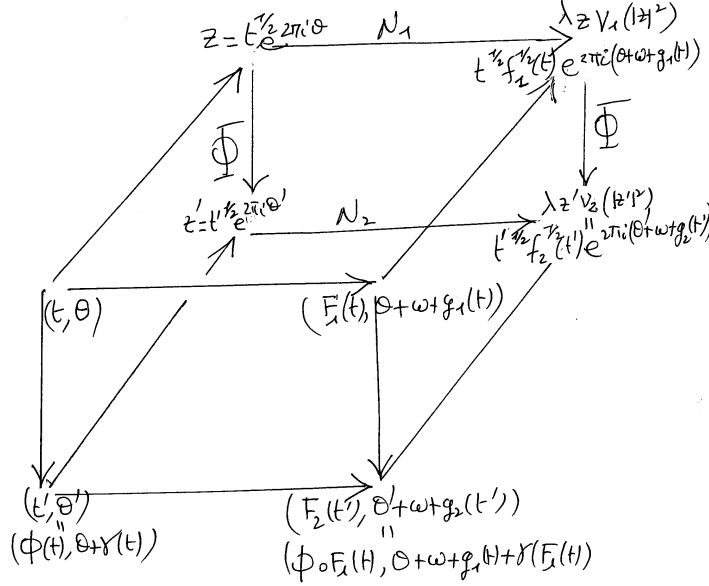


Figure 3 : Symplectic polar coordinates.

The conjugacy equation takes the form :

$$\begin{cases} F_2 \circ \phi = \phi \circ F_1, & (CE1) \end{cases}$$

$$\begin{cases} \gamma \circ F_1 - \gamma + g_1 = g_2 \circ \phi. & (CE2) \end{cases}$$

Equation (CE1) is the conjugacy equation corresponding to the 1-dimensional real normalization problem. According to the formal analogue of [Ch], the map $F_2(t) = t - bt^{r+1} + O(t^{r+2})$ is formally conjugate to $F_1(t) = t - bt^{r+1} + ct^{2r+1}$, with $c \in \mathbb{R}$ uniquely defined by the $(2r+1)$ -jet of F_2 . Hence it is also conjugate to any map $F_1(t) = t - bt^{r+1} + ct^{2r+1} + O(t^{2r+2})$, that is

Lemma 13 *For any $F_1(t) \in t - bt^{r+1} + ct^{2r+1} + t^{2r+2}\mathbb{R}[[t]]$, there exists a formal diffeomorphism $\phi(t) \in t + t^2\mathbb{R}[[t]]$ such that $F_2 \circ \phi = \phi \circ F_1$; moreover, the $(2r+1)$ -jet of ϕ does not depend on the choice of such a F_1 .*

Being able to cope with the $O(t^{2r+2})$ term is crucial. First, the following lemma will allow us to solve (CE2) as soon as the r -jet of g_1 coincides with the r -jet $G(t)$ of $g_2 \circ \phi$ which, by the lemma, depends only on N_2 and not on the precise choice of the $O(t^{2r+2})$ terms in F_1 , that is not on ϕ :

Lemma 14 *For any $F_1(t) \in t - bt^{r+1} + ct^{2r+1} + t^{2r+2}\mathbb{R}[[t]]$ with $b \neq 0$, the linear operator $\gamma \mapsto \gamma \circ F_1 - \gamma$ induces a bijection $t\mathbb{R}[[t]] \rightarrow t^{r+1}\mathbb{R}[[t]]$.*

Proof. Write $F_1(t) = t + tu(t)$ with $u(t) \in -bt^r + ct^{2r} + t^{2r+1}\mathbb{R}[[t]]$ and hence $u(t) \in -bt^r + t^{r+1}\mathbb{R}[[t]]$. Taylor formula yields

$$\gamma \circ F_1 - \gamma = u \cdot \left(E\gamma + \sum_{k \geq 2} T_k \gamma \right), \quad E = t \frac{d}{dt}, \quad T_k = \frac{1}{k!} u^{k-1} t^k \left(\frac{d}{dt} \right)^k.$$

The series of operators $\sum T_k$ is convergent in the following sense: when applied to a formal series, T_k increases its order by at least $(k-1)r$ units (because u is of order r). Now $E : t\mathbb{R}[[t]] \rightarrow t\mathbb{R}[[t]]$ is a bijection which does not change the order. It follows that $E + \sum_{k \geq 2} T_k$ is also a bijection whose inverse is defined by the convergent series of operators

$$\left(E + \sum_{k \geq 2} T_k \right)^{-1} = \left(Id + E^{-1} \sum_{k \geq 2} T_k \right)^{-1} \circ E^{-1} = \sum_{l \geq 0} (-1)^l \sum_{k \geq 2} (E^{-1} \circ T_k)^l \circ E^{-1}.$$

Finally, the conclusion follows from the fact that multiplication by $u = -bt^r + O(t^{r+1})$ is a bijection from $t\mathbb{R}[[t]]$ to $t^{r+1}\mathbb{R}[[t]]$ because $b \neq 0$.

Corollary 15 *There exists a polynomial $G(t)$ of degree r such that, given any two formal series $\rho(t)$ and $\sigma(t)$, there exists a formal diffeomorphism $\phi \in t + t^2\mathbb{R}[[t]]$ and $\gamma \in t\mathbb{R}[[t]]$ which define a formal conjugacy of N_2 to*

$$N_1(t, \theta) = t^{\frac{1}{2}} \left[1 - bt^r + ct^{2r} + t^{2r+1} \rho(t) \right]^{\frac{1}{2}} e^{2\pi i(\theta + \omega + G(t) + t^{r+1} \sigma(t))},$$

that is

$$N_1(z) = \lambda z \left[1 - b|z|^{2r} + c|z|^{4r} + |z|^{4r+2} \rho(|z|^2) \right]^{\frac{1}{2}} e^{2\pi i(G(|z|^2) + |z|^{2r+2} \sigma(|z|^2))}.$$

Proof. One defines $G(t)$ as the r -jet of $g_2 \circ \phi$ which, by the remark preceding lemma 14, is independent of ϕ . Setting $g_1(t) = G(t) + t^{r+1} \sigma(t)$, one can solve (CE1) and (CE2).

Proof of Proposition 12. It remains to notice that ρ and σ may be chosen so that N_1 be the polynomial $N_1(z) = \lambda z P(|z|^2)$, where $P(t)$ is the $2r$ -jet of $[1 - bt^r + ct^{2r}]^{\frac{1}{2}} e^{2\pi i G(r)}$.

Remark. In the same way, one can achieve a normal form $N(z) = \lambda z Q(|z|^2) e^{2\pi i G(|z|^2)}$, where $Q(t)$ is the $2r$ -jet of $[1 - bt^r + ct^{2r}]^{\frac{1}{2}}$.

4 Topological theory

In this section we do suppose that F is not conservative and write

$$F(z) = \lambda z(1 + f(|z|^2))e^{2\pi i g(z)}, \quad f(u) = au^d + O(u^{d+1}), \quad d \geq 1, \quad a < 0,$$

with $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{C}, (0, 0)) \rightarrow (\mathbb{R}, 0)$ real analytic germs.

While in the formal theory Lemma 4 restrains the possible choices of n , for the topological theory any continuous $n : [0, R^2[\rightarrow \mathbb{R}$ vanishing at 0 is admissible and it is in this general setting that we recall Sternberg's theorem (section 4.1).

Nevertheless, in section 4.2 we come back to normal forms N satisfying the restrictions given in Lemma 4 when studying the existence of a special type of topological semi-conjugacies of F to N .

4.1 Sternberg's theorem

Here a normal form is a local homeomorphism of $(\mathbb{R}^2, 0)$ which commutes with the group of rotations, that is which sends each small circle centered at 0 to another such circle by a rotation. If the local contraction F preserves the foliation \mathcal{F}_0 , we call a normal form N F -special if it sends each small circle C onto the image $F(C)$ of this same circle by F .

One deduces from [S] that any two local contractions are topologically conjugate one to the other in the neighborhood of 0. Moreover, if F preserves the foliation \mathcal{F}_0 , the proof in [S] gives naturally a local topological conjugacy Φ to *any* F -special normal form N such that, as in section 2.1, Φ preserves individually each circle. Indeed, let us choose any F -special normal form N ; if \mathcal{D} is a small disk centered at 0 and \mathcal{C}_0 is its boundary, we may define Φ on \mathcal{C}_0 and its image $\mathcal{C}_1 = F(\mathcal{C}_0)$ by

$$\Phi|_{\mathcal{C}_0} = Id \quad \text{and} \quad \Phi|_{\mathcal{C}_1} = N \circ F^{-1}.$$

(\mathcal{C}_0 and \mathcal{C}_1 are disjoint if F is a contraction because F preserves the foliation by circles, see figure 4).

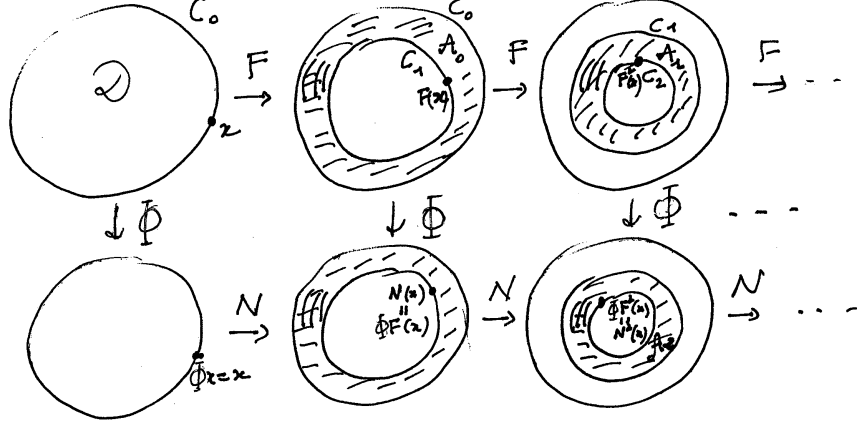


Figure 4 : Topological conjugacy.

It is then possible to extend Φ to the annulus $\mathcal{A}_0 = \mathcal{D} \setminus \text{int}F(\mathcal{D})$ by an interpolation which preserves the foliation by circles centered at 0. Indeed, it is always possible to choose an analytic (if N is chosen analytic) family of analytic diffeomorphisms of the circle which interpolates between the Id and $N \circ F^{-1}|_{C_1}$ after C_0 and C_1 have both been identified by radial projection with the standard circle. One then extends Φ to the images $F^n(\mathcal{A}_0) = N^n(\mathcal{A}_0) = \mathcal{A}_n$ of \mathcal{A}_0 by $\Phi|_{\mathcal{A}_n} = N^n \circ \Phi|_{\mathcal{A}_0} \circ F^{-n}$. This defines a local homeomorphism Φ of $(\mathbb{R}^2, 0)$.

4.2 Special semi-normalizations, homological equation, Neumann series

4.2.1 Topological semi-conjugacies and the homological equation

Definition 16 *Given maps F and N , we say that Φ establishes a semi-conjugacy of F to N if Φ is surjective and $\Phi \circ F = N \circ \Phi$. We speak of topological semi-conjugacy when Φ and N are continuous.*

We are interested in *special semi-normalizations* given by *special semi-conjugacies* Φ , i.e. such that $|\Phi(z)| = |z|$, of F to special normal forms N , i.e. such that $N(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n(|z|^2)}$. We write $\Phi(z) = ze^{2\pi i \varphi(z)}$, meaning that φ is a priori defined only on a punctured neighborhood $D_R^* = \{z \in \mathbb{C}, 0 < |z| < R\}$ of 0 and that $\Phi(0) = 0$. The map φ can be chosen continuous on D_R^* if Φ is continuous on $D_R = \{z \in \mathbb{C}, |z| < R\}$ and conversely, if φ is continuous on D_R^* , the map $\Phi : D_R \rightarrow D_R$ is continuous and surjective.

From now on we are interested only in the case Φ is continuous.

The semi-conjugacy equation $\Phi \circ F = N \circ \Phi$ is equivalent to the equation

$$\varphi - \varphi \circ F = \tilde{g} \mod \mathbb{Z}, \text{ where } \tilde{g}(z) = g(z) - n(|z|^2).$$

As φ and n are continuous, this is equivalent to

$$\exists k \in \mathbb{Z}, \varphi - \varphi \circ F = \tilde{g} + k. \quad (HE)_k$$

Since \tilde{g} is continuous on D_R and vanishes at 0, we are led to single out a particular class of topological special semi-conjugacies:

Definition 17 *We say that $\Phi(z) = ze^{2\pi i\varphi(z)}$ is θ -tame or angularly tame if φ extends to a continuous function $\varphi : D_R \rightarrow \mathbb{R}$.*

Lemma 18 *Φ is angularly tame if and only if φ is continuous at 0 and satisfies*

$$\varphi - \varphi \circ F = \tilde{g}. \quad (HE)_0$$

Proof. φ and F being continuous on D_R with $F(0) = 0$, the limit when z tends to 0 of the left hand side of $(HE)_k$ is $\varphi(0) - \varphi(0) = 0$ while the limit of the right hand side is $\tilde{g}(0) + k = k$.

Definition 19 *We call Neumann series the series of the form $\sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}$.*

Lemma 20 *If Φ is an angularly tame semi-conjugacy of F to N , the Neumann series is pointwise convergent and $\varphi = \varphi(0) + \sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}$. In particular, there is at most one angularly tame semi-conjugacy of F to a given N up to a rotation.*

Proof. From $(HE)_0$ one gets, for each integer $M \geq 1$,

$$\varphi(z) = \sum_{m=0}^{M-1} \tilde{g} \circ F^{(m)}(z) + \varphi \circ F^{(M)}(z).$$

As $\varphi \circ F^{(M)}$ tends pointwise to 0 when M tends to infinity, we get that $\varphi - \varphi(0)$ is the pointwise limit of the series $\sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}$.

Conversely, if the Neumann series is pointwise convergent, its sum φ provides a solution to $(HE)_0$ and thus a semi-conjugacy Φ but a stronger property is needed to ensure continuity:

Lemma 21 *If the Neumann series $\varphi = \sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}$ is uniformly convergent on D_R , it defines an angularly tame semi-conjugacy of F to N .*

Note that injectivity is not granted.

4.2.2 Existence of an angularly tame semi-conjugacy

Proposition 22 *Let $F(z) = \lambda z(1 + f(|z|^2))e^{2\pi i g(z)}$ be such that f has valuation d and let $n^*(|z|^2) = \sum_{s=1}^d n_s |z|^{2s}$ be the polynomial determined by the formal theory (see Lemma 4). Let $N(z) = \lambda z(1 + f(|z|^2))e^{2\pi i n(|z|^2)}$ be a normal form.*

1) If $n(|z|^2) = n^*(|z|^2) + O(|z|^{2d+1})$, then there exists an angularly tame semi-conjugacy of F to N .

2) If n is analytic but not of this form—that is if N is not a normal form in the sense of the formal theory—, then a topological conjugacy exists thanks to Sternberg but no angularly tame semi-conjugacy exists.

Proof. Without loss of generality we can assume that

$$F(z) = \lambda z(1 + f(|z|^2))e^{2\pi i(n^*(|z|^2) + \check{g}(z))}, \quad \text{with } \check{g}(z) = O(|z|^{2d+1}).$$

Indeed, we can always perform a preliminary change of coordinate $z \mapsto ze^{2\pi i\varphi(z)}$ where $\varphi(z)$ is an appropriate *polynomial* function of (z, \bar{z}) so that F is normalized up to any arbitrary order.

In order to be able to apply lemmas 20 and 21, we shall estimate the size of the general term of the series $\sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}$ with

$$\tilde{g}(z) = n^*(|z|^2) - n(|z|^2) + \check{g}(z).$$

The main step is controlling the decrease of $|F^{(m)}(z)|$: as F preserves the foliation \mathcal{F}_0 by circles, the norm $|F^{(m)}(z)|$ of any iterate depends only on $r = |z|$:

$$|F(z)| = \nu(r) := r(1 + f(r^2)) \quad \text{and} \quad |F^{(m)}(z)| = \nu^{(m)}(r).$$

Lemma 23 *There exist r_0, C, D, K such that, for all $r \in]0, r_0[$ and $m \geq 1$,*

- (i) $0 < \nu^{(m)}(r) < r$,
- (ii) $\nu^{(m)}(r) \leq Cm^{-\frac{1}{2d}}$,
- (iii) $m \geq Kr^{-2d} \Rightarrow \nu^{(m)}(r) \geq Dm^{-1/2d}$.

Proof. The inversion

$$\mathcal{I} : r \mapsto U = r^{-p}$$

exchanging 0 and infinity, will allow us to estimate $\nu^{(m)}(r)$ by comparing the transform of $\nu(r)$ to a translation.

The positive integer p and the positive real number \tilde{a} being fixed, let

$$\nu_{p,\tilde{a}}(r) = r(1 + \tilde{a}r^p)^{-\frac{1}{p}}.$$

One checks immediately that

$$\mathcal{I} \circ \nu_{p,\tilde{a}} \circ \mathcal{I}^{-1}(U) = U + \tilde{a},$$

which implies that

$$\nu_{p,\tilde{a}}^{(m)}(r) = r(1 + m\tilde{a}r^p)^{-\frac{1}{p}}.$$

Now, as $a < 0$, if $0 < a_- < 2d|a| < a_+$, $\exists r_0 > 0$ such that for $0 < r < r_0$

$$\nu_{2d,a_+}(r) \leq \nu(r) = r + ar^{2d+1} + O(r^{2d+2}) \leq \nu_{2d,a_-}(r).$$

As ν_{a_-} and ν are increasing functions that preserve $]0, r_0[$, this implies that there exists $r_0 > 0$ such that

$$\nu_{2d,a_-}^{(m)}(r) \leq \nu^{(m)}(r) \leq \nu_{2d,a_+}^{(m)}(r) \quad \text{for all } r \in]0, r_0[\text{ and } m \geq 1.$$

The explicit formula for $\nu_{2d,\tilde{a}}^{(m)}(r)$ concludes the proof.

For the first part of proposition 22, we have $\tilde{g} = \check{g} + O(|z|^{2d+1}) = O(|z|^{2d+1})$, hence $|\tilde{g}(z)| \leq A|z|^{2d+1}$ on some disc near 0 and

$$\left| \check{g} \circ F^{(m)}(z) \right| \leq ACm^{-\frac{2d+1}{2d}},$$

which entails the uniform convergence of the series $\sum_{m=0}^{\infty} |\check{g} \circ F^{(m)}|$. We can thus conclude by lemma 21.

For the second part of proposition 22, we have

$$\tilde{g}(z) = \gamma|z|^{2k} + O(|z|^{2k+1}), \quad \text{with } k \leq d \text{ and } \gamma \neq 0.$$

In particular, choosing $\tilde{\gamma} = \frac{1}{2}|\gamma|$,

$\tilde{g}(z)$ does not change sign and $|\tilde{g}(z)| \geq \tilde{\gamma}|z|^{2k}$ for $|z|$ small enough.

Since $|F^{(m)}(z)| = \nu^{(m)}(|z|)$, it follows that the Neumann series $\sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}(z)$ and the series $\sum_{m=0}^{\infty} (\nu^{(m)}(|z|))^{2k}$ are of the same nature for each z close enough to 0.

By Lemma 23(iii), $m \geq Kr^{-2d}$ implies that $(\nu^{(m)}(|z|))^{2k} \geq Dm^{-k/d}$. As $k/d \leq 1$, one concludes to the divergence of both series, which, according to lemma 20, prevents the existence of an angularly tame semi-conjugacy.

5 Analytical theory

5.1 The conservative case

Recall (section 2.2) that this means that $F(z) = \lambda z e^{2\pi i g(z)}$ preserves individually each circle centered at 0.

As any formal conjugacy Ψ of F to a normal form N preserves the foliation \mathcal{F}_0 (section 2.2, Corollary 6), if N is convergent, Ψ will be divergent as soon as there exist arbitrary small radii r such that the restriction F_r of F to the circle $|z| = r$ is not analytically conjugate to a rotation, in particular for $F = A_{\lambda,0,d}$ and $F = B_{\lambda,0,d}$. Indeed if convergent such a conjugacy would provide conjugacies Ψ_r between F_r and $N_{r'}$ for all small enough radii r (figure 5).

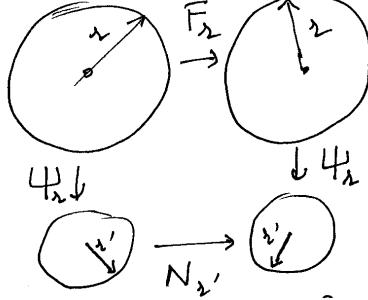


Figure 5: Conservative case.

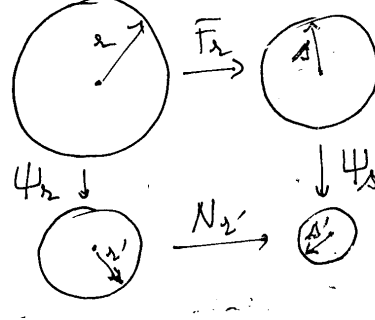


Figure 6: dissipative case.

5.2 Divergence implied by the holomorphic part F^0 of F

Contrarily to what happens in the conservative case, the existence of an analytic conjugacy Ψ of F to a normal form $N = \Psi \circ F \circ \Psi^{-1}$ only implies, for each $r > 0$, an identity $N_r = \Psi_s \circ F_r \circ \Psi_r^{-1}$ when restricting N to the circle $|z| = r$ (figure 6). As soon as $s \neq r$, this does not a priori contradict the non conjugacy of F_r to a rotation.

Nevertheless, divergence may occur as is shown by the following Theorems in which contraction could be present but does not play any part in the proofs.

5.2.1 A criterion of divergence

In this section and the following, we choose to consider real analytic maps like $F(z)$ as series in two variables z and \bar{z} and hence we change the notation, writing $F(z, \bar{z})$. We shall then note $F^0(z) = F(z, 0)$ its holomorphic part.

Theorem 24 *Let $F(z, \bar{z}) = \lambda z(1 + f(|z|^2))e^{2\pi i g(z, \bar{z})}$ be a local analytic diffeomorphism from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^2, 0)$ such that the complex holomorphic map in one variable, $F^0(z) = \lambda z e^{2\pi i g(z, 0)}$ be analytically non linearizable. Then any formal conjugacy Ψ of $F(z, \bar{z})$ to a normal form $N(z, \bar{z})$ is divergent.*

Proof. From section 2.2 we know that the most general conjugacy Ψ and normal form N have the form

$$\begin{aligned}\Psi(z, \bar{z}) &= z(1 + a(|z|^2))e^{2\pi i \varphi(z, \bar{z}) + b(|z|^2)}, \\ N(z, \bar{z}) &= \lambda z(1 + \alpha(|z|^2))e^{2\pi i \beta(|z|^2)},\end{aligned}$$

hence $\Psi^0(z) = ze^{2\pi i \varphi(z, 0)}$ and $N^0(z) = \lambda z$.

The proof consists in the following identities, true for the maps we consider but certainly not for general maps¹:

$$(\Psi \circ F)^0 = \Psi^0 \circ F^0 \quad \text{and} \quad (N \circ \Psi)^0 = N^0 \circ \Psi^0. \quad (*)$$

¹Thanks to Abed Bounemoura for insisting on this.

Indeed, as $\Psi \circ F = N \circ \Psi$, this implies that $\Psi^0 \circ F^0 = N^0 \circ \Psi^0$; convergence of $\Psi(z, \bar{z})$ (and hence $N(z, \bar{z})$) implying² that of Ψ^0 , we would conclude to the analytic linearizability of F^0 , a contradiction.

The proof of (*) consists in the following explicit computations which use the fact that $g(z, \bar{z})$ and $\varphi(z, \bar{z})$ are both real valued:

$$\begin{aligned}(N \circ \Psi)(z, \bar{z}) &= \lambda z (1 + a(|z|^2)) e^{2\pi i(\varphi(z, \bar{z}) + b(|z|^2))} (1 + \alpha(|\Psi(z, \bar{z})|^2)) e^{2\pi i\beta(|\Psi(z, \bar{z})|^2)}, \\ (\Psi \circ F)(z, \bar{z}) &= \lambda z (1 + f(|z|^2)) e^{2\pi i g(z, \bar{z})} (1 + a(|F(z, \bar{z})|^2)) e^{2\pi i(\varphi(F(z, \bar{z}), F(z, \bar{z})) + b(|F(z, \bar{z})|^2))}.\end{aligned}$$

Hence

$$(N \circ \Psi)^0(z) = \lambda z e^{2\pi i \varphi(z, 0)}, \quad (\Psi \circ F)^0(z) = \lambda z e^{2\pi i g(z, 0)} e^{2\pi i \varphi(F, \bar{F})^0(z)},$$

while

$$(N^0 \circ \Psi^0)(z) = \lambda z e^{2\pi i \varphi(z, 0)}, \quad (\Psi^0 \circ F^0)(z) = \lambda z e^{2\pi i g(z, 0)} e^{2\pi i \varphi(F^0(z), 0)}.$$

It only remains to prove that $\varphi(F, \bar{F})^0 = \varphi(F^0, 0)$:

If $\varphi(z, \bar{z}) = \sum_{jk} c_{jk} z^j \bar{z}^k$,

$$\varphi(F, \bar{F})(z, \bar{z}) = \sum_{jk} c_{jk} \lambda^j \bar{\lambda}^k z^j \bar{z}^k (1 + f(|z|^2))^{j+k} e^{2\pi i(j-k)g(z, \bar{z})},$$

hence $\varphi(F, \bar{F})^0(z) = \sum_j c_{j0} \lambda^j z^j e^{2\pi i j g(z, 0)} = \varphi(F^0(z), 0)$.

Corollary 25 *If ω is not a Brjuno number, any formal conjugacy of $A_{e^{2\pi i \omega}, a, d}$ to a normal form diverges.*

Proof. We have $A_{\lambda, a, d}^0(z) = \lambda z e^{\pi z}$ with $\lambda = e^{2\pi i \omega}$, ω an irrational number. Yoccoz had proved (see [Y, PM]) that if one replaces $F^0(z) = \lambda z e^{\pi z}$ by the beginning $z(1 + \pi z)$ of its Taylor expansion, the linearization converges if and only if ω is a Brjuno number. It was later on proved by Lukas Geyer (see [G])³ that the same is true for $F^0(z)$.

Remark. As $B_{\lambda, a, d}^0(z) = \lambda z$, the above result does not apply to $B_{\lambda, a, d}$. Notice that the sub-family of the Arnold family entering in the definition of $B_{\lambda, a, d}$ is much closer to a family of rotations than the one entering in the definition of $A_{\lambda, a, d}^0 = \lambda z$. Nevertheless the following strenghtening of Theorem 24 allows concluding also for the maps $B_{\lambda, a, d}$.

5.2.2 A more refined criterion of divergence

Recall the homological equation for a special normalization $\Phi(z) = z e^{2\pi i \varphi(z)}$ which conjugates the local (formal) diffeomorphism $F(z) = \lambda z (1 + f(|z|^2)) e^{2\pi i g(z)}$

²Consider Ψ as a function of two independent variables z and \bar{z} .

³Thanks to Ricardo Pérez-Marco for this reference

to the normal form $N(z) = \lambda z(1+f(|z|^2))e^{2\pi i n(|z|^2)}$, i.e. such that $\Phi \circ F = N \circ \Phi$:

$$\begin{aligned} g(z) - n(|z|^2) + \varphi \circ F(z) - \varphi(z) &= 0, \quad \text{that is} \\ \sum_{j+k \geq 1} g_{jk} z^j \bar{z}^k - \sum_{s \geq 1} n_s |z|^{2s} \\ + \sum_{j+k \geq 1} \varphi_{jk} \left[\lambda^j \bar{\lambda}^k (1+f(|z|^2))^{j+k} e^{2\pi i (j-k)g(z)} - 1 \right] z^j \bar{z}^k &= 0. \end{aligned}$$

This implies that, if $p \neq q$, the coefficient φ_{pq} satisfies

$$(\lambda^p \bar{\lambda}^q - 1) \varphi_{pq} = -g_{pq} + R,$$

where R is the sum of all coefficients of $z^p \bar{z}^q$ in the expression

$$\sum_{1 \leq j+k < p+q} \varphi_{jk} \left[\lambda^j \bar{\lambda}^k (1+f(|z|^2))^{j+k} e^{2\pi i (j-k)g(z)} - 1 \right] z^j \bar{z}^k.$$

Lemma 26 *Suppose there exists $\rho \leq 1$ such that $g_{pq} \neq 0$ implies $p-q \leq \rho(p+q)$. Then the same is true for the coefficients φ_{pq} of φ .*

Proof. Of course the lemma is empty if $\rho = 1$. We suppose by induction that for any couple (j, k) such that $1 \leq j+k < p+q$, $\varphi_{jk} \neq 0$ implies $(j-k) \leq \rho(j+k)$. The formula above shows that the property is still true for φ_{pq} . Indeed, each term of R is a product of terms $A_i z^{p_i} \bar{z}^{q_i}$ each of which satisfies $p_i - q_i \leq \rho(p_i + q_i)$.

Notations. Let $\rho = \frac{N}{M} = \sup_{g_{pq} \neq 0} \frac{p-q}{p+q}$. Supposing $\frac{N}{M}$ irreducible, and noting $z = r e^{2\pi i t}$, we define

$$Z = r^M e^{2\pi i N t}$$

and consider the pairs (p_k, q_k) of non negative integers such that

$$m_k = p_k + q_k = kM, \quad n_k = p_k - q_k = kN.$$

Notice that such pairs need not exist for all $k \geq 1$: for example, if $M = 2, N = 1$, (p_1, q_1) is not a pair of integers. Let $F^0(Z)$ and $\Phi^0(Z)$ be defined by

$$F^0(Z) = \lambda^N Z e^{2\pi i N g^0(Z)}, \quad \Phi^0(Z) = Z e^{2\pi i N \varphi^0(Z)},$$

where

$$g^0(Z) = \sum_k g_{p_k q_k} z^{p_k} \bar{z}^{q_k} = \sum_k g_{p_k q_k} Z^k, \quad \varphi^0(Z) = \sum_k \varphi_{p_k q_k} z^{p_k} \bar{z}^{q_k} = \sum_k \varphi_{p_k q_k} Z^k,$$

the sums being taken over the set of integers k such that (p_k, q_k) is defined.

Lemma 27 *If $\rho = \frac{N}{M}$, the homological equation implies*

$$g^0(Z) + \varphi^0 \circ F^0(Z) - \varphi^0(Z) = 0.$$

In other words, $\Phi^0(Z)$ linearizes $F^0(Z)$:

$$\Phi^0 \circ F^0 = L \circ \Phi^0,$$

where $L(Z) = \lambda^N Z$.

Proof. Developing the homological equation we get

$$\begin{aligned} & \sum_{j,k} g_{jk} z^j \bar{z}^k - \sum_s n_s |z|^{2s} \\ & + \sum_{j,k} \varphi_{jk} \left[\lambda^j \bar{\lambda}^k \left(1 + \sum_u f_u |z|^{2u} \right)^{j+k} \sum_n \frac{(2\pi i(j-k))^n}{n!} \left(\sum_{v,w} g_{vw} z^v \bar{z}^w \right)^n - 1 \right] z^j \bar{z}^k \\ & = 0. \end{aligned}$$

The general term $z^p \bar{z}^q$ in the last line has the form

$$p = j + u_1 + u_2 + \dots + u_{j+k} + v_1 + v_2 + \dots + v_n, \quad q = k + u_1 + u_2 + \dots + u_{j+k} + w_1 + w_2 + \dots + w_n.$$

As $j - k \leq \rho(j + k)$ and $\forall i, v_i - w_i \leq \rho(v_i + w_i)$, the only possibility for achieving $p - q = \rho(p + q)$ is

$$j - k = \rho(j + k) \quad , \forall i, u_i = 0 \quad \text{and} \quad \forall j, v_j - w_j = \rho(v_j + w_j).$$

Hence, restricting the summations to those pairs of indices which satisfy the above identities $j - k = \rho(j + k)$ and $v - w = \rho(v + w)$, we get

$$\sum_{j,k} g_{jk} z^j \bar{z}^k + \sum_{j,k} \varphi_{jk} \left[\lambda^j \bar{\lambda}^k \sum_n \frac{(2\pi i(j-k))^n}{n!} \left(\sum_{v,w} g_{vw} z^v \bar{z}^w \right)^n - 1 \right] z^j \bar{z}^k = 0,$$

that is

$$\sum_{l \geq 1} g_{p_l q_l} Z^l + \sum_{l \geq 1} \varphi_{p_l q_l} \left[\lambda^{lN} \exp \left(2\pi i l N \sum_{s \geq 1} g_{p_s q_s} Z^s \right) - 1 \right] Z^l = 0,$$

or

$$g^0(Z) + \varphi^0 (\lambda^N Z \exp (2\pi i N g^0(Z))) - \varphi^0(Z) = 0,$$

which is equivalent to the linearization equation

$$\Phi^0 \circ F^0(Z) = \lambda^N \Phi^0(Z).$$

Theorem 28 *Under the hypotheses of lemma 27, if the holomorphic germ $F^0(Z) = \lambda^N e^{2\pi i N g^0(Z)}$ is not holomorphically linearizable, any formal conjugacy Ψ of the germ $F(z) = \lambda z(1 + f(|z|^2))e^{2\pi i g(z)}$ to a normal form is divergent.*

Proof. Lemma 27 is still valid if Φ is replaced by any formal conjugacy Ψ of F to a normal form. Indeed, replacing $\Phi(z)$ by

$$\Psi(z) = H \circ \Phi(z) = z(1 + a(|z|^2))e^{2\pi i(\varphi(z) + b(|z|^2))}$$

does not change the proof because monomials of the form $|z|^{2s}$ never participate in the ones $z^p \bar{z}^q$ achieving the maximum of $\frac{p-q}{p+q}$.

Corollary 29 *If ω is not a Brjuno number, any formal conjugacy of $B_{e^{2\pi i \omega}, a, d}$ to a normal form diverges.*

Proof. We have $\rho = 1/3$, $Z = z^2 \bar{z}$ and $B_{e^{2\pi i \omega}, a, d}^0(Z) = Ze^{\pi Z}$.

A question. Here is a simple example for which Theorem 28 does not lead to a conclusion and hence leaves unsettled the question of divergence:

$$C_{\lambda, a, d}(z) = \lambda z(1 + a|z|^{2d})e^{2\pi i |z|^2(1 + \operatorname{Im} e^z)}.$$

Indeed,

$$\sup_{g_{pq} \neq 0} \frac{p-q}{p+q} = \sup_n \frac{n}{n+2} = 1.$$

Hence only Theorem 10 applies, but $F^0(z) = \lambda z$.

5.3 The case of strong contraction $|\lambda| < 1$

If $\rho = |\lambda| \neq 1$, Poincaré's theorem insures the existence of an analytic local conjugacy of F to its derivative $dF(0)$ but also to any convergent normal form $N(z) = \lambda z(1 + \sum_{k \geq 1} \alpha_k |z|^{2k})$ ($\alpha_k \in \mathbb{C}$). The difference with the case $|\lambda| = 1$ is the possibility of fixing arbitrarily the series $n(|z|^2)$ by choosing the coefficients φ_{pp} of a conjugacy $z \mapsto \Phi(z) = ze^{2\pi i \varphi(z)}$. But there is a unique formal diffeomorphism tangent to Identity which conjugates F to its derivative $dF(0)$: indeed, if Ψ is another one, the composition $h = \Psi \circ \Phi^{-1}$ is tangent to Identity and satisfies the equation $h(\lambda z) = \lambda h(z)$; a term by term identification of the series expansion of h then shows that, already at the formal level, h is the Identity. Hence, if $|\lambda| < 1$ the analytic linearization Φ of A_λ is of the form $\Phi(z) = ze^{2\pi i \varphi(z)}$ where φ is the convergent solution of the equations

$$\begin{cases} \varphi_{10} = \frac{1}{2i(1-\lambda)}, & \varphi_{01} = \frac{1}{2i(\bar{\lambda}-1)} = \bar{\varphi}_{10}, \\ \sum_{j+k \geq 2} \varphi_{jk}(\lambda^j \bar{\lambda}^k - 1) z^j \bar{z}^k + \sum_{j+k \geq 1, n \geq 1} \varphi_{jk} \lambda^j \bar{\lambda}^k \frac{\pi^n}{n!} (j-k)^n (z - \bar{z})^n z^j \bar{z}^k = 0. \end{cases}$$

Notice that in this case the formula $\varphi(z) = \sum_{m=0}^{\infty} \tilde{g} \circ F^{(m)}(z)$ which one deduces immediately by iterating the homological equation makes sense in the realm of power series while it makes sense only for each fixed z in the case of weak contraction (Lemma 20).

5.4 Always convergence or generic divergence

In [PM2], Ricardo Pérez-Marco showed that, for the Birkhoff normal form of an analytic Hamiltonian flow at a non-resonant singular point with given quadratic part, as well as for the normalizing transformation, the following alternative holds: either it is always convergent or it is generically divergent. We now show how to adapt the proof to the non conservative case in our context.

Let $\lambda = e^{2\pi i\omega}$ with real $\omega \notin \mathbb{Q}$. Consider the following families of local real analytic diffeomorphisms of \mathbb{R}^2 (the lower indices indicate the elements which are fixed in the family):

$$\begin{aligned}\mathcal{F}_\lambda &= \left\{ F \left| F(z) = \lambda z(1 + f(|z|^2)e^{2\pi i g(z)}), f, g \text{ arbitrary} \right. \right\}, \\ \mathcal{F}_{\lambda, \bullet, g} &= \left\{ F \left| F(z) = \lambda z(1 + f(|z|^2)e^{2\pi i g(z)}), f \text{ arbitrary} \right. \right\}, \\ \mathcal{F}_{\lambda, f, \bullet} &= \left\{ F \left| F(z) = \lambda z(1 + f(|z|^2)e^{2\pi i g(z)}), g \text{ arbitrary} \right. \right\}.\end{aligned}$$

with real analytic functions

$$f(u) = \sum_{j \geq 1} f_j u^j \quad (f_j \in \mathbb{R}) \quad \text{and} \quad g(z) = \sum_{j+k \geq 1} g_{jk} z^j \bar{z}^k \quad (g_{jk} = \bar{g}_{kj}). \quad (2)$$

At the end of this section, we will prove

Theorem 30 *Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$.*

- i) Let g be as in (2). The generic element of $\mathcal{F} = \mathcal{F}_{e^{2\pi i\omega}}$ or $\mathcal{F} = \mathcal{F}_{e^{2\pi i\omega}, \bullet, g}$ has no convergent normalization.*
- ii) Let $f \not\equiv 0$ be as in (2). If ω is not a Brjuno number, then the generic element of $\mathcal{F} = \mathcal{F}_{e^{2\pi i\omega}, f, \bullet}$ has no convergent normalization.*

Recall that a normalization Φ^* of F is called the basic normalization if

$$\Phi^*(z) = z e^{2\pi i \varphi^*(z)} \quad \text{with} \quad \varphi^*(z) = \sum_{p+q \geq 1} \varphi_{pq}^* z^p \bar{z}^q \quad \text{where} \quad \varphi_{pp}^* = 0 \quad \text{for} \quad p \geq 1,$$

and the corresponding normal form is called the basic normal form:

$$N^*(z) = \lambda z(1 + f(|z|^2)) e^{2\pi i n^*(|z|^2)}, \quad n^*(z) = \sum_{s \geq 1} n_s^* |z|^{2s}.$$

The main part of this section is devoted to the proof of

Theorem 31 *Let $\lambda = e^{2\pi i\omega}$ with $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and let \mathcal{F} be one of the three families of local real analytic diffeomorphisms defined above. Either the basic normalization of every $F \in \mathcal{F}$ is convergent (resp. its basic normal form is convergent), or the normalizations of a generic $F \in \mathcal{F}$ are divergent (resp. its basic normal form is divergent).*

The proof will require three lemmas.

Lemma 32 *Let $F(t; z) = \lambda z(1 + f(t, |z|^2))e^{2\pi i g(t, z)}$ be a family of local maps where*

$$f(t, u) = \sum_{j \geq 1} f_j(t) u^j, \quad f_j(t) = \overline{f_j(\bar{t})}, \quad g(t, z) = \sum_{j+k \geq 1} g_{jk}(t) z^j \bar{z}^k, \quad g_{jk}(t) = \overline{g_{kj}(\bar{t})},$$

and the coefficients $f_j(t)$ and $g_{jk}(t)$ are polynomial functions of $t \in \mathbb{C}$. Then the basic normalization $\Phi^(t; z)$ has the property that each $\varphi_{pq}^*(t)$ is polynomial in t with degree no larger than $p + q$, and the basic normal form $n^*(t; z)$ has its coefficients $n_s^*(t)$ polynomial in t with degree no larger than $2s$.*

Proof. Recall that the conjugacy equation $\Phi^* \circ F = N^* \circ \Phi^*$ is reduced to the homological equation

$$g(t; z) - n^*(t, |z|^2) + \varphi^*(t) \circ F(t; z) - \varphi^*(t; z) = 0.$$

Writing

$$(1 + f(t; |z|^2))e^{2\pi i g(t; z)} = 1 + \sum_{\alpha + \beta \geq 1} c_{\alpha\beta}(t) z^\alpha \bar{z}^\beta,$$

since $f_j(t)$ and $g_{pq}(t)$ are polynomial in t , the coefficient $c_{\alpha\beta}(t)$ is a polynomial in t of degree no larger than $\ell = \alpha + \beta$.

The conjugacy equation becomes

$$\begin{aligned} & - \sum_{j+k \geq 1} g_{jk}(t) z^j \bar{z}^k + \sum_{s \geq 1} n_s^*(t) |z|^{2s} \\ &= \sum_{p+q \geq 1} \varphi_{pq}^*(t) z^p \bar{z}^q [\lambda^p \bar{\lambda}^q (1 + \sum_{\alpha+\beta \geq 1} c_{\alpha\beta}(t) z^\alpha \bar{z}^\beta)^p (1 + \sum_{\alpha+\beta \geq 1} \overline{c_{\alpha\beta}}(t) z^\beta \bar{z}^\alpha)^q - 1] \\ &= \sum_{p+q \geq 1} \varphi_{pq}^*(t) z^p \bar{z}^q (\lambda^p \bar{\lambda}^q - 1) + A_{pq}(t) z^p \bar{z}^q, \end{aligned}$$

where $A_{pq}(t)$ given by summation and multiplication of $\varphi_{jk}^*(t)$ and $c_{jk}^*(t)$ with $j + k < p + q$, whence $A_{pq}(t)$ is a polynomial function of t .

The coefficients $\varphi_{pq}^*(t)$ and $n_s^*(t)$ are uniquely determined by induction on the degree $\ell := p + q$, once the φ_{pp}^* are chosen to be zero. By induction, we get $A_{pq}(t)$ of degree smaller than $p + q$, $\varphi_{pq}^*(t)$ polynomial function of t with degree no larger than $p + q$, and $n_s^*(t)$ polynomial with degree no larger than $2s$.

In the following, we will be using the notion of a polar set, the Green function V_E of a subset E of \mathbb{C} , and the Bernstein-Walsh lemma; the reader is referred to Pérez-Marco's paper [PM2] for all this.

Let (f_0, g_0) and (f_1, g_1) be as in (2). We will consider the affine subspace V consisting of the maps

$$F_t(z) = \lambda z(1 + (t f_0 + (1 - t) f_1)(|z|^2)) e^{2\pi i (t g_0(z) + (1 - t) g_1(z))}, \quad t \in \mathbb{C}.$$

Lemma 33 *Let E denote the set of parameters $t \in \mathbb{C}$ such that the basic normalization Φ_t^* (resp. the basic normal form N_t^*) is convergent. If E is not polar, then $E = \mathbb{C}$.*

Proof. Let E denote the set of parameters $t \in \mathbb{C}$ such that Φ_t^* is convergent and suppose that E is not polar. We have

$$E = \cup_{n \geq 1} E_n,$$

where E_n is the set of $t \in E$ such that the power series $\varphi_t^*(z)$ is convergent and bounded by 1 for $|z| \leq 1/n$. We can thus find $n \geq 1$ such that E_n is not polar. According to Lemma 32, we have

$$\varphi_t^*(z) = \sum_{j+k \geq 1} \varphi_{jk}^*(t) z^j \bar{z}^k$$

where $\varphi_{jk}^*(t)$ depends polynomially on t with degree no larger than $j+k$. The Cauchy inequalities (viewing φ_t^* as a function of two independent variables (z, \bar{z})) yield

$$|\varphi_{jk}^*(t)| \leq n^{j+k}.$$

By the Bernstein-Walsh lemma, we get that if $K \subset \mathbb{C}$ is compact and $j+k \geq 2$, then

$$\max_{t \in K} \|\varphi_{jk}^*(t)\| \leq C^{j+k} n^{j+k}, \quad \text{where } C = \exp \left(\max_{t \in K} V_{E_n}(t) \right).$$

Hence $\varphi_t^*(z)$ is convergent for any $t \in \mathbb{C}$. The argument for the set of parameters t such that N_t^* is convergent is similar.

Lemma 34 *If there exists $t \in \mathbb{C}$ such that Φ_t^* (resp. N_t^*) is divergent, then the set of parameters $t \in \mathbb{C}$ (resp. $t \in \mathbb{R}$) with convergent normalization Φ_t^* (resp. basic normal form N_t^*) has Lebesgue measure zero.*

Proof. It follows from the fact that a polar subset of \mathbb{C} is of Lebesgue measure zero, and the intersection of a polar subset of \mathbb{C} with \mathbb{R} is of Lebesgue measure zero.

Proof of Theorem 31: Let $\mathcal{F} = \mathcal{F}_\lambda$. Suppose that there exists $F_0 \in \mathcal{F}$ the basic normalization of which is divergent. For $n \geq 1$, denote by $E_n \subset \mathcal{F}$ the set of $F \in \mathcal{F}$ which have convergent basic normalization with $\varphi^*(z)$ convergent and bounded by 1 for $|z| \leq 1/n$. It is easy to check that each E_n is a closed set. Now

$$E = \cup_{n \geq 1} E_n$$

is the set of F in \mathcal{F} having a convergent basic normalization.

Let $n \geq 1$. We claim that the set $\mathcal{F} - E_n$ is dense. Otherwise, there exists a map F_1 in the interior of E_n . Consider the subspace

$$V = \{ F_t \mid F_t(z) = \lambda z (1 + (t f_0 + (1-t) f_1)(|z|^2)) e^{2\pi i (t g_0(z) + (1-t) g_1(z))}, \\ t \in \mathbb{C} \text{ (resp. } \mathbb{R}) \}$$

By Lemma 34, the set of parameters t giving rise to a convergent basic normalization has measure zero. But on the other hand it contains a neighborhood of 0, contradiction. Therefore, the set of maps F in \mathcal{F} with divergent basic normalization

$$\mathcal{F} - E = \bigcap_{n \geq 1} (\mathcal{F} - E_n)$$

is a countable intersection of open dense set.

Finally, recall that by Lemma 9, if the basic normalization of a map $F \in \mathcal{F}$ is divergent, then all normalizations of F are divergent.

An analogous argument works for the families $\mathcal{F}_{\lambda, f, \bullet}$ and $\mathcal{F}_{\lambda, \bullet, g}$, by taking $g_0 = g_1$ and $f_0 = f_1$ respectively, which ends the proof of Theorem 31.

Proof of Theorem 30: Theorem 31 gives an alternative: total convergence of the basic normalization or generic divergence of the normalizations. In case (i) it's generic divergence, in view of the existence of divergent conservative examples. In case (ii) too, in view of Corollary 25 (or, more accurately, its analogue where we replace $a|z|^{2d}$ with an arbitrary $f(|z|^2) \neq 0$).

5.5 More questions.

1) Nature of the special normal forms in the conservative case By section 3, polynomial normal forms always exist. As, in the conservative case we know that they correspond to non conservative conjugacies, this leaves open the question of the nature of the special normal forms, namely: is divergence of special normal forms N generic in the conservative case? Recall that we know that the conjugacy itself to the special normal form is in general divergent.

2) Nature of the special normal forms and special conjugacies in case of weak contraction Is divergence of the special normal form, the conjugacy Φ and more generally of any conjugacy Ψ to a normal form generic when ω is not a Brjuno number?

3) What about the role of translated objects? In this case, there are no more invariant objects but only translated objects. Indeed, in the simple case that we are considering, the circle of radius r centered at 0 is radially *translated* by F onto the circle centered at 0 of strictly smaller radius $s = r(1 + f(r^2))$. Let us call $\rho(r) \in \mathbb{R}/\mathbb{T}$ the rotation number of the diffeomorphism g_r of \mathbb{R}/\mathbb{T} defined by the restriction to the circle of radius r of the argument $2\pi g$ of F . The values of r such that $\rho(r) = p/q \in \mathbb{Q}/\mathbb{T}$ define *resonant annuli*. One can show that inside each such annulus there is a curve of translated periodic orbits of rotation number p/q , the translation depending on the orbit⁴

⁴A translated orbit is an orbit whose image under F is obtained by a radial translation by some constant. They exist independently of the hypothesis that F preserves the foliation \mathcal{F}_0 .

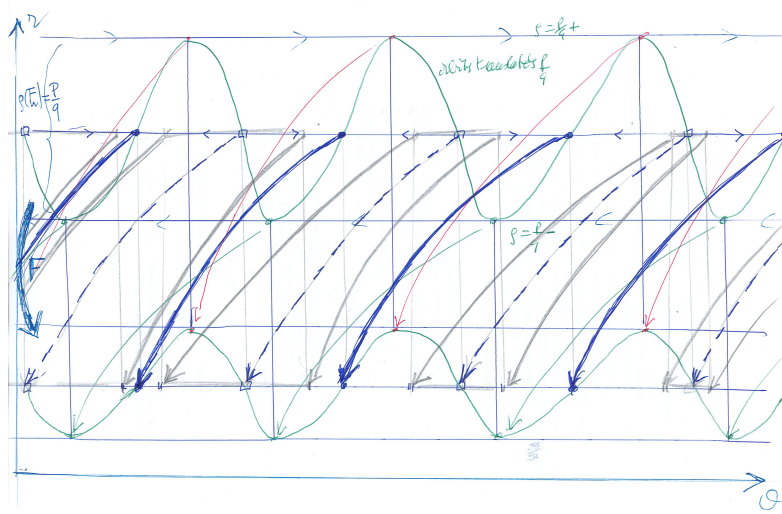


Figure 6 : Resonant zones.

Is the relation between the strength of attraction and the measure, in some system of local coordinates, of the set of translated circles whose rotation number is rational (the resonant zones) relevant to the conjugacy problem? In particular, are the results for $A_{\lambda,a,d}$ and $B_{\lambda,a,d}$ different?

The problem is, of course, that translated objects are not invariant under conjugacy ; in particular, in the case of a strong contraction the existence in some local coordinates of resonant zones, does not prevent analytical conjugacy to a rotation (see section 5.3)!

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