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Simple non-planar periodic solutions of the n-body problem

Alain Chenciner

Astronomie et Systèmes Dynamiques, IMCCE, UMR 8028 du CNRS,
77, avenue Denfert-Rochereau, 75014 Paris, France
et
Département de Mathématiques, Université Paris VII-Denis Diderot
16, rue Clisson, 75013 Paris, France

Abstract. A well-known result – going back to Lagrange in 1772 in the case of 3 bodies – asserts that a homographic solution of the n-body problem in 3-space must live in a fixed plane. Recent progress in the analysis of the "action minimizing" method, reported in [C4], proves the existence of a "simple" non-planar periodic solution for any set of at least 4 masses.

1 - Homographic motions and central configurations.

The only "explicit" solutions of the Newtonian n-body problem in $\mathbb{R}^3$ are the so-called *homographic* (or Kepler-like) solutions: the configuration which the bodies define at each instant changes only by similitudes and each body follows a similar Keplerian orbit. When the common eccentricity of these orbits is equal to 0 (resp. 1), one speaks of a *relative equilibrium* (resp. *homothetic*) solution: each body rotates uniformly on a circle centered on the center of mass of the system (resp. the whole configuration homothetically collapses towards its centre of mass). Only very special configurations, the *central configurations*, may support this kind of motion. Central configurations are very poorly understood theoretically as soon as the number of bodies exceeds 3, but whatever they be, the following result holds:

Theorem 1 -[L] [Pi]. *A homographic motion in $\mathbb{R}^3$ which is not homothetic takes place in a fixed plane.*

Proving that a relative equilibrium motion must take place in a fixed plane is a simple exercise. On the contrary, proving that any homographic motion which is not homothetic must lie in a fixed plane is difficult. For 3 bodies, it is, according to Lagrange himself, one of the main achievements of his great "Essai sur le problème des trois corps" [L], based on the systematic use of mutual distances as coordinates. A proof for any number n of bodies in $\mathbb{R}^3$ is given in [W] (par. 371 to 374) and in [AC], where the n-body problem is studied in a space of arbitrary dimension (in which case the result is that a homographic motion must take place in a fixed subspace of even dimension). On the other hand, any central configuration, planar or not, supports a homothetic motion and

Theorem 2 -[Pa][M]. *For any $n \geq 4$, there exists a central configuration of n bodies in R^3 which is non-planar.*

To explain the idea of the proof, we must recall briefly the equations of the n-body problem in $\mathbb{R}^3$ (for more, see [C1][C3]). A *configuration* is an n-tuple $x = (\vec{r}_1, \vec{r}_2, \dots \vec{r}_n) \in (\mathbb{R}^3)^n$.

A *non-collision* configuration is one such that no two $\vec{r}_i$ coincide. The *configuration space* is the quotient of the set of non-collision configurations by the action of translations (see [AC]). We shall identify it as in [C3] with the set $\hat{\mathcal{X}}$ of non-collision configurations whose center of mass $\vec{r}_G = (\sum_{i=1}^n m_i)^{-1} \sum_{i=1}^n m_i \vec{r}_i$ is at the origin. The closure $\mathcal{X}$ of the configuration space (which includes the collision configurations) is endowed with the "mass scalar product"

$$(\vec{r}_1, \ldots, \vec{r}_n) \cdot (\vec{s}_1, \ldots, \vec{s}_n) = \sum_{i=1}^n m_i \langle \vec{r}_i, \vec{s}_i \rangle,$$

where $\langle , \rangle$ is the standard euclidean scalar product in $\mathbb{R}^3$.
The basic isometry-invariants, defined on the *phase space* (=tangent space $\hat{\mathcal{X}} \times \mathcal{X}$ of the configuration space) whose elements are noted (x, y), are

$$I = x \cdot x,\ J = x \cdot y,\ K = y \cdot y.$$

They are respectively the *moment of inertia* of the configuration with respect to its center of mass, half its time derivative and twice the *kinetic energy* in a galilean frame which fixes the center of mass. The *potential function* (opposite of the potential energy), the *Hamiltonian* (=total energy) and the *Lagrangian* are respectively defined by

$$U = \sum_{i<j} m_i m_j ||\vec{r}_i - \vec{r}_j||^{-1},\ H = \frac{1}{2}K - U,\ L = \frac{1}{2}K + U.$$

The equations of the n-body problem

$$m_i \ddot{\vec{r}}_i(t) = \sum_{j \neq i} m_i m_j \frac{\vec{r}_j(t) - \vec{r}_i(t)}{|\vec{r}_j(t) - \vec{r}_i(t)|^3}, \quad i = 1, \ldots, n,$$

can be written $\ddot{x} = \nabla U(x)$, where the gradient is for the mass scalar product.

The central configurations may be defined (see [AC]) as those which admit *homothetic motions*, that is homographic motions with eccentricity equal to 1. This means that, at each instant, the forces $\nabla U(x)$ must be proportional to the configuration $x = \frac{1}{2}\nabla I(x)$. In other words, the configuration x must be a critical point of the restriction of the potential function U to the "spheres" $I =$ constant. It is more convenient to introduce the *scaled potential* $\tilde{U} = \sqrt{I}U$, homogeneous of degree 0. A central configuration is simply a critical point of $\tilde{U}$ on $\hat{\mathcal{X}}$. As $\tilde{U}$ is positive and tends to $+\infty$ at collisions, it possesses at least a minimum on $\hat{\mathcal{X}}$. What is proved in [Pa] for equal masses and in [M] for the general case is the following theorem, which implies Theorem 2:

Theorem 2'. *Let x_0 be a planar central configuration. There exists a normal variation $z_0 = (z_1, \cdots, z_n)$ (that is for any i, z_i is orthogonal to the plane spanned by the configuration x_0) such that $d^2\tilde{U}(x_0)(z_0, z_0) < 0$. Hence x_0 is never a local minimizer of $\tilde{U}$.*

2 - The variational method and the "italian" symmetry.

The equations of the n-body problem are the Euler-Lagrange equations of the action, which to a path $x(t)$ associates the real number

$$\mathcal{A}_T\big(x(t)\big) = \int_0^T L\big(x(t), \dot{x}(t)\big)dt.$$

This implies that periodic solutions of the n-body problem of a given period T are critical points of the action, considered as a function on the space of all regular enough (say, belonging to the Sobolev space H^1) loops of period T [C3]. Since the Lagrangian is positive, the action is positive and its minimum is attained "at infinity" by bodies moving infinitely slowly infinitely far from each other. To force *coercivity* of the action functional, i.e. to get rid of critical points at infinity, the following constraint on the loops was introduced by the italian school [C-Z][DGM][SeT]. The bodies were forced to occupy after half a period, a position symmetrical of the original one with respect to the center of mass of the system:

$$x(t + T/2) = -x(t).$$

Note that this selects relative equilibrium motions among elliptic homographic motions. To please at the same time analysts and geometers, I proposed in [C3] to call *(anti)symmetry* this constraint.

It happens that the (anti)symmetry constraint not only solves the problem of coercivity but also the much more serious problem of collisions posed by the weakness of the Newton force [C3]. Indeed, if $x(t)$ is a minimizer in this class, and if $t_0 \in [0, T]$ is arbitrary, the restriction of x to the time interval $[t_0, t_0 + T/2]$ (we consider here x as defined on $\mathbb{R}$ and T-periodic) minimizes the action among paths defined on $[t_0, t_0 + T/2]$, which start and end at the same configurations as x does (the *fixed-ends* problem). If this was not the case, the (anti)symmetrization of a path with lesser action would lead to an (anti)symmetric loop with lesser action than x.

Now, the absence of collision is a direct consequence of the following theorem, whose main step was given by Christian Marchal (for a complete proof, see [C4]):

Theorem 3 [Ma1][C4]. *A minimizer of the action in the space of H^1 paths joining two given configurations (eventually with collisions) between times T_1 and T_2 is collision-free on the whole open interval $]T_1, T_2[$.*

This leaves only the possibility of collisions at the ends t_0 or $t_0 + T/2$ of a minimizing path. But this is ruled out by the freedom in the choice of t_0.

We shall see now that the minimizers are not the same in the planar and the spatial problem as soon as $n \geq 4$.

(i) The planar problem. The absence of collision was already known as a consequence of the identification of minimizers. The following theorem is proved in [CD] (corrected in [C3]):

Theorem 4. *For the planar n-body problem, a relative equilibrium solution whose configuration minimizes $\tilde{U}$ is always a minimizer of the action among (anti)symmetric loops; moreover all minimizers are of this form provided there exists only a finite number of similitude classes of n-body central configurations.*

Remark *(not needed for what follows).* If similitude classes of central configurations minimizing $\tilde{U}$ were not isolated, other minimizers among (anti)symmetric loops could possibly exist [C3]. They would necessarily correspond to a uniform motion on a round circle in the metric configuration space $\mathcal{X}$, that is $x(t) = x_1 \cos \frac{2\pi t}{T} + x_2 \sin \frac{2\pi t}{T}$, with two orthogonal configurations x_1, x_2 of the same norm $\sqrt{I}$. This implies the constancy of I and K along the solution. As $\ddot{x}(t) = -\frac{4\pi^2}{T^2} x(t)$, each $x(t)$ has to be a central configuration, that is $\nabla\tilde{U}(x(t)) = 0$, from which the constancy of $\tilde{U}$ follows (it has in fact to be equal to the minimum of this function on $\mathcal{X}$). Hence, if an affine straight line (D) of non-similar central configurations minimizing $\tilde{U}$ did exist in $\mathcal{X}$, a minimizer of the action would be provided by a circle of well-chosen radius in the 2-dimensional vector subspace containing (D). In the corresponding solution, necessarily without collision, each body would be running around an ellipse centered at the center of mass.

(ii) The spatial problem. If $n = 2$ or $n = 3$, it follows from [CD] that a minimizer of the action among (anti)symmetric loops is necessarily a (planar) relative equilibrium motion (with equilateral configuration if $n = 3$). On the contrary, for $n \geq 4$,

Theorem 5. *For the spatial problem, if $n \geq 4$, a minimizer of the action among (anti)-symmetric loops is a collision-free* non-planar *solution.*

Proof. We have already seen that a minimizer is collision-free as a consequence of Theorem 3. The non-planarity assertion is a consequence of Theorem 2' and Theorem 4. Let us choose a relative equilibrium motion $x(t)$ which minimizes the action among (anti)symmetric loops of period T. We identify the plane where the motion takes place with the complex line $\mathbb{C}$, which allows to write $x(t) = x_0 e^{\frac{2\pi i t}{T}}$. We compute the value $d^2\mathcal{A}(x(t))\big(z(t), z(t)\big)$ of the Hessian of the action at $x(t)$ on a variation $z(t)$:

$$d^2 A(x(t))(z(t), z(t)) = \int_0^T \left[|\dot{z}(t)|^2 + d^2 U(x_0)(z(t), z(t))\right] dt.$$

If $z(t) = z_0 \cos \frac{2\pi t}{T}$, where z_0 is normal to x_0 in the sense introduced in Theorem 2' ($x(t) \cdot z(t) = 0$ for any t) and if $I_0 = x_0 \cdot x_0$, one gets

$$d^2\tilde{U}(x_0)(z, z) = I_0^{\frac{1}{2}} d^2 U(x_0)(z, z) + I_0^{-\frac{1}{2}} U(x_0)|z|^2.$$

Finally,

$$d^2 A(x(t))(z(t), z(t)) = \int_0^T \Big(|\dot{z}(t)|^2 - I_0^{-1} U |z(t)|^2 + I_0^{-\frac{1}{2}} d^2\tilde{U}(x_0)(z(t), z(t))\Big)\, dt.$$

But $x(t) = x_0 e^{\frac{2\pi i t}{T}}$ is a solution of $\ddot{x} = \nabla U(x)$, hence $-\frac{4\pi^2}{T^2} x = \nabla U(x)$ and taking the scalar product with x,

$$\frac{4\pi^2}{T^2} I_0 = -x \cdot \nabla U(x) = U(x) = U(x_0).$$

Hence (Kepler's third law)

$$I_0^{-1}U(x_0)=\frac{4\pi^2}{T^2}.$$

As $z(t)=z_0\cos\frac{2\pi t}{T}$, this gives

$$d^2A(x(t))(z(t),z(t))=I_0^{-\frac{1}{2}}\int_0^T d^2\tilde{U}(x_0)(z(t),z(t))=I_0^{-\frac{1}{2}}d^2\tilde{U}(x_0)(z_0,z_0)\int_0^T\cos^2\frac{2\pi t}{T}dt.$$

Now, Theorem 2' asserts that one can always choose z_0 such that $d^2\tilde{U}(x_0)(z_0,z_0)<0$. Hence, a relative equilibrium which minimizes in $\mathbb{R}^2$ ceases being a minimizer in $\mathbb{R}^3$. This ends the proof because other possible planar minimizers would have the same action as a relative equilibrium (thanks to A. Venturelli for this remark).

Comments.

1) I proposed in [C4] to call *generalized Hip-Hops* these minimizers. They somehow replace in $\mathbb{R}^3$ the non-existing relative equilibria of non-planar central configurations minimizing $\tilde{U}$. Recall [AC] that in $\mathbb{R}^4$, such relative equilibria would exist.

2) The actual determination of these non-planar minimizers is certainly a very difficult question. They should consist in well-chosen vertical oscillations of the bodies superimposed to a more or less rigid rotation (with a little periodic variation of the size as in the Hip-Hop) of a planar configuration which minimizes $\tilde{U}$. The idea is that such a minimizer should be a compromise between a planar relative equilibrium motion which does not minimize the action and a deformation in the direction of a non-planar central configuration which does minimize $\tilde{U}$ but cannot rotate in $\mathbb{R}^3$. But even in the case of 4 equal masses, where one may conjecture that a minimizer is necesarily the Hip-Hop with a $\mathbb{Z}/4\mathbb{Z}$-symmetric configuration at each instant [CV], no proof is in sight. This makes the situation a little worse than for the determination of central configurations, where Albouy's symmetry theorem [A1] leads to a complete classification of 4-body central configurations [A2].

3) Among the few assertions about central configurations minimizing $\tilde{U}$, we have:

(i) a configuration which minimizes $\tilde{U}$ among planar configurations cannot be collinear. Indeed, the collinear central configurations are local maxima in the directions normal to the set of collinear configurations [M] ;

(ii) for $2n\geq 6$ equal masses, the regular $2n$-gon is not a local minimizer among planar central configurations [SW] ;

(iii) the limit distribution of a minimizer for $n\to+\infty$ nearly equal masses among planar or spatial configurations is studied in [Li].

4) Previous studies of "simple" non-planar periodic solutions for $n\geq 4$ bodies make essential use of the symmetries attached to the equal-mass case: the generalized Lagrange solutions of [DTW], the "rosettes" and "pelotes" of [H], the Hip-Hop and its generalizations in [CV] [V], all belong to this case. In [MeS], the equal masses surround a much bigger central mass. Reference [G] deals only with the equal-mass n-ion problem: the masses are the same but not all the charges have the same sign. In this case, non-planar relative equilibria do exist. In [V] one proves the existence of a non-planar 4-body *choreography*, that is a solution such that all bodies stay on a given non-planar curve.

3 - Non-planar periodic solutions for 3 bodies.

It is noticed in [CV] that the proof given there of the absence of collision in the Hip-Hop solution for 4 equal masses works in exactly the same way for the isosceles spatial 3-body problem where the central mass is supposed to stick to the vertical line containing the center of mass. It is well known [M] that, if x_0 is the Euler collinear central configuration and z_0 a deformation among isosceles triangles in the direction of the Lagrange central configuration, the Hessian $d^2\tilde{U}(x_0)(z_0, z_0)$ is negative. As the only planar solution among (anti)symmetric loops of a given period in the configuration space of the isosceles problem considered above is a relative equilibrium solution with Euler configuration, one concludes that a minimizer of the action among such loops is a non-planar periodic solution. Note that, using Liapunov theorem, one can prove the existence of non-planar periodic orbits of long periods near the collinear relative equilibrium solution [Ca]; they differ from the action minimizers in that 1) the mutual inclinations are small, 2) the two bodies of equal masses make a high number of revolutions around the symmetry axis.

In the planetary problem, simple enough non-planar solutions were found by Poincaré (his "solutions de la troisième sorte" [P] vol. I, chap. III, par. 48). For 3 general masses, simple non-planar periodic solutions do exist – for example, an almost Keplerian binary with a third body very far away [Ma2] – but I do not know how to characterize the "simplest" ones which should continue those found above in the isosceles case.

Aknowledgements

1) The present work has its origin in my reading the title of the lecture proposed by Shiqing Zhang for the Warwick symposium on "Classical n-body systems" (april 2002). It immediately occured to me that I could give a very simple proof of the result which was annouced in this title (existence of non-planar periodic solutions of the n-body problem) and I gave an idea of the above proof of Theorem 4 in my Warwick lecture. Unfortunately, the proof proposed by Shiqing Zhang was found to be in error. This is because, needing a high degree of temporal symmetry in his solutions to be able to guarantee the absence of collision in a minimizer, he was obliged to consider vertical deformations of the form $z_0 \cos \frac{2\pi m t}{T}$ *with uncontroled* m*. But, as immediately noticed by Susanna Terrracini, this increases the kinetic part in the Hessian and may well make it always positive. Hence it could be that in most of the cases Shiqing Zhang's solutions are indeed planar.*

2) As explained in [CD] and [C3], it is a question of Vitorio Coti Zelati about the case of 4 bodies in R^3 *which led to the Hip-Hop in [CV] : its existence is based on the fact that the minimum of* $\tilde{U}$ *is attained at the regular tetrahedron which cannot have a relative equilibrium motion in* R^3*. Theorem 4 is the general answer to Coti Zelati's question about the minimizers of the action among (anti)symmetric loops of spatial n-body configurations.*

3) Many thanks to Hildeberto Cabral and Eduardo Leandro for a careful reading of the manuscript.

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e.mail address: chencine@bdl.fr

Web address: http://www.bdl.fr/Equipes/ASD/person/chenciner/chenciner.htm