A note by Poincaré

On November 30th 1896, Poincaré published a note entitled "On the periodic solutions and the least action principle" in the "Comptes rendus de l'Académie des Sciences". He proposed to find periodic solutions of the planar Three-Body Problem by minimizing the Lagrangian action among loops in the configuration space which satisfy given constraints (the constraints amount to fixing their homology class). For the Newtonian potential, proportional to the inverse of the distance, the "collision problem" prevented him from realizing his program; hence he replaced it by a "strong force potential" proportional to the inverse of the squared distance.

In the lecture, the nature of the difficulties met by Poincaré is explained and it is shown how, one century later, these have been partially resolved for the Newtonian potential, leading to the discovery of new remarkable families of periodic solutions of the planar or spatial n-body problem.

The Three-Body Problem: a topic dear to Poincaré.

In 1883, Poincaré publishes his first short note dedicated to the Three-Body Problem, entitled *On some particular solutions of the Three-Body Problem.* He applies a generalisation, due to Kronecker, of the intermediate value theorem to the proof of the existence of the three kinds of relative periodic solutions* of the planetary Three-Body Problem. Results will then follow each other, culminating in the three volumes of the *New Methods of Celestial Mechanics* (1892,1893,1899). In this outstanding book, Poincaré develops the memoir *On the Three-Body Problem and the equations of Dynamics*, which had won in 1889 the prize of the king of Sweden **; he founds a great part of the theory of Dynamical Systems (existence and stability of the periodic solutions, integral invariants, the recurrence theorem, homoclinic solutions, ...). Although he uses some global arguments, these works are mostly dedicated to the perturbative theory, planetary or lunar, in which one of the masses dominates the other two, or even to the "restricted problem" in which one of the masses vanishes. The search for periodic solutions plays an important part: as early as in 1884, in the conclusion of the paper in the *Bulletin astronomique* entitled *On some particular solutions of the Three-Body Problem* which expands the 1883 note, he explains the importance of periodic solutions as "intermediate orbits": an arbitrary solution will stay close to such a solution during a long time if it corresponds to close enough initial conditions. This assertion is made more precise in 1892 in the famous conclusion of section 36 in the first volume of the *New Methods of Celestial Mechanics*:

> *"There is even more: here is a fact that I could not prove rigorously, but which nevertheless seems very likely to me.*

* i.e. modulo rotation or, what amounts to the same, in a rotating frame.

** see the book by June Barrow-Greene "Poincaré and the Three Body Problem", American Mathematical Society and London Mathematical Society, 1997.

Given equations of the form defined in $n^0 13$ *and an arbitrary solution of these equations, one can always find a periodic solution (with a period which,admitedly, may be very long), such that the difference between the two solutions be arbitrarily small. In fact, what makes these solutions so precious to us, is that they are, so to say, the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible."*

The principle of least action: a great principle of physics.

"Now, what does the principle of least action tell us ? It teaches us that in order to move from the initial situation it occupies at time t_0 *to the final situation it occupies at time* t_1*, the system must follow a path such that, in the time interval from* t_0 *to* t_1 *the mean value of the "action" (i.e. of the difference between the two energies* T *and* U*) be as small as possible. The first of these two principles [energy conservation] is indeed a consequence of the second. If one knows the two functions* T *and* U*, this principle is sufficient to determine the equations of motion." (Science and Hypothesis, chapter XII, 1902.)*

As Poincaré just said, each solution $x(t) = \big(\vec{r}_1(t), \vec{r}_2(t), \vec{r}_3(t)\big)$, $t \in [t_0, t_1]$ of the Three-Body Problem , and more generally each solution of a problem in conservative mechanics, is an extremum of the Lagrangian action $\int L(x(t), \dot{x}(t))dt$ (in this formula, the *Lagrangian* is the difference $L(x, \dot{x}) = T(\dot{x}) - U(x)$ between kinetic and potential energies). *Extremum* and not minimum as we are reminded of by this delightful sentence of the *New Methods*:

"Until now, when I said, this integral is minimum, *I used an abridged but incorrect way of speaking, which of course could not fool anybody; I wanted to say,* the first variation of this integral vanishes; *this condition is necessary for a minimum, but it is not sufficient." (New Methods of Celestial Mechanics, volume III, chapter XXIX, n^o 341, 1899.)*

In *The value of Science*, Poincaré puts this principle together with the great conservation principles (energy, mass, action-reaction), the principle of degradation of energy and the relativity principle. Due to its global character it may appear at first sight nearer to theology than to physics. I do not resist quoting some sentences from chapter VIII of *Science and Hypothesis* which show up to what point Poincaré was as much a physicist as a mathematician:

"The very statement of the principle of least action has something shocking to the mind. To go from one point to another one, a material molecule, taken away from the action of any force, but constrained to move on a surface, will follow the geodesic line, i.e. the shortest path. It seems that this molecule knows the point where one wants it to go, that it anticipates the time needed to reach it along such or such path, and then chooses the most convenient path. In a sense, the statement presents this molecule as a free animated being. It is clear that it would

be better to replace it by a less shocking statement where, as philosophers would say, the final causes would not appear to replace the efficient ones."

... and echoing it, Feynman's questioning in [Fe1] *(volume I, Chap. 26, par. 5: A more precise statement of Fermat's principle)* with this time the answer given by quantum electrodynamics, that is the *principle of stationary phase* (see also the superb *Light and Matter* [Fe2] by the same author):

"Instead of saying it is a causal thing, that when we do one thing, something else happens, and so on, it says this: we set up the situation, and light decides which is the shortest time, or the extreme one, and chooses that path. But what does it do, how does it find out ? Does it smell the nearby paths, and check them against each other ? The answer is, yes, it does in a way."

The C.R.A.S. note of November 30th 1896.

This note is somewhat exotic with respect to the main stream of researches of Poincaré on the Three-Body Problem. Dedicated to the search for global, non perturbative, solutions, it takes the principle of least action in its litteral meaning: one looks for *minima* and not only for extrema of the action among paths in the configuration space which satisfy given constraints ! More precisely, Poincaré proposes to find relative periodic solutions of the planar Three-Body Problem (with arbitrary masses) with the following property: after time T (the period) the first side of the triangle defined by the three bodies has turned by some total angle θ, the second one has turned by an angle $\theta + 2k\pi$ and the third by an angle $\theta + 2l\pi$, where k and l are signed integers. For this, he minimizes the Lagrangian action among all paths in the configuration space with given period T and the said behaviour. With an appropriate choice of k and l, he obtains an infinity of solutions, most of which are new, *not for the Newtonian force, proportional to the inverse of the squared distance, but for a "strong force" proportional to the inverse of the cube of the distance.* Note that fixing the period T is harmless because the homogeneity of the potential implies the existence of a scaling symmetry: if $x(t) = \big(\vec{r}_1(t), \vec{r}_2(t), \vec{r}_3(t)\big)$ is a solution of the Three-Body Problem with a force in $1/r^{\alpha+1}$, the same is true of $\lambda^\beta x(\lambda t) = \big(\lambda^\beta \vec{r}_1(\lambda t), \lambda^\beta \vec{r}_2(\lambda t), \lambda^\beta \vec{r}_3(\lambda t)\big)$, where $\beta = -2/(\alpha + 2)$, whatever be the positive real number λ. But if the period of x is T, the period of x_λ is $\frac{1}{\lambda}T$.

To impose constraints was necessary of course : the unconstrained minimum is trivially realized "at infinity" by motionless bodies infinitely remote one from the other. To fix k and l non zero forbids that at some time the bodies be too far from each other because this would force the path to be very long, hence the kinetic part of the action would be very large without compensation from the potential part. This makes the minimization problem *coercive.* Another advantage of such a constraint is to garantee that, if $(k, l) \neq (0, 0)$, the corresponding solutions will be “non trivial”; in particular, they will be different from the only explicit solutions, the *homographic* ones (curiously, it seems that Poincaré was never interested in these homographic solutions).

Figure 1 (A relative loop of Hill type: $k = -1, l = 0$
and a homographic solution: $k = l = 0$)

What Poincaré does amounts to recognizing that a triangle has a“shape”. This is a fundamental distinction between the three (and more)-body problem and the two-body problem: a segment has no shape, just a size. Fixing k and l is indeed equivalent to fixing the *homology class* of the loops defined in the *space of oriented triangles* by the paths among which one minimizes the action. By *space of oriented triangles*, I mean here the configuration space of the planar Three-Body Problem “reduced” by the oriented isometries of the plane. This space is obtained from $(\mathbb{R}^2)^3$ deprived from three four-dimensional collision subspaces (triples $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ of distinct points in the plane) by two successive quotients: the first, by the translations, which can be realized for example by the choice of *Jacobi coordinates* ($\vec{r}_2 - \vec{r}_1$ and $\vec{r}_3 - \frac{1}{2}(\vec{r}_1 + r_2)$), results in $\mathbb{R}^4$ deprived from three planes; the second, by the diagonal action of the rotations, can be realized by the Hopf map from $\mathbb{R}^4 \equiv \mathbb{C}^2$ to $\mathbb{R}\times\mathbb{C}$: $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\bar{z}_1 z_2)$. One obtains $\mathbb{R}^3$ deprived from three half-lines. The homology (or the homotopy) of this space is the same as the one of the sphere minus three points, the set of oriented triangles with “fixed size” (and with distinct vertices, i.e. without collision). Hence, the first homology group of the space of oriented triangles is isomorphic to $\mathbb{Z}^2$, each component being represented by the algebraic number of turns accomplished in one period by two sides of the triangle with respect to the third.
Remarks. If one is interested in absolute periodic solutions (Poincaré is not), this homology becomes $\mathbb{Z}^3$ and it is represented by the algebraic number of turns accomplished in one period by the three sides of the triangle. Moreover, Poincaré contrains the homology but he could well have chosen to constrain the homotopy, i.e. the type of the braid described by the three bodies in space-time: he had invented the *fundamental group* in 1895; this group, isomorphic to the free group on two generators $\mathbb{Z} * \mathbb{Z}$, is indeed much richer. Finally, if one considers the Three-Body Problem in space, the notion of orientation of a triangle disappears and with it all this topology: the sphere minus three points is replaced by a disk deprived of three points on its boundary, that is by

Figure 2 (The space of oriented triangles: the loops of figure 1
correspond respectively to the loop γ and the point L_+)

a contractible space. Notice that if the number of bodies is greater than three, the "space of forms" becomes singular (in the case of the planar problem, it is the cone over a complex projective space).

Poincaré's boldness is clear in this note:

1) first he admits without discussion the existence of a minimum; but, as was shown by later history, this is not without risk. In fact, it is in 1925 that this existence will be rigorously proved by par Leonida Tonelli to be a consequence of coercivity. At this occasion, Tonelli discovers the key role of the lower semi-continuity of the action functional (a well-known property of the length: it can suddenly decrease at the limit – as in the exemple of a broken line which converges uniformly to a straight line – but it cannot suddenly increase at the limit !);

2) on the other hand, he discovers the true problem, that is collisions: an elementary computation* shows that when two bodies $\vec{r}_i(t)$ and $\vec{r}_j(t)$, which interact according to Newton's law, collide at time t_0, they satisfy estimates of the form: $|\vec{r}_i(t)-\vec{r}_j(t)| \sim \alpha|t-t_0|^{\frac{2}{3}}$ and $|\dot{\vec{r}}_i(t)-\dot{\vec{r}}_j(t)| \sim \beta|t-t_0|^{-\frac{1}{3}}$. Morever, about fifteen years later, Sundman will show that the same estimates hold for collisions of an arbitrary number of bodies in a space of arbitrary dimension. But these estimates imply the convergence of the action integral in the neighborhood of collisions. Hence a minimizing path could a priori consist in the concatenation of a – possibly infinite – number of segments of solutions linked one to the next through a collision;

3) Finally, not being able to conclude in the Newtonian case, he does not hesitate "cheating" by replacing the Newtonian force in $1/r^2$ by a "strong force" in $1/r^3$ for which the action integral diverges at collisions. Some eighty years will go by before this direction of research is resumed.

Here we have a beautiful illustration of the way Poincaré works: he clears the way, he goes on and he leaves aside questions to which he will come back ... if time allows.

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* even more elementary if one considers the Kepler problem (one fixed centre) with zero energy on a line.

Beyond Poincaré: first results for the Newtonian potential.

The first results on minimization with fixed homology for the Newtonian potential are obtained by William Gordon [G] in 1977 in ignorance of Poincaré's note. They concern absolute (i.e. in the fixed frame) periodic solutions of the Kepler problem (i.e. the problem of one fixed center, to which one can reduce the two-body problem) in the plane. They were generalized in 2001 by Andrea Venturelli [V] to the planar Three-Body Problem. The statements are parallel: the first homology group of the configuration space is isomorphic to $\mathbb{Z}$ for Gordon, and to $\mathbb{Z}^3$ for Venturelli. In both cases, one notes that, as Poincaré feared, collisions may appear for the Newtonian potential when the action is minimized under some homological constraint: if imposing to the homology the value ± 1 in Gordon's case (resp. $\pm(1,1,1)$ in Venturelli's case), leads to the elliptical solutions (resp. the equilateral *homographical* solutions) of the given period, every homology class different from 0 and ± 1 in the first case (resp. from $\pm(1,1,1)$ or from a class with one zero component in the second one), admits collision orbits as the sole minima (*homothetic* collapse of an equilateral triangle on its centre of mass in the second case).

Venturelli's work says nothing on the homology classes with one zero component; in particular, even in a case apparently as simple as the class $(1,0,1)$, the minima have not yet been identified in spite of the fact that a serious candidate is known, a solution found numerically in the seventies by Roger Broucke [B] and (independantly) Michel Hénon [He]. On the other hand, Venturelli's proof is based on the decomposition of the three-body action as the sum of three two-body actions and hence cannot be generalized to a larger number of bodies.

Finally, as we already noted, one can try to fix homotopy instead of homology. This amounts to fixing the braid type which the solution describes in space-time. This is what Cris Moore proposed to do in 1993 [Mo], also in ignorance of Poincaré's note. He found a great number of periodic solutions by numerically minimizing the action. Among those was the *Eight*, which we evoke in the next paragraph. His achievement was made possible by the strong symmetry constraints he was imposing to the paths from which he was starting the minimization process and by the fat that these symmetries were preserved by the process. In the absence of such choices, the minima should have presented collisions most of the time.

Beyond Poincaré: symmetry constraints.

First introduced by the italian school ([D-G-M], [CZ]) in order to insure the coercivity of the action functional, symmetry constraints are the key of he recent successes in the application of the variational method to the search for periodic solutions of the Newtonian n-body problem. Already in the Kepler problem, imposing the *Italian symmetry* $x(t+T/2) = -x(t)$ selects the circle among ellipses and hence excludes collision orbits. For three bodies in the plane or in space, the minimizing solutions for this symmetry are the equilateral *relative equilibria* but I showed in 2002 [C1] that, as soon as the number of bodies is at least four, one obtains non trivial (because *non planar*) solutions of the spatial problem. The first example of these *generalized Hip-Hops* (four bodies with the same mass) had been obtained with Venturelli in 2000 [C-V]. In a way these solutions are the simplest non planar solutions of the n-body problem.

The *Eight* [C-M], whose existence was proved with Richard Montgomery at the end of 1999, is another example of minimization under symmetry constraints. Here the symmetry group is that of the space of oriented triangles which was described in the latter paragraph, that is the dihedral group D_6, with 12 elements. The equilateral relative equilibrium and the Eight are the first examples of a family of periodic solutions of the equal mass n-body problem in which the bodies chase each other on one and the same closed curve with constant time shift. Admiring their evolutions on the screen of his computer, Carles Simó, their main discoverer[S1], named them *choreographies.* Animations can be contemplated on his website http://www.maia.ub.es/dsg/nbody.html

That the minimization under symmetry constraints often leads to collision-free solutions is explained by the absence of collision in the minimization with fixed ends *(Marchal's theorem)* [Ma]: indeed, one gets back to this question by restricting a symmetric loop to a time interval which is a fundamental domain of the action of the symmetry group on the time circle. For an overview, see [C3]; for more details, see my lectures at ICM (Beijing 2002) [C1], at ICMP (Lisbon 2003) [C2] and the references given there.

Back to Poincaré: the strong force potential and the Jacobi-Maupertuis metric.

The $1/r^2$ potential, introduced by Poincaré in order to avoid the collision problem, plays a very special role among potentials of the form $1/r^\alpha$. It is the only one for which the scaling symmetry originating from the homogeneity is symplectic, and this implies the existence of an additional first integral of the n-body problem. The Lagrange-Jacobi identity, also a consequence of the homogeneity of the potential, reads $\ddot{I} = 4H$ (I is the *moment of inertia* of the configuration with respect to its center of mass and H is the energy, normalized to zero when the bodies are at rest at infinity). In particular, a bounded collision-free solution – e.g. a periodic solution, a relative periodic solution or more generally a quasi-periodic solution – must satisfy I=constant and $H = 0$. This implies a reduction of the Jacobi-Maupertuis metric (the form given by Jacobi to the Maupertuis principle, i.e. the least action principle with fixed energy) to the sphere of oriented triangles of given size (or inertia). Montgomery [M2] recently showed that, in case the three masses are equal, the curvature of the corresponding metric on the sphere minus three points is everywhere negative, except at the Lagrange points where it vanishes. In particular he deduces that each homotopy class for which the minimum length of a loop is not attained at infinity* contains exactly *one* relative periodic solution. This implies the unicity of the Eight for this potential, while this unicity, even if very likely, is not proved for the Newtonian potential.

Figure 3 (The space of oriented triangles with the Jacobi metric)

The $1/r^2$ potential contains more surprises: the works of Fujiwara *et al.* [Fu] have revealed a surprizing *triangle geometry* associated to the correpsonding Eight solution; one can admire beautiful animations at the address
http://www.clas.kitasato-u.ac.jp/~fujiwara/nBody/IeqConstLeq0/centers
GIF.html
At each instant, the tangents to the curve at the positions of the three bodies meet at a common point and the same is true of the three normals. The intersection of the tangents, which follows from the vanishing of the angular momentum, is still true for the Newtonian potential; it is connected to the existence of what Aurel Wintner calls a *center of force* for the Three-Body Problem with Newtonian attraction: at each instant the forces applied to the three bodies meet at a common point (Hargrave 1858, Schiaparelli 1864); as the angular momentum vanishes, one can replace the accelerations by the velocities in the computation. On the contrary, the intersection of the normals, which follows from the constancy of the moment of inertia I, holds only for the $1/r^2$ potential.

Back to Poincaré: the question of stability.

The instability of the periodic solutions which locally minimize the action is announced by Poincaré in a note (C.R.AS., vol. 124, pages 713-716) untitled *Periodic solutions and least action principle.* The details of the proof are given in 1899 in the Chapter XXIX of volume III (*Various forms of the principle of least action*) of the New Methods. Poincaré distinguishes to kinds of unstable solutions according to whether or not the neighboring solutions intersect the given solution. He announces that only the first case happens for minima and that this property is characteristic.

> *"To summarize, in order that a closed curve corresponds to an action smaller than the one of any closed curve which is infinitely close, it is*

* These classes are identified in [M1].

necessary and sufficient that this closed curve corresponds to a periodic solution which is unstable of the first category." (New Methods, end of paragraph 358.)

This assertion holds only in Poincaré's setting, that is for a mechanical system with *two* degrees of freedom. Indeed, examples found by Marie-Claude Arnaud in 1998 [A] show that in the higher dimensions a locally action minimizing periodic solution may possess only "two directions of instability " transverse to the flow in its energy level.

Symmetry constraints modify the stability question. It came as a surprize when Simó [S2] numerically showed the stability of the Eight in the plane but with other symmetry constraints, action minimizers appear to be mostly unstable.

Conclusion. Few directions of investigation of the Three-Body Problem have been left aside by Poincaré: this note is a good example as the methods proposed there were rediscovered independantly only much later on, with the works of Gordon and those of the Italian school. Poincaré recognizes the main obstacle – that collision orbits have finite action — and he was only in need of more time. In section XVIII "Three-Body Problem; qualitative properties" of his analysis of his own scientific works, he writes:

I came back to these periodic solutions and I studied them in detail. The methods I used to prove their existence are very simple and can be reduced to the calculus of limits.

But one can arrive at this proof by a completely different path, which it will be often useful to follow, but from which I did not yet draw all the conclusions. Let us suppose for example that one looks for the geodesics of an indefinite surface which presents the same general shape as a one-sheeted hyperboloid. One can be sure that there exists a closed geodesic (corresponding to a periodic solution) because among all the closed curves which can be drawn on the surface and which go around, one must be shorter than the others.

The same principles may be applied to various problems of Mechanics thanks to the principle of least action. This principle can be used under the form it was given by Hamilton, or under the one given by Jacobi. I have only sketched this method from which there is still probably much to get.

In fact, Poincaré came back at least once to this method. He did that, not directly about the Three-Body Problem, but in a problem both simpler, because there is no collision problem, and harder, because it concerns the sphere, which possesses no "hole" around which one can go. In the 1905 paper *On the geodesic lines of convex surfaces, Transactions AMS 6, p. 237-274*, he studies the problem of periodic geodesics on convex surfaces as a caricature of the corresponding problem in the Planar Circular Restricted Three-Body

Problem. Having probably in mind the periodic solutions of the planetary or lunar type, in particular the *Hill solutions* of the lunar problem, and having maybe forgotten his 1896 note, he writes in the introduction that

> *"...it is not to the geodesics of the surfaces with opposite curvatures that the trajectories of the Three-Body Problem may be compared; it is on the contrary to the goedesics of convex surfaces.*
> *Hence I took up studying the geodesics of convex surfaces; unfortunately, the problem is much harder than the one solved by Mr. Hadamard [the case of surfaces with opposite curvatures]. I had to be content with some partial results, essentially on closed geodesics, which play here the role of the periodic solutions of the Three-Body Problem".*

These "partial" results are nevertheless impressive: applying in a daring manner the continuation method, Poincaré obtains the existence of at least one closed geodesic which is *embedded* (i.e. without self-intersection) on any convex surface in $\mathbb{R}^3$ endowed with the metric induced by the euclidean one. At the end of the paper, he sketches a second proof of this existence in a very "physical" way with fluids and ribbons. He then uses this new proof to discuss stability: as both homology and homotopy are trivial and hence cannot be used as constraints for the minimization of the length (i.e. of the action), he introduces the *Gauss-Bonnet constraint*: one minimizes the length among all embedded closed curves which part the surface into two pieces on each of which the integral of the curvature is the same (this is exactly the assertion of the Gauss-Bonnet theorem for a geodesic). A complete (and nice) proof along the lines suggested by Poincaré was only given in 1994 by Joel Hass and Frank Morgan [H-M].
René Garnier, one of the editors of volume VI of the complete works (the one which contains this paper), recalls in his commentary the spectacular advances accomplished in the Calculus of Variations by Morse, Birkhoff, Lusternik, Schnirelmann. He writes:

> *"The researches of all these authors constitute without doubt one of the most important accomplishments of the modern technique in the Calcul of Variations; but acknowledging that, one shoud not forget that, according to M. Morse's word, H. Poincaré was one of the first geometers who have anticipated the existence of a macro-analysis and, without doubt, the one who contributed the most efficiently to the constitution of such a subject."*

I shall close with a sentence of Hadamard. In my opinion, it describes very accurately the works of Poincaré. This sentence is quoted from the text – collected by E. Terradas and B. Bassegoda – of a conference [H] given at the "Institute of catalan studies":

> *"Faced with a discovery of Hermite, one is inclined to say:*
> *– Admirable to see how a human being could arrive at such an extraordinary way of thinking !*

But, reading a memoir of Poincaré, one says:
– How is it possible that one has not arrived much earlier to things so deeply natural and logical ?."

Thanks to Robert McKay who taught me the existence of Poincaré's note during a conference in Rio de Janeiro where I was presenting the Eight; thanks to Anne Robadey for clarifications on the 1905 paper of de Poincaré; thanks to Alain Albouy for a discussion on the center of forces and to Jacques Laskar for the Schiaparelli reference. Thanks at last to Sebastià Xambó and Amadeu Delhsams for having invited me to speak on Poincaré in their beautiful city of Barcelona, and to Tere Seara for having helped me to do it, not, unfortunately, in catalan, but at least in castillan.

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