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Are Nonsymmetric Balanced Configurations of Four Equal Masses Virtual or Real?

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Abstract—Balanced configurations of N point masses are the configurations which, in a Euclidean space of high enough dimension, i. e., up to $2(N-1)$, admit a relative equilibrium motion under the Newtonian (or similar) attraction. Central configurations are balanced and it has been proved by Alain Albouy that central configurations of four equal masses necessarily possess a symmetry axis, from which followed a proof that the number of such configurations up to similarity is finite and explicitly describable. It is known that balanced configurations of three equal masses are exactly the isosceles triangles, but it is not known whether balanced configurations of four equal masses must have some symmetry. As balanced configurations come in families, it makes sense to look for possible branches of nonsymmetric balanced configurations bifurcating from the subset of symmetric ones. In the simpler case of a logarithmic potential, the subset of symmetric balanced configurations of four equal masses is easy to describe as well as the bifurcation locus, but there is a grain of salt: expressed in terms of the squared mutual distances, this locus lies [redacted] outside the set of true configurations (i. e., generalizations of triangular inequalities are not satisfied) and hence could lead [redacted] only to the bifurcation of a branch of virtual nonsymmetric balanced configurations. [redacted] This raises the question of the title, a question which, thanks to the explicit description given here, should be solvable by computer experts even in the Newtonian case: [redacted] is there a possibility for a bifurcating branch of virtual nonsymmetric balanced configurations to come back to the domain of true configurations.

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To the memory of Vladimir Arnold, with admiration

1. INTRODUCTION

We first recall how to characterize up to an isometry a configuration of N points $\vec{r}_1, \ldots, \vec{r}_N$ in a Euclidean space E by their mutual distances $r_{ij} = \|\vec{r}_i - \vec{r}_j\|$ (see [3–5]).

1.1. Real and Virtual N-body Configurations

Theorem (Borchart 1866). *The $N(N-1)/2$ real numbers s_{ij}, $i<j$, are the squared mutual distances r_{ij}^2 of N points $\vec{r}_i$ in a Euclidean space E (whose dimension is not imposed) if and only if the quadratic form $\beta = \sum_{i,j}\left(-\frac{1}{2}s_{ij}\right)\xi_i\xi_j$ on $\mathcal{D}^* = \{(\xi_1, \cdots, \xi_N) \in \mathbb{R}^N, \sum_{i=1}^N \xi_i = 0\}$ is nonnegative. Moreover, one can choose E of dimension k if and only if the rank of β is less than or equal to k.*

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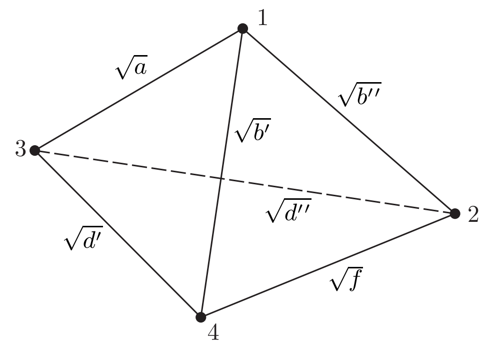


Fig. 1.

If $N = 4$, we shall use the same notations as in [7], that is,

$$s_{13} = a,\ s_{14} = b',\ s_{12} = b'',\ s_{34} = d',\ s_{23} = d'',\ s_{24} = f.$$

A convenient basis of $\mathcal{D}^*$, orthonormal for the standard Euclidean structure of $\mathbb{R}^4$, is

$$u_1 = \frac{1}{2}(1,-1,1-1),\ u_2 = \frac{1}{\sqrt{2}}(1,0,-1,0),\ u_3 = \frac{1}{\sqrt{2}}(0,1,0,-1);$$

Borchart's criterium then becomes the positivity of the symmetric matrix

$$B = \begin{pmatrix} u & z & y \\ z & v & x \\ y & x & w \end{pmatrix}, \quad \text{where}$$

$$\begin{cases} u = \frac{1}{4}(b'' + d'' + b' + d' - a - f), \quad v = \frac{1}{2}a, \quad w = \frac{1}{2}f, \\ x = \frac{1}{4}(d'' - d' + b' - b''), \\ y = \frac{1}{4\sqrt{2}}(b' - b'' + d' - d''), \quad z = \frac{1}{4\sqrt{2}}(b'' - d'' + b' - d'). \end{cases}$$

Definition. *If the condition of the theorem is not satisfied, we shall say that the s_{ij} are the squared mutual distances of a virtual N-body configuration.*

Remark. In case the vectors $\vec{r}_i$, $i = 1, \ldots, 4$, represent unit point masses, the matrix B contains the same spectral information as the classical inertia matrix of the configuration (see [5] where it is called a *subjective inertia matrix*), in particular, its trace is the moment of inertia of the configuration with respect to its center of mass; more generally (see [4]), if $vol_{i_1 \ldots i_k}$ is the volume of the $(k-1)$-dimensional parallelotope generated in E by the vectors $\vec{r}_2 - \vec{r}_1, \vec{r}_3 - \vec{r}_1, \ldots, \vec{r}_k - \vec{r}_1$, that is, $(k-1)!$ times the volume of the simplex defined by the points $\vec{r}_1, \ldots, \vec{r}_k$, one has

$$\det(\mathcal{I}d_{\mathcal{D}^*} - \lambda B) = 1 - \eta_1\lambda + \eta_2\lambda^2 - \eta_3\lambda^3, \text{ where } \eta_{k-1} = \frac{1}{4}\sum_{i_1 < \ldots < i_k} vol^2_{i_1 \ldots i_k}.$$

1.2. Balanced Configurations

Given a law of attraction defined by a homogeneous potential function $U = \sum_{1 \leqslant i < j \leqslant N} m_i m_j \Phi(r_{ij}^2)$, depending only on the mutual distances r_{ij} between N point masses m_i, *balanced configurations* were defined in [4] as the ones for which there exists a choice of initial velocities giving rise to a relative equilibrium motion of the configuration in some Euclidean space whose dimension is not specified (which means up to $2(N-1)$).

Using the notations introduced in the former paragraph, the equations of balanced configurations of four point masses take the following form, where $P_{123} - P_{124} + P_{134} - P_{234} \equiv 0$ (see [7]).

$$(P_{123}) \begin{cases} m_1(d'' - a - b'')\big[\varphi(a) - \varphi(b'')\big] - m_4(d'' + b')\big[\varphi(f) - \varphi(d')\big] \\ +m_2(a - b'' - d'')\big[\varphi(b'') - \varphi(d'')\big] - m_4(a + f)\big[\varphi(d') - \varphi(b')\big] \\ +m_3(b'' - d'' - a)\big[\varphi(d'') - \varphi(a)\big] - m_4(b'' + d')\big[\varphi(b') - \varphi(f)\big] = 0, \end{cases}$$

$$(P_{124}) \begin{cases} m_1(f - b' - b'')\big[\varphi(b') - \varphi(b'')\big] - m_3(f + a)\big[\varphi(d'') - \varphi(d')\big] \\ +m_2(b' - b'' - f)\big[\varphi(b'') - \varphi(f)\big] - m_3(b' + d'')\big[\varphi(d') - \varphi(a)\big] \\ +m_4(b'' - f - b')\big[\varphi(f) - \varphi(b')\big] - m_3(b'' + d')\big[\varphi(a) - \varphi(d'')\big] = 0, \end{cases}$$

$$(P_{134}) \begin{cases} m_1(d' - b' - a)\big[\varphi(b') - \varphi(a)\big] - m_2(d' + b'')\big[\varphi(d'') - \varphi(f)\big] \\ +m_3(b' - a - d')\big[\varphi(a) - \varphi(d')\big] - m_2(b' + d'')\big[\varphi(f) - \varphi(b'')\big] \\ +m_4(a - d' - b')\big[\varphi(d') - \varphi(b')\big] - m_2(a + f)\big[\varphi(b'') - \varphi(d'')\big] = 0, \end{cases}$$

$$(P_{234}) \begin{cases} m_2(d' - f - d'')\big[\varphi(f) - \varphi(d'')\big] - m_1(d' + b'')\big[\varphi(a) - \varphi(b')\big] \\ +m_3(f - d'' - d')\big[\varphi(d'') - \varphi(d')\big] - m_1(f + a)\big[\varphi(b') - \varphi(b'')\big] \\ +m_4(d'' - d' - f)\big[\varphi(d') - \varphi(f)\big] - m_1(d'' + b')\big[\varphi(b'') - \varphi(a)\big] = 0. \end{cases}$$

In these equations, the function $\varphi(s)$ of one real variable s is the derivative of the function $\Phi(s)$ defining the potential. It is convenient to use the following normalizations for the potential:

$$\varphi(x) = -x^{-1} \text{ (Log)},\ \varphi(x) = -x^{-3/2} \text{ (Newtonian)},\ \varphi(x) = -x^{-1} \text{ (Strong)}.$$

Remark. These configurations also admit a variational characterization (see [4]), which in the case of four equal masses becomes the following: they are the critical points of the restriction of the potential function, considered as a function of the subjective inertia matrix B, to the isospectral manifold, that is, to the set of (isometry classes of) configurations with a given moment of inertia with respect to the center of mass, a given sum of the squared areas of the faces and a given volume. This generalizes the definition of central configurations as critical points of the potential in restriction to the (isometry classes of) configurations with a given moment of inertia with respect to the center of mass.

2. BALANCED CONFIGURATIONS WITH A SYMMETRY

The precise result of [1] was that planar central configurations of four equal masses necessarily possess a symmetry axis containing two of the bodies. In contrast, spatial balanced configurations may either have a symmetry plane containing two of the bodies or a symmetry axis. With the above notations, both cases can be reduced, *after possible renumbering of the bodies*, to the equality $b' + d' = b'' + d''$, the first case corresponding to $b' = b''$, $d' = d''$ and the second to $b' = d''$, $d' = b''$.

2.1. Equations in the Case of a Symmetry Plane

Let x_S be a four-body configuration with a symmetry plane and squared mutual distances

$$a = r_{13}^2,\ b = r_{12}^2 = r_{14}^2,\ d = r_{23}^2 = r_{34}^2,\ f = r_{24}^2.$$

Note that this excludes the colinear configurations.

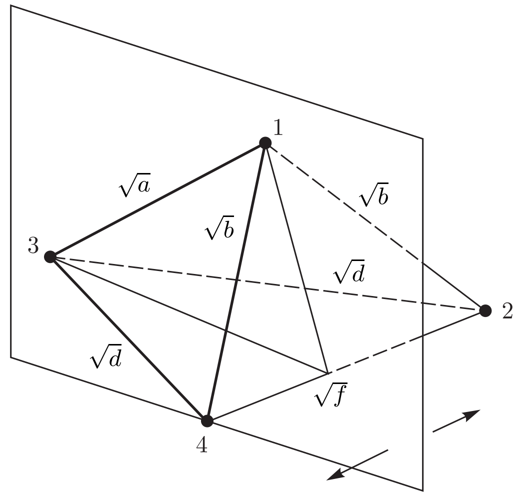


Fig. 2. Symmetry with respect to a plane.

The four equations above reduce to a single one: P_{234} and P_{124} are identically satisfied, while $\frac{1}{m}P_{123} = -\frac{1}{m}P_{134} = 0$ becomes

$$-\begin{pmatrix} 1 & 1 & 1 \\ (d-a-b) & (a-b-d) & (b-d-a) \\ \varphi(d) & \varphi(a) & \varphi(b) \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ d+b) & a+f & b+d \\ \varphi(b) & \varphi(f) & \varphi(d) \end{pmatrix} = 0,$$

that is,

$$\big(\varphi(b)-\varphi(d)\big)(3a+f-2b-2d) + \big(2\varphi(a)-\varphi(b)-\varphi(d)\big)(d-b) = 0. \tag{2.1}$$

This defines a subset of $\mathbb{R}^4_+$ (see [6]), parts of which do not depend on φ:

The rhombus configurations: the equation of symmetric balanced configurations is trivially satisfied (both terms equal 0) if $b = d$; The projection of the configuration on a plane parallel to the sides 13 and 24 is then a rhombus and the configuration has a $\mathbb{Z}/4\mathbb{Z}$ symmetry. These configurations were more generally studied in [7].

The equilateral configurations: the equation is also trivially satisfied if

$$a = d,\ b = f, \quad \text{or} \quad a = b,\ d = f.$$

The unique spatial central configuration: whatever the masses are, there is a unique truly $(N-1)$-dimensional central configuration of N bodies, the regular simplex. This is because the mutual distances are independent coordinates for the space of isometry classes of such configurations.

2.2. Equations in the Case of a Symmetry Axis

We use the following notations for the squared mutual distances:

$$r_{13}^2 = a,\ r_{14}^2 = r_{23}^2 = b,\ r_{12}^2 = r_{34}^2 = d,\ r_{24}^2 = f.$$

Note that a and f now play similar roles. With the above notations, each of the four equations P_{ijk} of balanced configurations reduces to the single equation

$$2(b-d)\big(\varphi(a)-\varphi(f)\big) + (f-a)\big(\varphi(b)-\varphi(d)\big) = 0, \tag{2.2}$$

that is,

$$(b-d)(a-f)\big(2\psi(a,f)-\psi(b,d)\big) = 0, \quad \text{where} \quad \psi(x,y) = \frac{\varphi(x)-\varphi(y)}{x-y}\cdot$$

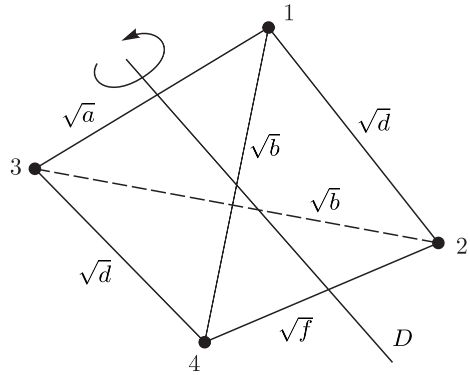


Fig. 3. Symmetry with respect to a line.

Solutions are $b = d$ or $a = f$ or $2\psi(a, f) = \psi(b, d)$. The first two correspond, respectively, to rhombus configurations, already encountered, and twisted rectangle configurations (which project onto rectangles); the last one, which contains the colinear balanced (and hence central) configurations becomes particularly simple for the log potential ($\varphi(s) = -s^{-1}$): indeed, it reduces to $2bd = af$, the colinear case corresponding to $\dfrac{b}{d} = 2 \pm \sqrt{3}$.

Note that the only intersection with the solutions previously studied, which are symmetric with respect to a plane containing some side, are the rhombus configurations.

3. LOOKING FOR NONSYMMETRIC BALANCED CONFIGURATION OF FOUR EQUAL MASSES IN THE NEIGHBORHOOD OF SYMMETRIC ONES

Let $\mathcal{B}$ be the set of balanced configurations (up to similarity) of 4 equal masses and let $\mathcal{S} \subset \mathcal{B}$ be the subset of the symmetric ones.

In this section, we compute the derivative of the equations of $\mathcal{B}$ along every branch of $\mathcal{S}$. The only places where branches of nonsymmetric balanced configurations could possibly bifurcate are the ones where the rank of this derivative is strictly lower than its maximum, that is, 3. This, of course, occurs at the singularities of $\mathcal{S}$, but also at smooth points, which are then the natural candidates for being the sought-for bifurcation points. Hence our first task is to identify the singularities of $\mathcal{S}$, paying attention to the intersections of branches corresponding to a permutation of the vertices.

3.1. Singularities of the Set $\mathcal{S}$ of Symmetric Balanced Configurations

1) Sets P_i or L_i of balanced configurations with, respectively, a given plane or line symmetry:

$$(P_1')\ b' = b'', d' = d'', \qquad (P_1'')\ b' = d', b'' = d'', \qquad (L_1)\ b' = d'', b'' = d';$$
$$(P_2')\ a = b'', d' = f, \qquad (P_2'')\ a = d', b'' = f, \qquad (L_2)\ a = f, b'' = d';$$
$$(P_3')\ a = b', d'' = f, \qquad (P_3'')\ a = d'', b' = f, \qquad (L_3)\ a = f, b' = d''.$$

2) Singularities coming from intersections of the sets corresponding to different symmetries:

$$(P_1' \cap P_1'') = (P_1' \cap L_1) = (P_1'' \cap L_1)\ b' = b'' = d' = d'',$$
$$(P_1' \cap P_2') = (P_1' \cap P_3')\ b' = b'' = a, d' = d'' = f,$$
$$(P_1' \cap P_2'') = (P_1' \cap P_3'')\ b' = b'' = f, d' = d'' = a,$$
$$(P_1' \cap L_2) = (P_1' \cap L_3)\ b' = b'' = d' = d'', a = f.$$

It turns out that the intersections correspond only to rhombus or equilateral configurations.

3.2. The Derivative at a Configuration Symmetric w.r.t. to a Plane ($b := b' = b''$, $d := d' = d''$)

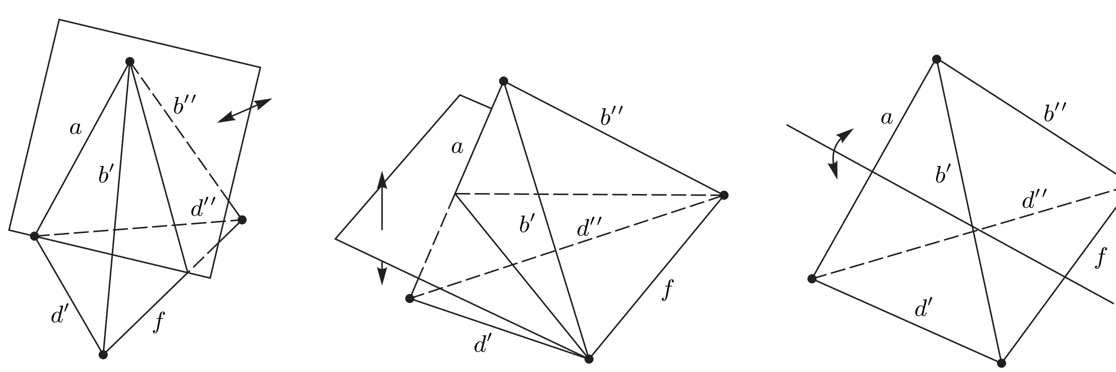
$2(d-b)\varphi'(a)+3(\varphi(b)-\varphi(d))$	$(a+f-b-d)\varphi'(b)+\varphi(d)-\varphi(f)$	$2(a-d)\varphi'(b)+2(\varphi(d)-\varphi(a))+\varphi(f)-\varphi(b)$	$(d+b-a-f)\varphi'(d)+\varphi(f)-\varphi(b)$	$2(b-a)\varphi'(d)+2(\varphi(a)-\varphi(b))+\varphi(d)-\varphi(f)$	$\varphi(b)-\varphi(d)$
0	$2(f-b)\varphi'(b)+2(\varphi(b)-\varphi(f))+\varphi(a)-\varphi(d)$	$2(b-f)\varphi'(b)+2(\varphi(f)-\varphi(b))+\varphi(d)-\varphi(a)$	$(f+a-b-d)\varphi'(d)+\varphi(d)-\varphi(a)$	$(b+d-f-a)\varphi'(d)+\varphi(a)-\varphi(d)$	0
$2(b-d)\varphi'(a)+3(\varphi(d)-\varphi(b))$	$2(d-a)\varphi'(b)+2(\varphi(a)-\varphi(d))+\varphi(b)-\varphi(f)$	$(b+d-a-f)\varphi'(b)+\varphi(f)-\varphi(d)$	$2(a-b)\varphi'(d)+2(\varphi(b)-\varphi(a))+\varphi(f)-\varphi(d)$	$(a+f-d-b)\varphi'(d)+\varphi(b)-\varphi(f)$	$\varphi(d)-\varphi(b)$
0	$(d+b-f-a)\varphi'(b)+\varphi(a)-\varphi(b)$	$(f+a-d-b)\varphi'(b)+\varphi(b)-\varphi(a)$	$2(d-f)\varphi'(d)+2(\varphi(f)-\varphi(d))+\varphi(b)-\varphi(a)$	$2(f-d)\varphi'(d)+2(\varphi(d)-\varphi(f))+\varphi(a)-\varphi(b)$	0

Using appropriate line and column operations, one finds that the rank of the derivative of the equations of balanced configurations at a configuration symmetric with respect to a plane of type (P_1') (i. e., $b' = b'', d' = d''$) is strictly less than 3 if and only if one of the following conditions holds:

$$b = d \quad \text{and} \quad \begin{cases} (3a+f-4b)\varphi'(b) + 2\big(\varphi(b)-\varphi(a)\big) = 0, & (I_1) \\ \text{or} \\ (3f+a-4b)\varphi'(b) + 2\big(\varphi(b)-\varphi(f)\big) = 0, & (I_2) \\ \text{or} \\ (a-f)\varphi'(b) + 2\big(\varphi(f)-\varphi(a)\big) = 0, & (I_3) \end{cases}$$

$$\text{or} \quad \det \begin{pmatrix} 2(b-f)\varphi'(b) + 2\big(\varphi(f)-\varphi(b)\big) & (a+f-d-b)\varphi'(d) \\ +\varphi(d)-\varphi(a) & +\varphi(d)-\varphi(a) \\ ---------- & ---------- \\ (a+f-b-d)\varphi'(b) & 2(d-f)\varphi'(d)+2\big(\varphi(f)-\varphi(d)\big) \\ +\varphi(b)-\varphi(a) & +\varphi(b)-\varphi(a) \end{pmatrix} = 0. \qquad (3.1)$$

The first case corresponds to singularities of the set $\mathcal{S}$ of balanced configurations with a symmetry, namely, of intersections of the submanifold of rhombus configurations $b' = b'' = d' = d''$ with other branches of $\mathcal{S}$ as depicted in Fig. 4.



I_1 $\begin{pmatrix} b' = b'' \\ d' = d'' \end{pmatrix}$ I_2 $\begin{pmatrix} b' = d' \\ b'' = d'' \end{pmatrix}$ I_3 $\begin{pmatrix} b' = d'' \\ b'' = d' \end{pmatrix}$

Fig. 4.

In the second case one must add Eq. (2.1):

$$\big(\varphi(b)-\varphi(d)\big)(3a+f-2b-2d)+\big(2\varphi(a)-\varphi(b)-\varphi(d)\big)(d-b)=0 \quad \text{and} \quad b\neq d.$$

3.3. The Derivative at a Configuration Symmetric w.r. to a Line ($b:=b'=d''$, $d:=d'=b''$)

$2(b-d)\varphi'(a)+\varphi(d)-\varphi(b)$	$(a+f-2d)\varphi'(b)+\varphi(d)-\varphi(f)$	$2(a-b)\varphi'(d)-2\varphi(a)+\varphi(b)+\varphi(f)$	$(2b-a-f)\varphi'(d)+\varphi(f)-\varphi(b)$	$2(d-a)\varphi'(b)+2\varphi(a)-\varphi(d)-\varphi(f)$	$2(d-b)\varphi'(f)+\varphi(b)-\varphi(d)$
$2(b-d)\varphi'(a)+\varphi(d)-\varphi(b)$	$2(f-d)\varphi'(b)+\varphi(a)+\varphi(d)-2\varphi(f)$	$2(b-f)\varphi'(d)+2\varphi(f)-\varphi(a)-\varphi(b)$	$(f+a-2b)\varphi'(d)+\varphi(b)-\varphi(a)$	$(2d-f-a)\varphi'(b)+\varphi(a)-\varphi(d)$	$2(d-b)\varphi'(f)+\varphi(b)-\varphi(d)$
$2(b-d)\varphi'(a)+\varphi(d)-\varphi(b)$	$2(d-a)\varphi'(b)+2\varphi(a)-\varphi(d)-\varphi(f)$	$(2b-a-f)\varphi'(d)+\varphi(f)-\varphi(b)$	$2(a-b)\varphi'(d)-2\varphi(a)+\varphi(b)+\varphi(f)$	$(a+f-2d)\varphi'(b)+\varphi(d)-\varphi(f)$	$2(d-b)\varphi'(f)+\varphi(b)-\varphi(d)$
$2(b-d)\varphi'(a)+\varphi(d)-\varphi(b)$	$(2d-f-a)\varphi'(b)+\varphi(a)-\varphi(d)$	$(f+a-2b)\varphi'(d)+\varphi(b)-\varphi(a)$	$2(b-f)\varphi'(d)+2\varphi(f)-\varphi(a)-\varphi(b)$	$2(f-d)\varphi'(b)-2\varphi(f)+\varphi(a)+\varphi(d)$	$2(d-b)\varphi'(f)+\varphi(b)-\varphi(d)$

Using appropriate line and column operations, one finds that the rank of the derivative of the equations of balanced configurations at a configuration symmetric with respect to a line of type (L_1) (that is, $b'=d'', b''=d'$) is strictly less than 3 if and only if one of the following conditions holds (we disregard the case $b=d$ already studied):

$$a=f \quad \text{and} \quad \begin{cases} 2(b-d)\varphi'(a)+\varphi(d)-\varphi(b)=0, & (J_1)\\ \text{or} & \\ 2(a-d)\varphi'(b)+\varphi(d)-\varphi(a)=0, & (J_2)\\ \text{or} & \\ 2(a-b)\varphi'(d)+\varphi(b)-\varphi(a)=0, & (J_3)\end{cases}$$

$$\text{or} \quad \det\begin{pmatrix} (2a+2f-4b)\varphi'(d) & (a-f)\varphi'(b)\\ +2\varphi(b)-\varphi(a)-\varphi(f) & +\varphi(f)-\varphi(a)\\ ----------- & -----------\\ (a-f)\varphi'(d) & (2a+2f-4d)\varphi'(b)\\ +\varphi(f)-\varphi(a) & +2\big(\varphi(d)-\varphi(a)\big)-\varphi(f)\end{pmatrix}=0. \tag{3.2}$$

The first case corresponds to the intersection of the component $f=a$ with the three branches of line symmetric balanced configurations:

In the second case one must add Eq. (2.2):

$$2(b-d)\big(\varphi(a)-\varphi(f)\big)+(f-a)\big(\varphi(b)-\varphi(d)\big)=0.$$

4. THE CASE OF A LOGARITHMIC POTENTIAL

When the potential is logarithmic ($\varphi(u)=-u^{-1}$), it is well known that the equations of central configurations become simpler: for example, the isosceles and equilateral central configurations coalesce into a unique equilateral one. Equations of balanced configurations also become simpler in the sense that they factorize:

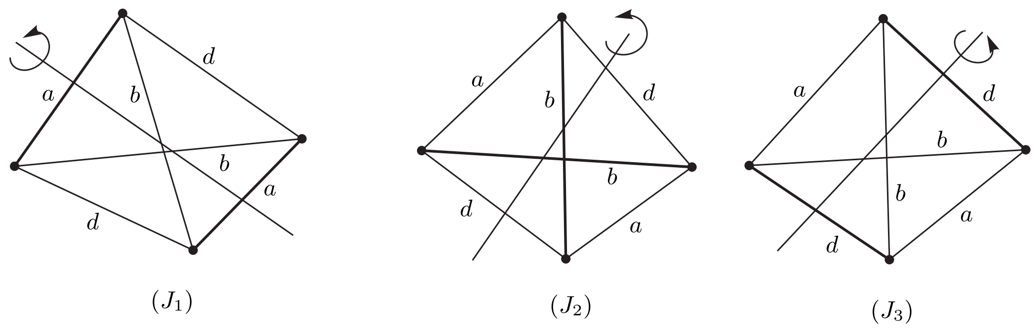


Fig. 5.

4.1. Singularities of $\mathcal{B}$ at a Configuration with Plane Symmetry

The Equation (2.1) of balanced configurations with plane symmetry (such that $b' = b''$, $d' = d''$) becomes:

$$(b-d)\big[2bd - 3a(b+d) + a(3a+f)\big] = 0.$$

We have seen in Section 3.2 that, independently of the potential, the singularities of the set $\mathcal{B}$ of balanced configurations at a configuration of rhombus type ($b' = b'' = d' = d''$) are also singularities of the set $\mathcal{S}$ of symmetric balanced configurations (either $b' = b'', d' = d''$ or $b' = d''; d' = b''$). Hence we shall suppose that $b' = b'' := b \neq d := d' = d''$. After scaling the configuration by setting $a = 1$, the equation of balanced configurations becomes

$$2bd - 3(b+d) + f + 3 = 0. \tag{4.1}$$

As we have already understood the case $f = a$, we need deal only with condition (3.1) of Section 3.2, which becomes

$$\begin{aligned}(4bdf - 2df^2 - 2b^2d + b^2df - b^2f)(4bdf - 2bf^2 - 2bd^2 + bd^2f - d^2f)\\ -bdf^2(b^2 + f - d - 2b + 1)(d^2 + f - b - 2d + 1) = 0.\end{aligned} \tag{4.2}$$

The equilateral solutions $b = a = 1, d = f$ or $b = f, d = a = 1$ are factored out in the following way: Eqs. (4.1) and (4.2) can be written

$$(2d-3)(b-1) + (f-d) = 0, \text{ or equivalently } (2b-3)(d-1) + (f-b) = 0, \tag{4.3}$$

and

$$\begin{aligned}\big[-2d(b-f)^2 + b^2f(d-1)\big] \times \big[-2b(d-f)^2 + d^2f(b-1)\big]\\ -bf\big[(b-1)^2 + (f-d)\big] \times df\big[(d-1)^2 + (f-b)\big] = 0.\end{aligned} \tag{4.4}$$

Using (4.1) in order to eliminate f, one transforms Eq. (4.2) into

$$6(b-1)^2(d-1)^2A = 0, \tag{4.5}$$

where

$$\begin{aligned}A = {}& 4b^2d^2(b+d)^2\\ &- 4bd(3b^3 + 8b^2d + 8bd^2 + 3d^3)\\ &+ 9b^4 + 36b^3d + 82b^2d^2 + 36bd^3 + 9d^4\\ &- (9b^3 + 63b^2d + 63bd^2 + 9d^3)\\ &+ 48bd.\end{aligned}$$

4.2. Singularities of $\mathcal{B}$ at a Configuration with Line Symmetry

After scaling the configuration by setting $a = 1$, the Eq. (2.2) of balanced configurations with line symmetry ($b' = d''$ and $d' = b''$) becomes

$$2bd = f, \tag{4.6}$$

while the singularities of equation $\mathcal{B}$ are defined by the equation (after multiplying by $b^3d^3f^2$)

$$\begin{aligned}\left[2bf(1+f-2b)+d^2(-2f+bf+b)\right]\times\left[2df(1+f-2d)+b^2(-2f+df+d)\right]\\ -bd(1-f)^2(f-d^2)(f-b^2)=0.\end{aligned}\tag{4.7}$$

Taking (2.2) into account and dividing by b^2d^2, we get

$$\begin{aligned}\left[4b(1+2bd-2b)+d(-4d+2bd+1)\right]\times\left[4d(1+2bd-2d)+b(-4b+2db+1)\right]\\ -(1-2bd)^2(2b-d)(2d-b)=0.\end{aligned}\tag{4.8}$$

4.3. *Virtual or Real?*

Recall that the reality of a configuration is equivalent to the positivity of the symmetric matrix B introduced in Section 1.1.

1) *Case of plane symmetry.* When $b'=b''=b$, $d'=d''=d$, the condition becomes

$$B=\frac{1}{4}\begin{pmatrix}2(b+d)-(a+f) & \sqrt{2(b-d)} & 0\\ \sqrt{2(b-d)} & 2a & 0\\ 0 & 0 & 2f\end{pmatrix}\geqslant 0,$$

that is,

$$a\left[2(b+d)-(a+f)\right]-(b-d)^2\geqslant 0,\quad 2(b+d)-(a+f)+2a\geqslant 0,\quad\text{and}\quad f\geqslant 0.$$

Setting $a=1$ and using the equation of nonrhombus balanced configurations with plane symmetry, $f=-2bd+3(b+d)-3$, this becomes

$$\begin{cases}\left[(\sqrt{3}-1)b-(\sqrt{3}+1)d+1\right]\left[(\sqrt{3}+1)b-(\sqrt{3}-1)d-1\right]\leqslant 3\\ (2d-3)(2b-3)\leqslant 3.\end{cases}$$

Figure 6 represents the locus in the (b,d) plane of singularities (for the log potential) of the set $\mathcal{B}$ of balanced configurations at regular points of the set (2.1) of balanced configurations with a plane symmetry. This locus is entirely outside the set (shaded) of true configurations, that is, of configurations such that $B>0$.

2) *Case of line symmetry.* When $b'=d''=b$, $d'=b''=d$, the condition becomes

$$B=\frac{1}{4}\begin{pmatrix}2(b+d)-(a+f) & 0 & 0\\ 0 & 2a & 2(b-d)\\ 0 & 2(b-d) & 2f\end{pmatrix}\geqslant 0,$$

that is,

$$2(b+d)-(a+f)\geqslant 0,\quad a+f\geqslant 0,\quad\text{and}\quad af-(b-d)^2\geqslant 0.$$

Setting $a=1$ and using the equation of balanced configurations with line symmetry, $f=2bd$, this becomes

$$2(b+d)-(1+2bd)\geqslant 0,\quad 4bd-b^2-d^2\geqslant 0.$$

On figure 7 is represented the locus in the (b,d) plane of singularities (for the log potential) of the set $\mathcal{B}$ of balanced configurations at regular points of the set of balanced configurations with a line symmetry. Remarkably, this locus consists in only 3 points, two of which ($(b=0.5,d=1)$ and the symmetric $(b=1,d=0.5)$ lie in the domain of true configurations but are rhombus configurations hence already singular points of S.

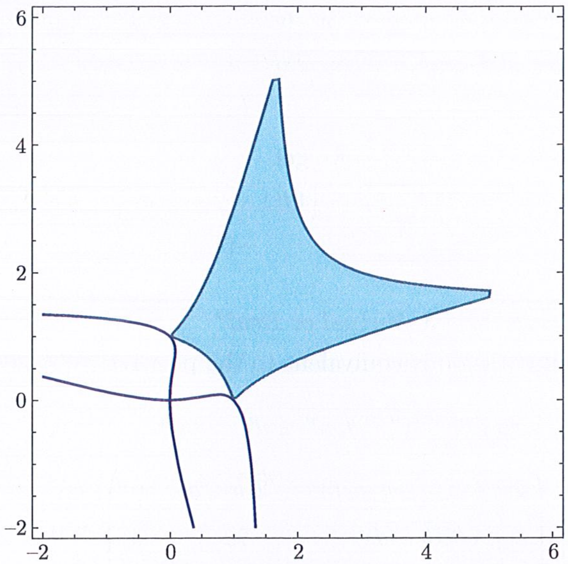


Fig. 6. Bifurcation locus in the case of plane symmetry.

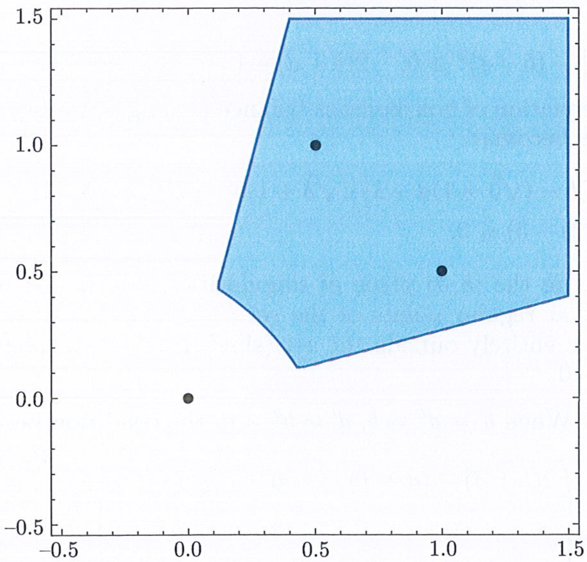


Fig. 7. Bifurcation locus in the case of line symmetry.

Two questions remain for numerical computation: 1) does a branch of nonsymmetric virtual balanced configurations truly bifurcate from the locus in Fig 6 in the case of the log potential and does a similar phenomenon occur for the Newton potential?

2) Is it possible for both potentials that a branch of virtual nonsymmetric balanced configurations starting from virtual symmetric balanced configurations returns at some place within the set of true configurations? These would be examples of nonsymmetric balanced configuration which seem to come "from the blue".

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REFERENCES

1. Albouy, A., Symétrie de configurations centrales de quatre corps, *C. R. Acad. Sci. Paris Ser. 1 Math.,* 1995, vol. 320, no. 2, pp. 217–220.
2. Albouy, A., The Symmetric Central Configurations of Four Equal Masses, in *Hamiltonian Dynamics and Celestial Mechanics (Seattle, Wash., 1995),* Contemp. Math., vol. 198, Providence, R.I.: AMS, 1996, pp. 131–135.
3. Albouy, A., Mutual Distances in Celestial Mechanics: Lectures at Nankai Institute (Tianjin, Chine), June 2004.
4. Albouy, A. and Chenciner, A., Le problème des n corps et les distances mutuelles, *Invent. Math.,* 1998, vol. 131, no. 1, pp. 151–184.
5. Chenciner, A., The Lagrange Reduction of the N-Body Problem: A Survey, *Acta Math.,* 2013, vol. 38, no. 1, pp. 165–186.
6. Chenciner, A. Symmetric 4-Body Balanced Configurations in the Case of Equal Masses, *unpublished manuscript*.
7. Chenciner, A., Non-Avoided Crossings for N-Body Balanced Configurations in $\mathbb{R}^3$ near a Central Configuration, *Arnold Math. J.,* 2016, vol. 2, no. 2, pp. 213–248.