Perturbing a planar rotation : normal hyperbolicity and angular twist

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1 Introduction

Perturbing the germ at the origin of a planar rotation $re^{2\pi i\theta} \mapsto re^{2\pi i(\theta+\omega)}$ leads to two celebrated results which describe geometrically the *dynamical* behaviour of the iterates of the perturbed diffeomorphism F, that is the structure of the *orbits* $\mathcal{O}(z) = \{z, F(z), F^2(z), \ldots, F^n(z), \ldots\}$: the *Andronov-Hopf-Neimark-Sacker bifurcation* of invariant curves under a generic radial hypothesis of weak attraction (or repulsion) and the *Moser invariant curve* theorem under an angular twist hypothesis in the area preserving case. The invariant curves whose existence is proved are normally hyperbolic with generic induced dynamics in the first case, with a dynamics smoothly conjugate to a diophantine rotation in the second one.

Statements and proofs illustrate the notion of *normal form*, introduced by Poincaré in his thesis in 1879. Closely related to the "averaging of perturbations" used by astronomers since the eighteenth century, it generalizes the Jordan normal form of a matrix to the nonlinear world. Namely, by introducing local coordinates which reveal an approximate geometry underlying the situation, it sets the scene for the application of refined analytic tools to the determination of which features of this geometry do really exist.

After recalling these two classical contexts, say the one of nonlinear self-sustained oscillations (Lord Rayleigh, Van der Pol) and the one of the 3-body problem (Poincaré), I shall describe an old result of mine which in some sense makes the two worlds meet: in generic 2-parameter families of germs of diffeomorphisms of the plane near a fixed point, the tension between radial and angular (or hyperbolic and elliptic) behaviour leads to phenomena where the whole wealth of the area preserving situation is unfolded along some direction of the parameter space.

2 Elliptic fixed points

Let $F : (S, p) \to (S, p)$ be a local C^∞ (or analytic) diffeomorphism of a surface S defined in the neighborhood of a fixed point $p = F(p)$. The fixed point is said to be elliptic if the spectrum of the derivative $dF(p)$ is of the form $\{2\pi i\omega, -2\pi i\omega\}$ with $\omega \neq \pm 1$. This is equivalent to the existence of a linear conjugation of $dF(p)$ with the rotation of angle $2\pi\omega$. Hence, after choosing good coordinates, one can suppose that $p = 0$ and that $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is such that

$$F(\zeta) = \lambda\zeta + O(|\zeta|^2), \quad \text{with} \quad \lambda = e^{2\pi i\omega}.$$

In other words, F is a perturbation of a rotation.[1] Now, a rotation preserves each circle centered at the origin. This is a very strong property, very likely to be destroyed by the non-linear terms in the Taylor expansion of F. Nevertheless, reality is subtler and the study of the fate of these invariant circles is the starting point of two famous theories which correspond roughly to the dichotomy between *dissipative* and *conservative* dynamics:

1) *Andronov-Hopf-Neimark-Sacker bifurcation theory* which analyzes what happens when one considers a *generic*[2] diffeomorphism F with an elliptic fixed point at 0. The local behaviour of F itself is quite dull: indeed, the radial behaviour of the nonlinear terms turns the fixed point into an attractor or a repulsor and no other invariant object persists in its neighborhood. It is only when considering "generic" 1-parameter families F_μ of local diffeomorphisms stemming from $F_0 = F$ that the whole richness of the dynamics is regained (see [A1, A2]): each small enough circle invariant under the rotation $dF(0)$ becomes a *normally hyperbolic*[3] closed curve invariant under some F_μ (figure 3).

2) *Kolmogorov-Arnold-Moser (K.A.M.) theory* which analyzes the case when F is area preserving, a hypothesis which is natural for diffeomorphisms with a mechanical origin, the paradigmatic example being *first return maps*[4] in the *restricted three body problem* first studied by Poincaré (see [C1, C2]). In this case, it is the angular behaviour of the non-linear terms which plays the key part, the result being that "many" of the circles invariant under the rotation $dF(0)$ persist in the form of closed curves invariant under the action of F itself. Moreover the restriction of F to such an invariant closed curve is smoothly conjugated to a rotation whose angle is of the form $2\pi\alpha$ with α not rational and even "far from the rationals" in a precise sense.

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[1] Beware that the notation $F(\zeta)$ does not mean that F is complex analytic, its expression depends on ζ and $\overline{\zeta}$

[2] We shall not give a formal definition of this word; it means essentially that what is described is the general situation and that only special hypotheses could prevent the description to be correct.

[3] Roughly speaking this mean that any attraction or repulsion normal to the curve under the iterates of F_μ dominates any attraction or repulsion inside the curve; this condition insures the robustness of the curve

[4] see section 1.4 of [C0] for a brief introduction

3 Preparation: Poincaré's theory of normal forms

The idea, which goes back to Poincaré's thesis in 1879, is the following: being a rotation, the derivative of F commutes with the whole group $SO(2)$ of rotations. This is shown to imply that, provided some conditions on ω are satisfied, a high order approximation of F is locally invariant by an action of $SO(2)$ close to the standard one. Equivalently, one proves the existence of local coordinates which reveal the approximate geometry of the map, in a spirit similar to the Jordan form of a matrix:

Theorem 1 *If $\lambda = e^{2\pi i\omega}$ is such that $\lambda^q \neq 1$ for all integers $q \in \mathbb{N}$ such that $q \leq 2n+2$, there exists a local diffeomorphism*

$$H : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad \zeta \mapsto z = H(\zeta) = \zeta + O(|\zeta|^2)$$

such that

$$H \circ F \circ H^{-1}(z) = N(z) + O(|z|^{2n+2}), \quad \textit{where} \quad N(z) = z\left(1 + f(|z|^2)\right) e^{2\pi i(\omega + g(|z|^2))},$$

with f and g real polynomials of degree n such that $f(0) = g(0) = 0$. If moreover $\lambda^{2n+3} \neq 1$, one can achieve a rest which is $O(|z|^{2n+3})$.

The so-called *normal form* N, is characterized by the fact that it commutes with the whole group $SO(2)$ of rotations:

$$\forall \alpha, N(e^{2\pi i\alpha} z) = e^{2\pi i\alpha} N(z).$$

Proof. Let us start with a local diffeomorphism of degree 2,

$$H_2 : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad z = H_2(\zeta) = \zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \bar{\zeta}^j.$$

The direct computation of $H_2 \circ F \circ H_2^{-1}$ is illustrated on the diagram below:

$$\begin{tikzcd}
\zeta \arrow[r, "F"] \arrow[d, "H_2"'] & \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + O(|\zeta|^3) \arrow[d, "H_2"] \\
\zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \bar{\zeta}^j = z \arrow[r, "H \circ F \circ H^{-1}"] & \begin{array}{l} \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + \sum_{i+j=2} \gamma_{ij} \lambda^i \bar{\lambda}^j \zeta^i \bar{\zeta}^j + O(|\zeta|^3) \\ = \lambda z + \sum_{i+j=2} \left(\alpha_{ij} + \gamma_{ij}(\lambda^i \bar{\lambda}^j - \lambda)\right) z^i \bar{z}^j + O(|z|^3) \end{array}
\end{tikzcd}$$

Figure 1. Changing coordinates.

Supposing that $F(\zeta) = \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + O(|\zeta|^3)$, we get

$$H_2 \circ F \circ H_2^{-1}(z) = \lambda z + \sum_{i+j=2} \left(\alpha_{ij} + (\lambda^i \bar{\lambda}^j - \lambda)\gamma_{ij}\right) z^i \bar{z}^j + O(|z|^3).$$

Hence, if no *resonance* relation of the form $\lambda^i\overline{\lambda}^j - \lambda = 0$ is satisfied with indices i, j such that $i + j = 2$, that is if $\lambda^3 \neq 1$ (otherwise $\overline{\lambda}^2 - \lambda = 0$), the choice of $\gamma_{ij} = -(\lambda^i\overline{\lambda}^j - \lambda)^{-1}\alpha_{ij}$ kills all degree 2 terms in the Taylor expansion of the transformed map $H_2 \circ F \circ H_2^{-1}$.
If one tries in the same way to simplify the terms of degree 3 in the Taylor expansion of $H_2 \circ F \circ H_2^{-1}$, one stumbles upon an *unavoidable resonance*

$$\lambda^2\overline{\lambda} - \lambda = 0$$

which merely reflects that $|\lambda| = 1$. Hence , if no other resonance of order 3 exists, which amounts to saying that $\lambda^4 \neq 1$ (otherwise $\overline{\lambda}^3 - \lambda = 0$), a local diffeomorphism H_3 of the form $H_3(z) = z + \sum_{i+j=3} \gamma_{ij} z^i \overline{z}^j$ can be found such that[5]

$$H_3 \circ H_2 \circ F \circ H_2^{-1} \circ H_3^{-1}(z) = \lambda z + c_1 z|z|^2 + O(|z|^4).$$

Now, if $\lambda^q \neq 1$ for all $q \leq 2n+3$, one finds by induction a local diffeomorphism $H = H_{2n+2} \circ H_{2n+1} \circ H_3 \circ H_2$ tangent to Id at 0 such that

$$H \circ F \circ H^{-1}(z) = \lambda z + \sum_{k=1}^{n} c_k z|z|^{2k} + O(|z|^{2n+3}).$$

If $\lambda^{2n+3} = 1$, there is possibly a monomial $\gamma\overline{z}^{2n+2}$ which cannot be canceled. Finally, chosing polar coordinates, one writes $H \circ F \circ H^{-1}$ as in the conclusion of the theorem.

Remark. Resonances of the form $\lambda^q = 1$ for $1 \leq q \leq 4$ are called *strong resonances*. They are characterized by the fact that the resonant monomial $\overline{z}^{q-1}$ is of smaller or comparable order to the first unvoidable resonant monomial $z|z|^2$ and hence could play a role in the geometry of the normal form N which could become invariant only by rotations by an angle multiple of $2\pi/q$. *In the sequel, the hypotheses always exclude strong resonances.*

Remark on notations. : *Theorem 1 allows us to suppose from the start that local coordinates z have been chosen so that F is in the form given, by Theorem 1. In other words, from now on we shall write $F(z)$ instead of $H \circ F \circ H^{-1}(z)$.*

4 The dissipative case

4.1 Andronov-Hopf-Neimark-Sacker bifurcation

The first two names are attached to the "continuous" case of a differential equation, the last two to the present "discrete" case of a map (see [A1, A2, I, C6]).

[5] in order to avoid too cumbersome notations we still call z the transformed coordinate $H_3(z)$.

In general, the polynomial $f(s) = \sum_{k=1}^{n} a_k s^k$ is such that $a_1 \neq 0$. If $a_1 < 0$, one can scale the coordinates so that $a_1 = -1$ which, provided $\lambda^q \neq 1$ for all integers $1 \leq q \leq 4$, puts F into the form

$$F(z) = N(z) + O(|z|^4), \quad \text{where} \quad N(z) = z\left(1 - |z|^2\right) e^{2\pi i\left(\omega + b_1|z|^2\right)}.$$

As well as the rotation $dF(0)$, the normal form N still leaves invariant the *foliation* by circles centered at 0 but it sends the circle of radius r onto the circle of radius $r(1-r^2)$. This implies not only that $\lim_{m\to\infty} N^m(z) = 0$ but also that $\lim_{m\to\infty} F^m(z) = 0$ as soon as $|z|$ is small enough. Indeed,

$$\text{if } |z| < \epsilon, \quad |F(z)| < \epsilon \left|1 - \frac{1}{2}\epsilon^2\right| < \epsilon, \text{ hence by induction } |F^m(z)| < \epsilon \left|1 - \frac{1}{2}\epsilon^2\right|^m.$$

One says that 0 is a *weak attractor* (figure 2), the adjective "weak" recalling that the attraction is due to a non-linear term.

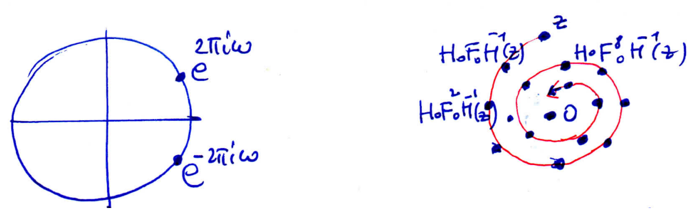

Figure 2. Weak attraction.

Hence we completely understand the dynamics of F in some neighborhood $\mathcal{V}$ of the fixed point 0. Things become much more interesting if one perturbs F by including it in a smooth one parameter family of local diffeomorphisms F_μ such that $F_0 = F$. A direct application of the implicit function theorem shows that, in the neighborhood of 0, the equation $F_\mu(z) - z = 0$ has a unique solution z_μ depending smoothly on μ and such that $z_0 = 0$. Hence, after a translation by z_μ of the coordinates, one can suppose that for all μ near 0, one has $F_\mu(0) = 0$. For values of μ such that the spectrum of $dF_\mu(0)$ is not on the unit circle, there is no resonance and one could get a normal form which is linear up to any order. However, this would not be of much use: on the one hand the domain of definition of the conjugating diffeomorphism H_μ tends to 0 when the spectrum of $dF_\mu(0)$ tends to the unit circle and interesting phenomena occur outside of this domain, on the other hand, this would break the continuity with respect to μ of the coordinate change H_μ. In consequence, one chooses to eliminate in F_μ only the same terms as the ones we have eliminated in F_0, that is we mimic for H_μ the construction of H in section 3. Doing so one gets a smooth family H_μ of local diffeomorphisms of $(\mathbb{C}, 0)$ defined in a fixed neighborhood of 0 which put F_μ into the form $F_\mu(z) = z(1 + f_\mu(|z|^2))e^{2\pi i(\omega + g_\mu(|z|^2))} + \cdots$ given by Theorem 1 except that $f_\mu(s) = \sum_{i=0}^{n} a_k s^k$ and $g_\mu(s) = \sum_{k=0}^{n} b_\mu(s) s^k$ now start with terms of degree 0. Finally, we shall suppose that $a_0(\mu)$ is monotone (say increasing) for μ close enough to zero. This is also a "generic" condition which amounts to saying that the spectrum of the derivative $dF_\mu(0)$ crosses transversally the unit circle when μ crosses the value 0. It allows us to change parameters and suppose that $a_0(\mu) = \mu$. At the end, we are reduced to study a family F_μ of local diffeomorphisms of the form

$$\begin{cases} F_\mu(z) = N_\mu(z) + O(|z|^4), \text{ where} \\ N_\mu(z) = z\left(1 + \mu + a_1(\mu)|z|^2\right) e^{2\pi i\left(b_0(\mu) + b_1(\mu)|z|^2\right)}, \text{ and} \\ a_1(\mu) = -1 + O(|\mu|),\ b_0(\mu) = \omega + O(|\mu|). \\ \text{The rest can be made } O(|z|^5) \text{ except if } \lambda^5 = 1, \text{ which can leave a term } \gamma\bar{z}^4. \end{cases}$$

Due to the commutation of N_μ with the group $SO(2)$ of rotations, the study of its dynamics reduces to an elementary question in dimension 1. The results are summarized in figure 3: the origin, which is a strong (=linear) attractor when $\mu < 0$, becomes a strong repellor when $\mu > 0$. But points far enough from the origin are still attracted and in between appears an invariant circle C_μ of radius the unique solution r_μ of the equation $\mu + a_1(\mu) r_\mu^2 = 0$.

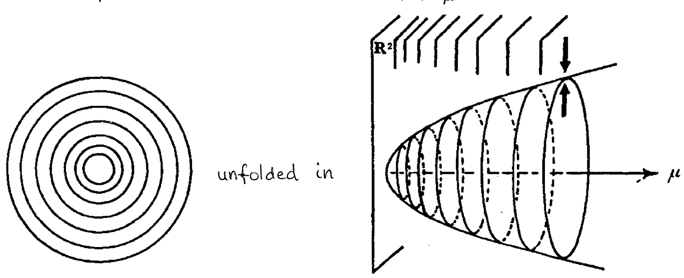

Figure 3. Dynamics of the family of normal forms N_μ.

Theorem 2 (Neimark 1959, Sacker 1964) *Under the above hypotheses, for each $\mu > 0$ small enough, F_μ possesses an invariant closed curve Γ_μ, close to C_μ, which attracts a uniform (that is independent of μ) neighborhood $\mathcal{V}$ of 0 (with 0 deleted). If the local diffeomorpisms F_μ are of class C^∞, these curves are of class C^k with k going to infinity when μ tends to 0.*

The proof proceeds in two steps: 1) One encloses the invariant circle $C\mu$ in an annulus A_μ of width $O(|\mu|)$, say the one bounded by the circles whose radii $r_\mu^\pm$ are the two solutions of the equation $\mu + a_1(\mu)r^2 \pm r^3 = 0$. One checks that every point $z \neq 0$ in some uniform (i.e. independent of μ) neighborhood $\mathcal{V}$ of 0 is eventually sent inside A_μ under the iterates of F_μ.

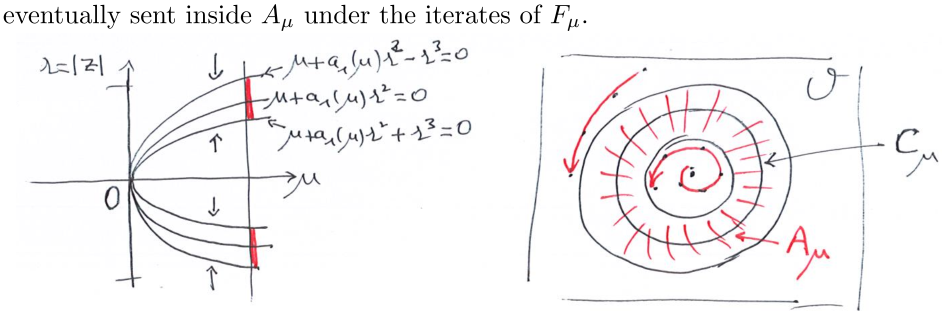

Figure 4. The attracting annulus A_μ.

2) One shows that under the iterates of F_μ, every point inside the annulus tends asymptotically to some invariant curve Γ_μ close to the circle C_μ. For this, we choose coordinates in A_μ centered on C_μ of the form:

$$z = r_\mu(1 + \sqrt{\mu}\sigma)\, e^{2\pi i\theta}.$$

If $\lambda^5 \neq 1$, the map F_μ becomes (we keep the same notation F_μ for convenience)

$$F_\mu(\sigma, \theta) = \Big((1 - 2\mu)\sigma + O(\mu^{3/2}), \theta + b\mu + O(\mu^{3/2})\Big).$$

(If $\lambda^5 = 1$ and the term $\gamma\bar{z}^4$ is present, a circle is not a good enough approximation of the invariant curve and a further change of variables is necessary to get to the above form, see [I].) Let $(\theta, \psi(\theta))$ be the graph of a function $\theta \mapsto \sigma = \psi(\theta)$ from the circle $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$. If ψ is small enough, its graph Γ_ψ is contained in the annulus A_μ and the image by F_μ of its graph, also contained in A_μ, is the graph of a function $\mathcal{F}_\mu\psi$:

$$F_\mu(\Gamma_\psi) = \Gamma_{\mathcal{F}_\mu\psi}.$$

The map $\psi \mapsto \mathcal{F}_\mu\psi$ is called the *graph transform*. Thanks to the contracting factor $1 - 2\mu$ which dominates any contraction along the angular direction (a manifestation of the fact that the *normal hyperbolicity* of C_μ dominates the perturbation), one shows that $\mathcal{F}_\mu$ is a contraction in a well chosen Banach space of C^k functions provided μ is close enough to 0 (a condition more and more stringent when k tends to $+\infty$). The attracting invariant curve $\Gamma_\mu \subset A_\mu$ we are looking for is the graph of the unique fixed point of this contraction.

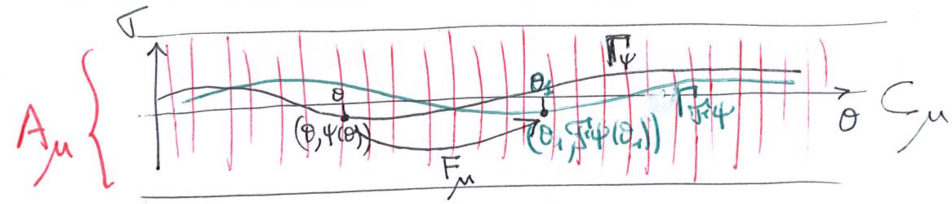

Figure 5. Graph transform.

4.2 Dynamics on the invariant curves

In conclusion, from the "radial" hypothesis $a_1(0) < 0$ we have obtained a complete control on the radial dynamics of F_μ in a uniform neighborhood $\mathcal{V}$ of 0 (i.e. figure 3 is still pertinent to describe the normal dynamics of F_μ), but we have no control of the dynamics restricted to the invariant curves. Indeed, this dynamics may be a "generic" dynamics of a diffeomorphism of the circle (see section 4 of [C0]). To be more precise we should add another "generic" assumption, this time on the "angular" part of F, namely that $b_1(0) \neq 0$, for example $b_1(0) > 0$. This implies that, for μ close enough to 0, the restriction of the normal form N_μ to its invariant circle C_μ is a rotation whose angle increases with μ. The two-parameter family $f_{\omega,\mu}$ of diffeomorphisms of the circle defined by the restriction of F_μ to its invariant curve Γ_μ, the other parameter being ω, behaves in general as does Arnold's family $T_{\omega,\mu}$, described in [A3]:

$$T_{\omega,\mu}(\theta) = \theta + \omega + \mu \cos 2\pi\theta.$$

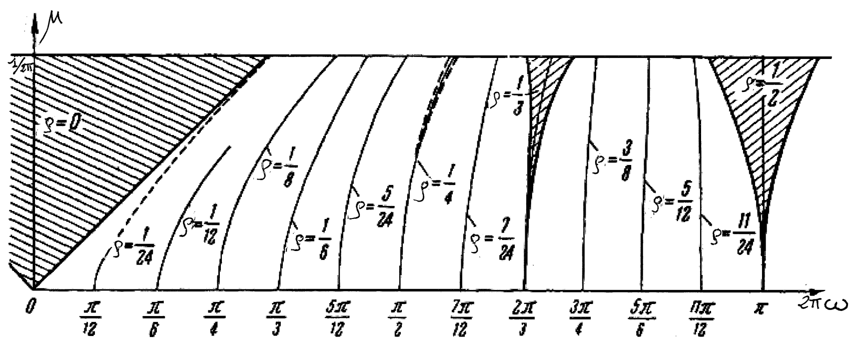

Figure 6 : Typical behavior of a 2-parameter family of circle diffeomorphisms (figure adapted from [A3]).

In the interior of each of the so-called *Arnold's tongues* – values of the parameters for which the rotation number is rational – $f_{\omega,\mu}$ is in general a diffeomorphism of the circle with two periodic orbits of the same period q if the root of the tongue is the rotation of angle $2\pi\omega = 2\pi\frac{p}{q}$). One orbit is attracting, the other repelling. Such periodic orbits cannot be destroyed by a small enough perturbation and hence persist over an interval of values of ω for each $\mu \neq 0$; the complement of the union of all these intervals is a Cantor set of values of ω for which $f_{\omega,\mu}$ is topologically (but not always smoothly) conjugated to a rotation. Moreover, for any $\mu \neq 0$, the set of ω for which the rotation number of $f_{\omega,\mu}$ is rational is in general big in the sense of topology, namely it is open and dense, but its complement is always big in the sense of measure, namely, its measure tends to 1 when $\mu \to 0$ (see [H]).

5 The area preserving case

5.1 Moser's invariant curve theorem ([M])

We now suppose that, in addition to satisfying $\lambda^q \neq 1$ for all integers $1 \leq q \leq 4$, F is area preserving. It follows that the radial component f of the normal form N vanishes identically and one can show that it is possible to choose H area preserving. Hence, one is reduced to the study in the neighborhood of its elliptic fixed point 0 of an area preserving diffeomorphism of $\mathbb{C}, 0$ of the form

$$F(z) = N(z) + O(|z|^4), \quad N(z) = ze^{2\pi i(\omega + b_1|z|^2)}.$$

The normal form N is called a *truncated Birkhoff normal form.* Dynamically, it is an *integrable monotone twist*: as well as the rotation $dF(0)$, it leaves invariant each circle C_r centered at 0 but the angle of rotation $2\pi(\omega+b_1r^2)$ on C_r varies now monotonically with the radius r of this circle

Poincaré, while studying the three body problem, became aware of a fundamental difference between the invariant circles on which N induces a periodic ($\omega+b_1r^2$ rational) or non periodic ($\omega+b_1r^2$ irrrational) rotation: in the first case (angle $2\pi\omega=2\pi p/q$) the invariant circle is simply the union of a continous family of q-periodic points z (i.e. of points z such that $N^q(z)=z$); in consequence, a small perturbation should in general break such a circle, with only a finite number of periodic points surviving the perturbation. On the other hand, if ω is irrational, the invariant circle being the closure $\overline{\cup_{n\geq 0}N^n(z)}$ of an orbit has a dynamical origin and hence has more chance to resist a perturbation. In the first volume of his famous book *The New Methods of Celestial Mechanics*, Poincaré even ventured to write that some arithmetic condition on ω could perhaps grant resistance to perturbations of such an invariant circle but that he considered such a possibility as quite improbable.

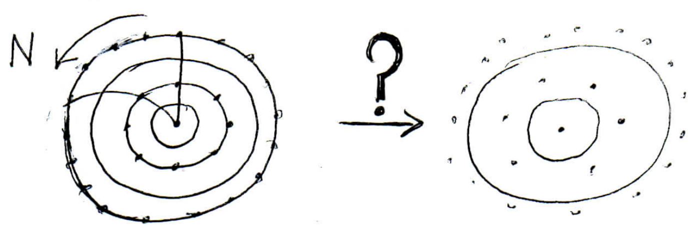

Figure 7. Perturbation of a monotone twist ???

Nevertheless, after the pioneering work of Kolmogorov in 1954, the so-called *K.A.M. theory* (from the names of Kolmogorov, Arnold and Moser) showed that indeed, what Poincaré deemed improbable was in fact a dominant phenomenon. In the present case, the pertinent statement is the following

Theorem 3 (Moser 1962) *Given an area preserving diffeomorphism F as above, given $C>0$ and $\beta>0$, there exists $\epsilon(C,\beta)>0$ such that each invariant circle C_{r_0} of the normal form N such that its rotation angle $2\pi\omega_{r_0}=2\pi(\omega+b_1r_0^2)$ satisfies the diophantine condition*

$$\forall\ \frac{p}{q}\in\mathbb{Q},\ \left|\omega_{r_0}-\frac{p}{q}\right|\geq\frac{C|\omega_{r_0}-\omega|}{|q|^{2+\beta}}\quad \textit{and}\quad |\omega_{r_0}-\omega|<\epsilon(C,\beta)$$

will give rise to a smooth (resp. analytic) closed curve Γ_{r_0} invariant under F and such that the restriction $F|_{\Gamma_{r_0}}$ of F is smoothly conjugate to the rotation of angle $2\pi\omega_{r_0}$.

The most transparent proof of theorem 2 is based on a version of the so-called "hard implicit function theorem" adapted to the problem of *small denominators* well known to astronomers since eighteenth century. The following consequence of area preservation, named *intersection property*, is crucial: the image $F(\Gamma)$ of a curve Γ surrounding the origin cannot be disjoint from Γ. Note that such a property is preserved even under changes of coordinates which do not preserve area. Fixing $r = r_0$ satisfying the hypotheses of the theorem, one chooses coordinates centered on C_{r_0} of the form:

$$z = r_0\sqrt{1+\sigma}\, e^{2\pi i\theta}.$$

The map F is now (as before we keep the same notation F)

$$F(\sigma,\theta) = \left(\sigma + O(r_0^4),\ \theta + \omega_{r_0} + b_1 r_0^2\sigma + O(r_0^4)\right).$$

As a further simplification, one replaces σ by $\rho = \sigma + O(r_0^2)$ so that the formula for F takes the form

$$F(\rho,\theta) = \left(\rho + \varphi(\rho,\theta),\ \theta + \omega_{r_0} + b_1 r_0^2\rho\right),$$

where the perturbation φ is $O(r_0^4)$. Following Rüssmann, it is enough to look for a curve of the form $\rho = \psi(\theta)$ which is sent by F to the *translated curve* $\rho = \psi(\theta) + \tau$ for some $\tau \in \mathbb{R}$. This is because the intersection property, still valid after the changes of coordinates, implies that τ must be equal to 0. This leads to the equation

$$\psi\big(g(\theta)\big) + \tau = \psi(\theta) + \varphi\big(\psi(\theta),\theta\big), \quad \text{where} \quad g(\theta) = \theta + \omega_{r_0} + b_1 r_0^2\psi(\theta).$$

Recall that in the dissipative case, the radial hypothesis $a_1(0) \neq 0$ implied the existence of a curve invariant under F_μ with a prescribed normal dynamics. Having now an angular hypothesis $b_1 \neq 0$, it is natural to look for invariant curves of F with a prescribed angular dynamics. It turns out that the right constraint to impose to the (translated) curve we are looking for is the existence of a diffeomorphism h of the circle $\mathbb{R}/\mathbb{Z}$ such that $g(\theta) = h^{-1}\circ R_{\omega_{r_0}}\circ h(\theta)$. Finally, defining ψ by $\psi(\theta) = \frac{1}{b_1 r_0^2}\left[h^{-1}\circ R_{\omega_{r_0}}\circ h(\theta) - \theta - \omega_{r_0}\right]$, we must solve

$$\mathcal{F}(\varphi,\tau,h) := \psi(\theta) - \psi(h^{-1}\circ R_{\omega_{r_0}}\circ h(\theta)) - \tau + \varphi(\psi(\theta),\theta) = 0$$

in the neighborhood of the solution ($\varphi = 0$, $\tau = 0$, $h = Id$). This is typically a "hard implicit function problem" because even the best diophantine condition allows us only to invert the "derivative" of $\mathcal{F}$ in a weak sense (i.e. with loss of a finite number of derivatives on the target space of the inverse).

Warning. Examples in [AK] show that an area preserving C^∞-diffeomorphism of the disk D^2 with an elliptic fixed point such that ω is a *Liouville number*, too well approximated by rational numbers, may have a very wild dynamics, with dense orbits.

5.2 Periodic orbits, Aubry-Mather sets and homoclinic tangles

The curves Γ_{r_0} given by theorem 3 form a Cantor family for which 0 is a density point (the relative measure of the Cantor set in smaller and smaller neighborhoods of 0 tends to 1). Nevertheless, this is far from being the whole story. The dynamics of such a generic area preserving F in the complement of the invariant curves (the so-called *Birkhoff domains of instability*) is extremely complicated and, if the works of Birkhoff, Aubry, Mather, Herman, have shed considerable light on the way invariant circles of the normal form break (periodic points, invariant Cantor sets, see [C7]), many questions remain open.

Some of the complexity of a generic area preserving map of the disc is roughly suggested in figure 8. This figure, taken from [C1], originates from [C8]. It illustrates the dynamics of the monotone twist map of the annulus which arises when studying the *restricted three-body problem* at high values of the Jacobi constant (see [C2] for explanations). To the periodic points are attached invariant stable (resp. unstable) manifolds along which the images of a point under the positive (resp. negative) iterates of F converge exponentially fast to the periodic orbit. The homoclinic tangles (see [S, C1]) created by the intersections of such invariant manifolds produce invariant Cantor sets on which the dynamics of F is the same as the one of throwing a dice (more technically, a Bernoulli shift, see [C0, KH]) and hence possesses positive *topological entropy*. Also, orbits go from one boundary of a domain of instability to the other, but their *diffusion* is blocked by the invariant curves.

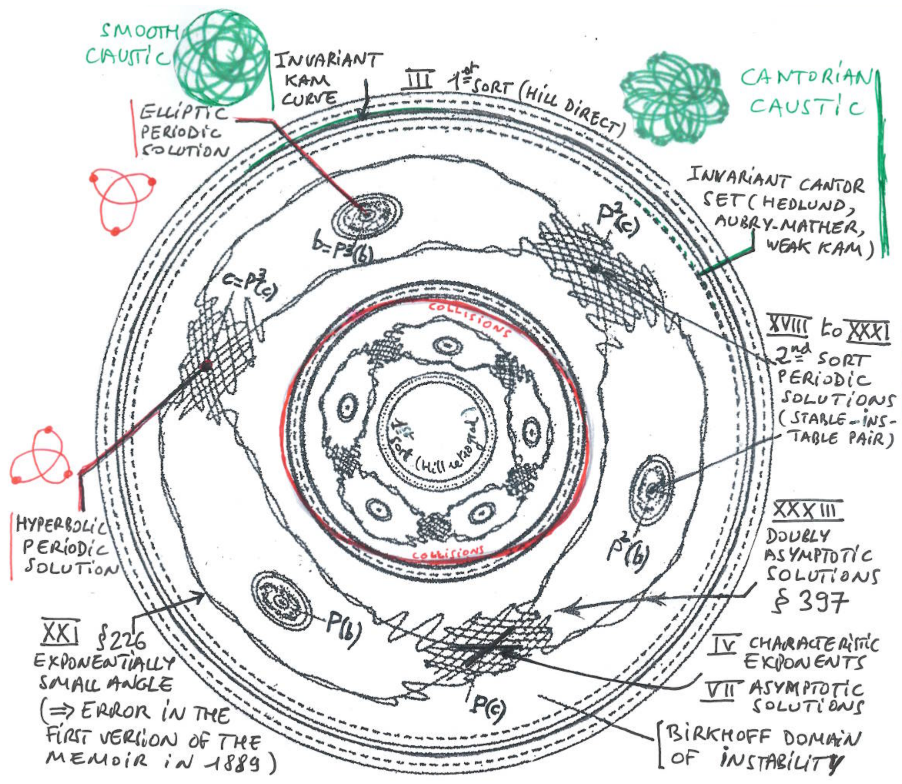

Figure 8. The return map of the restricted 3-body problem at high Jacobi constant (figure reproduced (slightly modified) with the kind permission of *Encyclopædia Universalis*).

Remark. One can check ([K, C2]) that Moser's invariant curve theorem applies to the Poincaré first return map on a surface of section of the planar circular restricted three body problem with any large enough energy in the rotating frame (i.e. *Jacobi constant*). This implies stability in a strong sense as the invariant tori corresponding to the invariant closed curves are of codimension 1 in the energy surface and hence serve as barriers confining the solutions. This is precisely because he lacked such a theorem that Poincaré tried to prove such a stability result using barriers made from invariant manifolds of periodic orbits, which lead to the famous error in the first version of his prize winning Memoir on the Three-body problem (see [C1]).

6 When radial and angular behaviours compete

Area preserving maps form a subspace of infinite codimension within the set of all smooth maps and the same is true of rotations. If one views Neimark-Sacker bifurcation as an unfolding, due to the nonlinear terms, of the continuum of circles invariant by the rotation along the parameter μ (fig. 3), the infinite codimension reflects the infinite number of events which happen for one and the same map while generically they happen for different values of μ. In a similar but subtler way one shows ([C3], summarized in [C4, C5, Y, AP]) that the whole complexity of the dynamics of an area preserving map happens unfolded along some curve Γ of the parameter space in generic 2-parameter families $F_{\mu,a}$ of diffeomorphisms of the plane in the neighborhood of a degenerate elliptic fixed point which is a *very weak attractor*. Along the lines of section 2, provided $F = F_{0,0}$ satisfies the non resonance relations $\lambda^k \neq 1$ for all integers $1 \leq k \leq 6$, such a family can be written

$$\begin{cases} F_{\mu,a}(z) = N_{\mu,a} + O(|z|^6), \quad \text{where} \\ N_{\mu,a}(z) = z\left(1+\mu+a|z|^2+a_2(\mu,a)|z|^4\right) e^{2\pi i\left(b_0(\mu,a)+b_1(\mu,a)|z|^2+b_2(\mu,a)|z|^4\right)}, \\ a_2(0,0)=-1,\ b_1(0,0)\neq 0,\ b_1(0,0)+2\dfrac{\partial b_0}{\partial a}(0,0)\neq 0. \end{cases}$$

Figure 9 shows the dynamics of $N_{\mu,a}$ in the different regions of the parameter plane around $(0,0)$. Along the curve Γ, $N_{\mu,a}$possesses a non normally hyperbolic invariant curve, attracting from the outside and repelling from the inside. This is in some sense the closest dissipative approximation to an invariant curve of an area preserving normal form[6].

[6]All figures in this section are reproduced with the kind permission of *Publications mathématiques de l'IHÉS*.

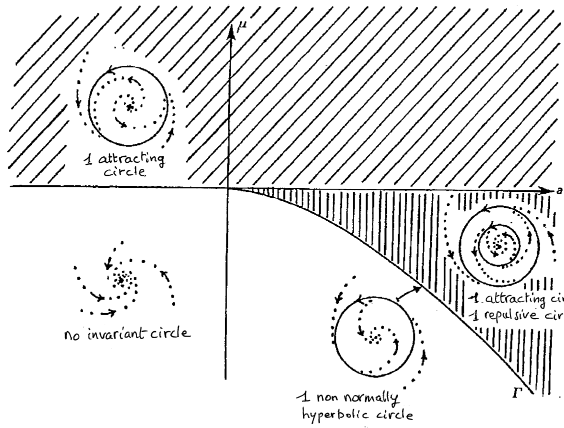

Figure 9. Dynamics of $N_{\mu,a}$

The complement of some cusp neighborhood of Γ belongs to the *hyperbolic domain*: here, the "normal" dynamics of $F_{\mu,a}$ is similar to the one of $N_{\mu,a}$ and the methods of proof are the ones of section 4. On the contrary, in the cusp domain along Γ, the control is more on the angular dynamics of $F_{\mu,a}$ and the methods of proof are the ones of section 5. This is a first approximation of the *elliptic domain*.

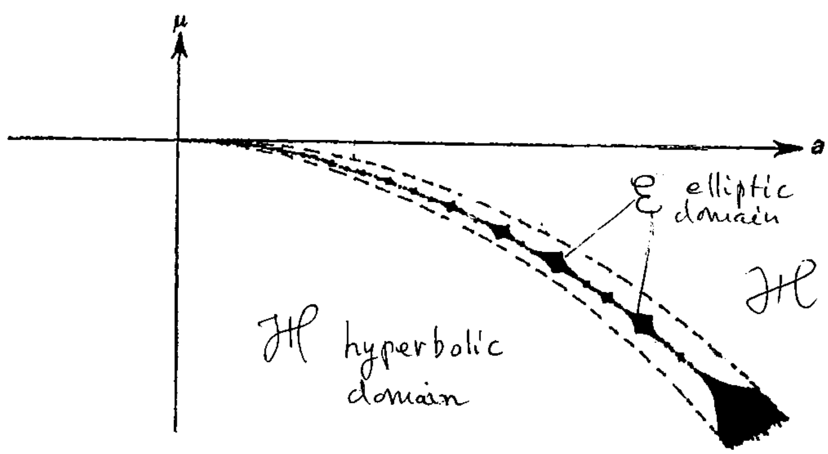

Figure 10. Hyperbolic and elliptic domains

More precisely, for a Cantor set of points (μ, a) near Γ the dynamics of $F_{\mu,a}$ is similar both in radial and angular directions to the one of $N_{\mu',a'}$ for some $(\mu', a') \in \Gamma$. Moreover, the hyperbolic domain extends to the complement of a countable number of bubbles having this Cantor set in their closure. The union of these bubbles is precisely the elliptic domain, the only place where complicated dynamics occurs. Figure 11, to be compared to figure 3, shows that one can describe heuristically the dynamics of $F_{\mu,a}$ along this elliptic domain as the unfolding of the dynamics of a generic area preserving map as represented in figure 8.

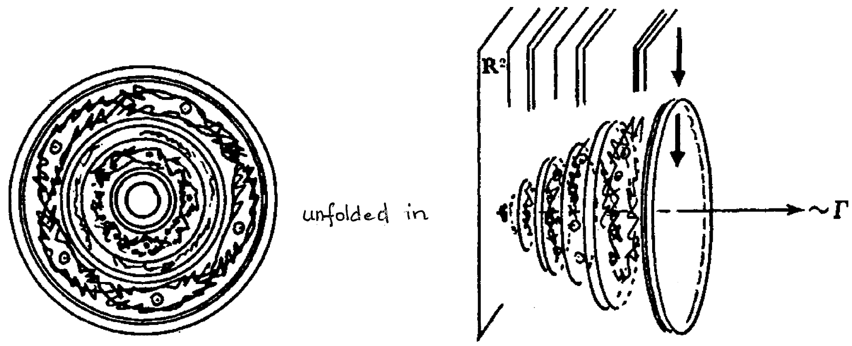

Figure 11. "Unfolding" the dynamics of a monotone twist

Finally, in the neighborhood of an elliptic fixed point, generic one-parameter families of planar diffeomorphims displaying the elimination of a pair of invariant closed curves, one repelling and one attracting, may be thought of as being the dissipative analogues of the invariant subsets of a generic area preserving diffeomorphism : in particular, to the Cantor set of KAM curves corresponds a Cantor set of families along which the elimination proceedes as simply as in the case of normal forms (or equivalently of time one maps of differential equations) with a single value of the parameter for which the diffeomorphism posesses a smooth invariant closed curve which is non normally hyperbolic and on which $F_{\mu,a}$ is smoothly conjugate to a diophantine rotation, while to the well ordered periodic orbits with rational rotation numbers p/q such that q is not too large with respect to the distance of the orbit to the fixed point 0, correspond one-parameter families along which the elimination process, much more complicated, is represented on figure 12.

The condition on p/q amounts to asking that in some annulus A containing the periodic points of rotation number p/q, the qth iterate $F^q_{\mu,a}$ of the map still be a small perturbation of the qth iterate $N^q_{\mu,a}$ of its normal form. The said periodic points are then interpreted as the trace left by a nearby resonant elliptic fixed point (compare section 4.2) and resonant normal forms provide local coordinates (θ, y) in the annulus A which make the one parameter subfamily $F_{\mu,a}$ depicted in figure 12 appear as a perturbation of the composition of the rotation $R_{p/q}$ (of angle $2\pi p/q$) with the time 1 map of a differential equation of the form

$$\frac{d\theta}{dt} = y, \quad \frac{dy}{dt} = \alpha + \gamma y^2 + \delta \cos 2\pi q\theta,$$

where $\gamma < 0$ and $\delta > 0$ are fixed and α is the parameter.

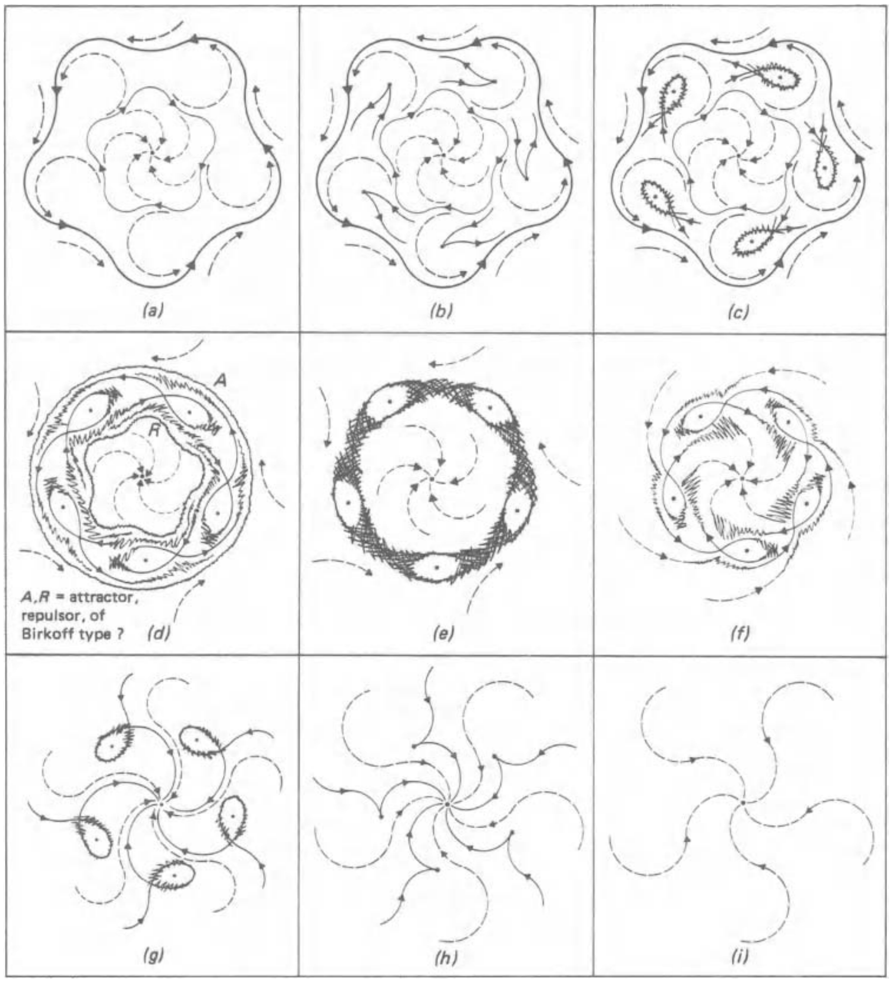

Figure 12. Resonant elimination of a pair on invariant curves (from [C3]III)

Finally, a surprizing consequence of this study is the strong organizing power of diophantine rotation numbers: if some $F_{\mu,a}$ possesses a closed invariant curve encircling 0 on which it induces a diffeomorphism with such a rotation number, it behaves like a normal form in a uniform (independent of (μ, a)) neighborhood of the origin, the sole possibly more complicated dynamics occuring in restriction to the second invariant closed curve when that curve exists.

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