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Some facts and more questions about the "Eight"

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Abstract. I discuss some properties of the "Eight" solution of the three-body problem, many of them conjectural. I describe in particular a simple approach to the P_{12} family, proposed by C. Marchal, which is a choreography in the rotating frame with the same 12-fold symmetry as the "Eight". .

I - Introduction

I-1 The equal mass 3-body problem in $\mathbb{R}^2$ ([C1],[C2]). We consider three bodies of unit mass in $\mathbb{R}^2$. As we are interested in periodic solutions, we suppose from the start that the center of mass is fixed at the origin. Hence, the configuration space is the open subset $\hat{\mathcal{X}}$ of

$$\mathcal{X} = \left\{x = (\vec{r}_1, \vec{r}_2, \vec{r}_3) \in (\mathbb{R}^2)^3,\ \sum_{i=1}^{3} \vec{r}_i = 0\right\}$$

defined by the condition of "no collision" : $\forall i \neq j,\ \vec{r}_i \neq \vec{r}_j$. The vector space $\mathcal{X}$ is endowed with the "mass metric", which coincides here with the standard euclidean metric:

$$I(x) = ||x||^2 = \sum_{i=1}^{3} ||\vec{r}_i||^2.$$

Newton-Lagrange equations can now be written $\ddot{x} = \nabla U(x)$, where U is the Newtonian potential

$$U(x) = \sum_{i<j} \frac{1}{||\vec{r}_i - \vec{r}_j||}.$$

It is well known that they are the Euler-Lagrange equations of the *action* $\mathcal{A}$ which associates to a C^1 path $x : [t_0, t_1] \to \hat{\mathcal{X}}$ the real number

$$\mathcal{A}\big(x(t)\big) = \int_{t_0}^{t_1} \left(\frac{||\dot{x}(t)||^2}{2} + U\big(x(t)\big) \right) dt.$$

I-2 The space of oriented triangles ([AC],[CM],[C4]). The squares a, b, c of the lengths of the sides of a triangle are a good set of parameters of the set of all triangles up to isometries. We fix the size by setting $a + b + c = 1$, which amounts to fixing the moment of inertia, with respect to the center of mass, of three equal masses located at the vertices of the triangle. Not all the points of the simplex $a + b + c = 1,\ a \geq 0, b \geq 0, c \geq 0$ represent triangles. The true triangles correspond to the points inside the disc defined by the inequation

$$16S^2 = 2ab + 2bc + 2ca - a^2 - b^2 - c^2 \geq 0,$$

which asserts the positivity of the squared area S of the triangle, a formula going back to Heron of Alexandria. The positivity of S^2 is equivalent to the three triangular inequalities.

Gluing 2 discs together, we represent all similitude classes of *oriented* triangles by a 2-sphere (a more “metric” construction is given in [CM]). This sphere – the so-called “shape sphere” – is represented on figure 1 with the action of its symmetry group, the dihedral group

$$D_6 = \{s, \sigma \mid s^6 = 1,\, \sigma^2 = 1,\, s\sigma = \sigma s^{-1}\}.$$

The North and South poles represent respectively the positively and negatively oriented equilateral triangle, the equator represents the flat triangles and the three vertical planes represent the isosceles triangles while C_1, C_2, C_3 are the collision half-lines.

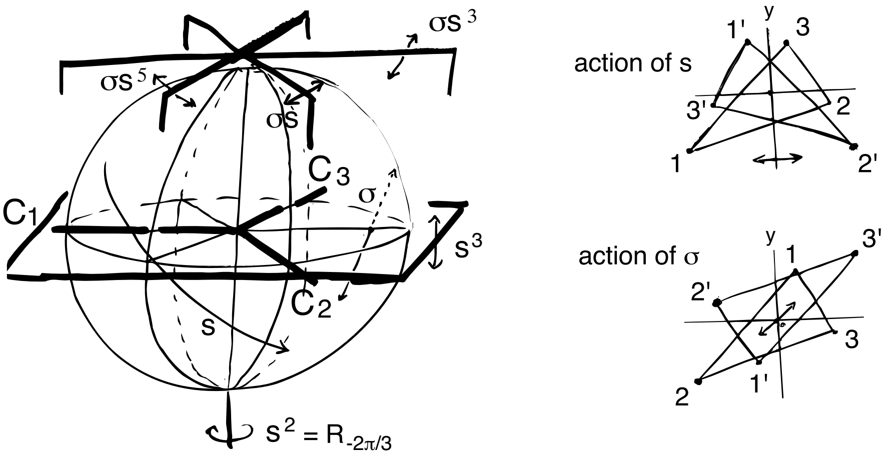


Figure 1 (the shape sphere and the action of D_6)

I-3 The action of the dihedral group D_6 on the loop space ([C2]). The actions of the two generators on a loop $x : \mathbb{R}/T\mathbb{Z} \to \mathcal{X}$ are defined by the following formulas: if $x(t) = \big(\vec{r}_1(t), \vec{r}_2(t), \vec{r}_3(t)\big)$,

$$(s \cdot x)(t) = \big(\Sigma\vec{r}_3(t - T/6), \Sigma\vec{r}_1(t - T/6), \Sigma\vec{r}_2(t - T/6)\big),$$
$$(\sigma \cdot x)(t) = \big(-\vec{r}_1(-t), -\vec{r}_3(-t), -\vec{r}_2(-t)\big),$$

where Σ is the symmetry with respect to the y axis in R^2.

I-4 Choreographies. As the subgroup $\mathbb{Z}/3\mathbb{Z}$ generated by s^2 acts by cyclically permuting the bodies after one-third of a period, each D_6-invariant loop is a *choreography*, that is a loop of the form $(q(t), q(t+T/3), q(t+2T/3)$, where $q : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ is a parametrized plane curve. More generally, in [CGMS] and [S2] the authors call (simple) *choreography* a loop in the configuration space of the n-body problem in $\mathbb{R}^k$ of the form

$$\big(q(t), q(t+T/n), \cdots, q(t+(n-1)T/n)\big),$$

where q is a parametrized closed curve in $\mathbb{R}^k$.

II - The “Eight” ([Mo,][CM],[C2],[Ch],[ZZ2]).

FACT 1: it exists. The existence of a minimizer of the Lagrangian action in the space Λ_{D_6} of D_6-invariant loops in the configuration space is a consequence of Tonelli’s theory:

coercivity comes from the D_6-invariance which forces the length – and hence the action – of an element of Λ_{D_6} to be big as soon as some part of the loop goes far away ("tied" class in the sense of Gordon, see [M1]). The main point is to show that such a minimizer is collision-free, which implies that it is a solution of the three-body problem and even a choreography supported by an eight-shaped curve. Accurate pictures were obtained numerically by Carles Simó

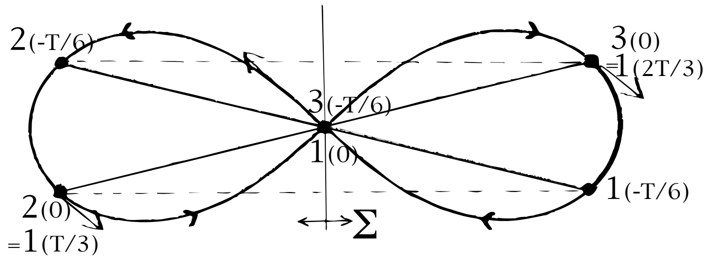


Figure 2 ("The" Eight)

The proof boils down to showing the inequality $\mathcal{A}_{coll} > \mathcal{A}_{test}$, where $\mathcal{A}_{coll}$ is a lower bound for the action of a D_6-invariant loop undergoing at least one collision and $\mathcal{A}_{test}$ is the action of a collision-free equipotential test path (see [CM]). The best estimate for $\mathcal{A}_{coll}$ (better than the ones in [CM]: $\mathcal{A}_{coll} = 2^{\frac{1}{3}} A_2$, or even in [Ch]) is found in [ZZ2]: thanks to a formula of Leibnitz, the action of a 3-body problem splits into the sum of three terms, each of which is one third of the action of the Kepler problem with attraction constant equal to the total mass $M = 3$ (see [V1],[ZZ1]):

$$\mathcal{A}\big(x(t)\big) = \frac{1}{3}\sum_{i<j}\int_0^T \left[\frac{||\dot{\vec{r}}_i(t) - \dot{\vec{r}}_j(t)||^2}{2} + \frac{3}{||\vec{r}_i(t) - \vec{r}_j(t)||}\right] dt.$$

As the configurations at t and $t+T/2$ are symmetric with respect to the $0y$ axis (compute the action of s^3), any collision which occurs at t_0 occurs also at $t_0 + T/2$. The lower bound of the Kepler action during a period T is then twice the minimum of the Kepler action of an ejection-collision wih attraction constant 3 and period $T/2$. This is equivalent to $\mathcal{A}_{coll} = \mathcal{A}(2L(T/2))$, where $2L(T/2)$ is the equilateral relative equilibrium which makes two complete turns in one period T. An estimate for $\mathcal{A}_{test}$ is given in [CM].

QUESTION 1: but is it unique ? Numerical evidence by Carles Simó in [S1] suggests unicity of the minimizer. From now on we shall indulge in speaking of "the" Eight.

QUESTION 2: is each lobe convex? in [CM] we prove only that each lobe of a minimizer is starshaped (the problem is near the crossing point, convexity outside a neighborhood of this point is easy to prove).

QUESTION 3: is it a transcendental curve? C. Simó ([S1],[S3]) computed numerically truncated Fourier expansions of the components of the Eight. We take the period $T = 2\pi$. Because of the symmetries, the curve $q(t)$ is of the form:

$$q(t) = \left(\sum_{j odd} a_j \sin jt, \sum_{j even} b_j \sin jt\right),$$

with $a_j = 0$ if $j \equiv 3$ mod. 6 and $b_j = 0$ if $j \equiv 0$ mod. 6. Because of the missing terms and the quick decrease with j of the non-zero ones, one gets a good approximation of the Eight by the *quartic* $t \mapsto \big(a_1 \sin t, b_2 \sin 2t + b_4 \sin 4t\big)$. Indeed, the next terms are $(a_5 \sin 5t, b_8 \sin 8t)$, with a_5 of the order of 2×10^{-2} and b_8 of the order of 10^{-4}. Going to order 30 is enough to have all harmonics larger than 10^{-16}. Finally, fits of the solution by algebraic curves $g(x, y) = 0$ up to degree 12 definitely show disagreements which, according to C. Simó, cannot be due to computational errors.

FACT 2: existence of a priori less symmetric "Eights" ([C3]). The $\mathbb{Z}/3\mathbb{Z}$ symmetry is certainly not enough to characterize the eight because it follows from [CD] that minimizing among choreographies of three equal masses leads to Lagrange equilateral relative equilibrium. The dihedral group D_6 has three subgroups which contain $\mathbb{Z}/3\mathbb{Z}$. The one generated by s^2 and σs, isomorphic to D_3, is not interesting for our purpose because every element acts on the shape sphere preserving the orientation of the triangles (in particular, the Lagrange equilateral relative equilibrium is symmetric under the action of this subgroup). The two others are respectively generated by s^2 and σ (isomorphic to D_3) and by s (isomorphic to $\mathbb{Z}/6\mathbb{Z}$). It is shown in [C3] that a minimizer of the action among D_3 or $\mathbb{Z}/6\mathbb{Z}$-invariant loops is collision-free.

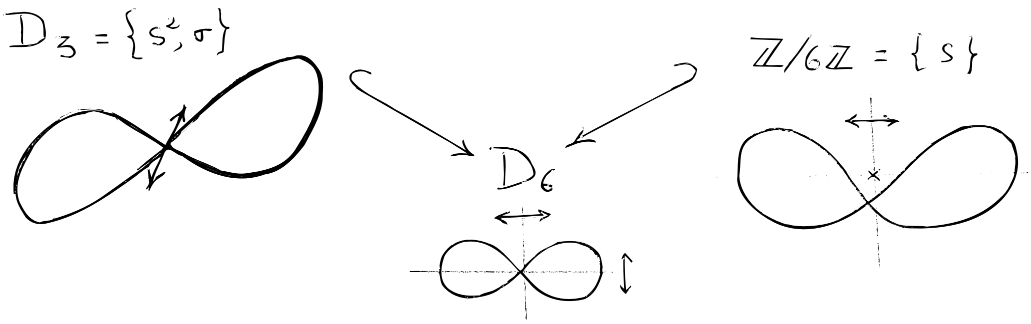


Figure 3 (less symmetric "Eights" ?)

QUESTION 4: all the same ? This is of course intimately linked to the unicity question. For more on possible extra-symmetries of action minimizers, see [C2],[C3],[V2].

III - The spatial case

III-1 Action of D_6 on spatial configurations ([C3]) Let Δ be the axis $0x$ and P, Σ be respectively the coordinate planes $0xz$ and $0xy$ (see figure 4). We extend to loops of spatial configurations of three bodies the action of D_6 by replacing the symmetry with respect to 0 (resp. 0x, resp. Oy) by the symmetry with respect to Δ (resp. P, resp. Σ):

$$(s \cdot x)(t) = \big(\Sigma\vec{r}_3(t - T/6), \Sigma\vec{r}_1(t - T/6), \Sigma\vec{r}_2(t - T/6)\big),$$
$$(\sigma \cdot x)(t) = \big(\Delta\vec{r}_1(-t), \Delta\vec{r}_3(-t), \Delta\vec{r}_2(-t)\big),$$

where Σ (resp. Δ) denotes the symmetry with respect to the horizontal plane Σ (resp. to the line Δ). The Eight in the plane orthogonal to Δ is an obvious example of an invariant loop, but there is also the equilateral relative equilibrium $2L(T/2)$ which makes two full turns in one period in the horizontal plane (figure 4).

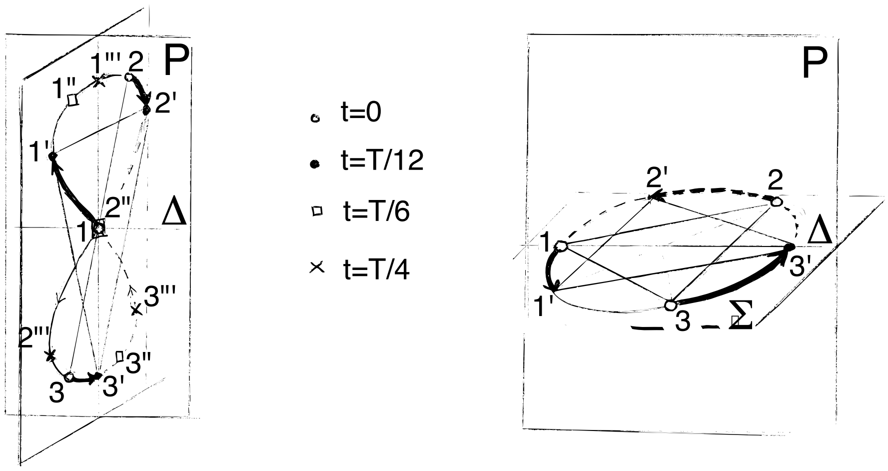


Figure 4 (Two examples of D_6-invariant spatial loops)

Moreover the lower bound given for the action D_6-symmetric loops with collision in the plane still holds in space.

QUESTION 5: is a minimizer in $\Lambda_{D_6}^{spatial}$ necessarily planar? i.e. of angular momentum zero, i.e. an "Eight"?

III-2 The P_{12} family ([Ma],[C3]). This family continues the Eight solution in three-space up to Lagrange equilateral solution, through choreographies in a rotating frame. It is described in detail in [C3]. It is parametrized by an angle u between 0 and $\frac{\pi}{6}$: the solution labeled by u is supposed to minimize the action in fixed time $T/12$ between configurations which are symmetric with respect to the line Δ with $0 \in \Delta$ and configurations which are symmetric with respect to a vertical plane P through the origin which contains body 2 and makes angle u with Δ. In a frame rotating around the vertical axis of an angle $-u$ in time $T/12$, one gets a family of D_6-symmetric choreographies of period T which connects the two examples depicted on figure 4 (the Eight and twice Lagrange $2L(T/2)$) via progressive folding in the direction of Δ (see [C3] fig. 1 and 2, and [N] fig. 5).

FACT 3: existence of a family. For all values of u, minimizers are collision-free

The surprise is that, using as a model the horizontal Lagrange family x_u (which satisfies the symmetry requirements), one can give a simple direct proof of the absence of collisions in a minimizer of the action in $\Lambda_{D_6}^{spatial}$, simpler than in the planar case:

1) the action of an admissible path undergoing a collision is bigger than the action $\hat{A}_2 = 2^{-\frac{5}{3}}3^{\frac{2}{3}}\pi^{\frac{2}{3}}T^{\frac{1}{3}}$ (masses $=1$) of the horizontal relative equilibrium solution x_0 of an equilateral triangle which rotates by $\frac{\pi}{3}$ in the same amount of time $T/12$;

2) this last action is, for any $u \leq \frac{\pi}{3}$, bigger than the one $A(u) = \hat{A}_2\big[\frac{3}{\pi}(\frac{\pi}{3} - u)\big]^{\frac{2}{3}}$ of the horizontal relative equilibrium solution x_u of an equilateral triangle which rotates by an angle $(\frac{\pi}{3} - u)$ during the given amount of time.

Finally, we prove that, for $0 \leq u < \frac{\pi}{6}$, the Lagrange solution x_u is not a minimizer. This is because the value $d^2\mathcal{A}(x_u)(\xi,\xi)$ of the Hessian of the action on the vertical variation

$$\xi = \left(\sin(\frac{2\pi t}{T}),\ \sin(\frac{2\pi t}{T} + \frac{2\pi}{3}),\ \sin(\frac{2\pi t}{T} + \frac{4\pi}{3})\right)$$

which "opens" x_u in the direction of the Eight, is negative for $u < \pi/6$ and positive for $u > \pi/6$. Indeed, the Hessian of x_u is positive when $\pi/6 < u \leq \pi/3$, which supports Marchal's claim that x_u is the minimizer when $\pi/6 \leq u \leq \pi/3$ (notice that its size increases to infinity and its action decreases to 0 when u tends to $\pi/3$). To be sure that this family really connects the Lagrange and Eight solutions, we need answering positively Questions 5 and 6.

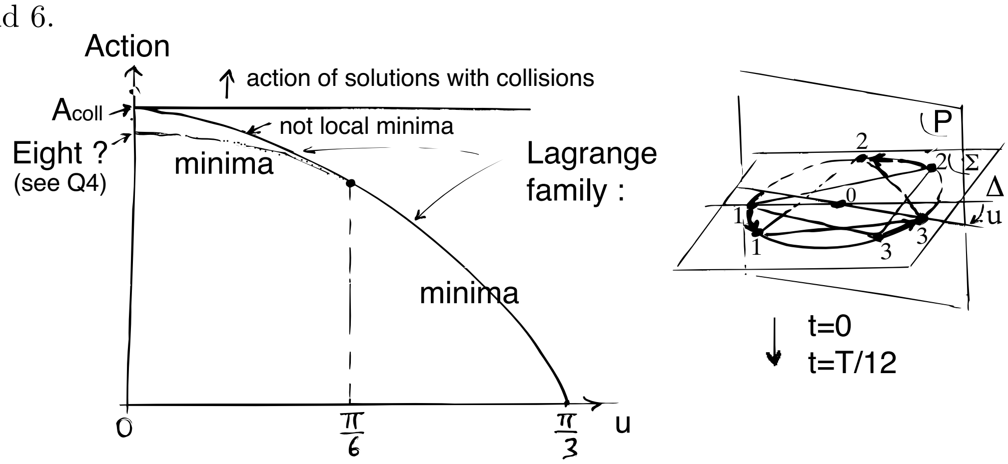


Figure 5 (Bifurcation of the P_{12} family from the Lagrange family)

QUESTION 6: unicity of minimizer for any value of the parameter u? or at least continuity of a minimizing family? Such continuity would imply the existence among the family of spatial (non-planar) 3-body choreographies in the fixed frame. Indeed, for well-chosen values of u, the period of the rotating frame and the one of the solution in the rotating frame would be resonant.

III-3 Other continuations in a rotating frame. The first continuation of the Eight into a family of rotating planar choreographies was given – up to the first orbit encountering a collision – by Michel Hénon [CGMS], using the same program as in [H]. The continuation beyond this orbit can be found is [S1]. A third family should exist, rotating around an axis orthogonal to the first two.

IV - Fixing homology

FACT 4: homology class of the eight is $(0,0,0)$. This means that, during a period, each side of the triangle has zero total rotation. Hence the eight does not minimize the action in its homology class: the minimum, equal to 0, is attained for still bodies infinitely far from each other (for the case of homology class $(1,1,1)$, see Poincaré [P] 1896 and Venturelli 2001 [V1], [ZZ1]).

QUESTION 7: does the eight minimize the action among choreographies in its homology class? What makes this question hard is the mixture of topological and symmetry constraints

V - Fixing homotopy. The homotopy class of a loop in the configuration space of the planar n-body problem may be thought of as the *braid* described by the bodies in (periodic) space time $\mathbb{R}^2 \times (\mathbb{R}/T\mathbb{Z})$. Knowing that each lobe of the Eight is starshaped is enough to imply the following:

FACT 5: the braid defined by the Eight is the "Borromean rings" ([Ber],[C3]). This is the signature of a truly triple interaction.

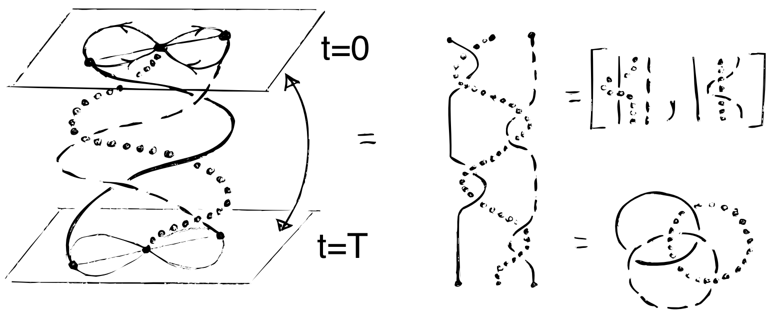


Figure 6 (The braid defined by the Eight)

FACT 6 ([C2],[M2]): the eight does not minimize the action in its homotopy class

QUESTION 8: does the eight minimize the action among choreographies in its homotopy class? And are there other choreographies in this homotopy class?

Remark. An interesting example of mixed conditions (topology and symmetry) for a minimization problem may be found in [V2] where generalizations of the Hip-Hop lead to spatial choreographies of 4 equal masses. But, as for most choreographies, no proof was found of the existence of Gerver's "supereight" with four equal masses [CGMS],[C2].

VI - Stability

FACT 7: numerically, the "Eight" is KAM stable. A numerical computation of a Poincaré map to high order around the fixed point corresponding to the "Eight" and the subsequent computation of the normal form shows that one can apply KAM theorem and, hence, that the "Eight" is KAM stable on the manifold of zero angular momentum ([S1]). For theoretical works on the stability properties of action minimizers, see [Ar],[Bi],[O].

QUESTION 9: Give a detailed proof of the KAM stability of the "Eight".

VII - Masses ([C5],[C6],[BCS])

FACT 8: a choreography with $n \leq 5$ bodies must have equal masses. This is proved in [C6] using the remark that if a choreography is solution of the n-body problem with masses $m_1, m_2, \ldots, m_n$, it is also solution of the n-body problem with masses $\mu, \mu, \ldots, \mu$, where $\mu = (m_1 + m_2 + \cdots + m_n)/n$.

QUESTION 10: is the same true for any number of bodies? in particular $n = 6$?

VIII - "Eights" with more bodies and limit when the number $n = 2p + 1$ of bodies tends to $+\infty$ ([S2],[C2]). Eight-shaped choreographies exist numerically with any number n of bodies. If n is odd, the eight-curve has the full $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ symmetry (see figure 7) but if n is even, it has only a $\mathbb{Z}/2\mathbb{Z}$ symmetry (the two lobes are unequal ([CGMS] figure 3a)).
In [C2], the action of D_6 on the loop space of the configuration space of the equal mass three-body problem is extended, for n *odd*, to an action of the dihedral group D_{2n} on the

loop space of the configuration space of the equal mass n-body problem. We call $\Lambda_{D_{2n}}$ the subspace of invariant loops under this action.

QUESTION 11: Is an Eight with n (odd) bodies an action minimizer in $\Lambda_{D_{2n}}$?

QUESTION 12: understand the limit of the Eight when n odd tends to $+\infty$. According to C. Simó [S3], the angle at the crossing point tends to $\pi/2$ and, for a given period, the size has a precise scaling law in n.

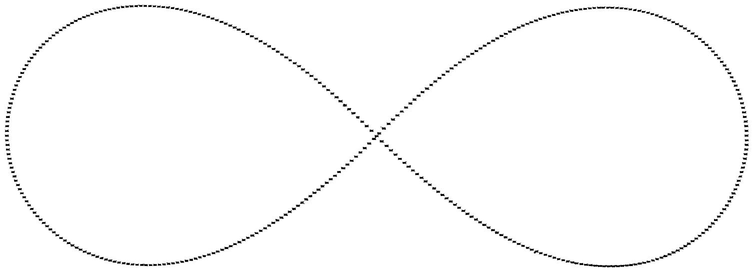
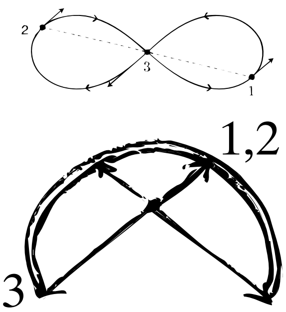


Figure 7 (399 bodies on a Eight, computed by C. Simó)

IX - Other potentials ([CGMS]). According to [Mo],[CGMS], the "Eight" exists for all potentials of the form r^α with $\alpha \in]-\infty, 0[$. When $\alpha = -2$ (Jacobi potential), it follows from the Lagrange-Jacobi identity that the energy of any periodic solution is necessarily equal to 0, and its moment of inertia $I = ||x(t)||^2$ is constant. For the Newtonian potential ($\alpha = -1$), it is a conjecture of D. Saari that a solution of the n-body problem can have a constant moment of inertia with respect to the center of mass only in the case it is a *relative equilibrium*, that is ([AC], Proposition 2.5) when the mutual distances stay constant along the motion (*rigid motion*). Numerically, the variations of I for the Newtonian Eight are of the order of 0.5% ([S3]).

QUESTION 13: show that the moment of inertia $I = ||x(t)||^2$ of the "Eight" stays constant only when $\alpha = -2$.

Two curiosities. Another nice property of the Eight, consequence of its high symmetry, is the shape of its hodograph (figure 8-1); also curious is the curve described by the *center of force* (see [W] p.) of the configuration (figure 8-2 from fig. 13 of [Br]).



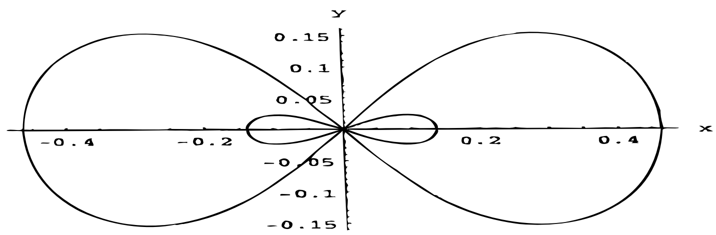


Figure 8-1 (the hodograph) *Figure 8-2 (the curve described by the center of force)*

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