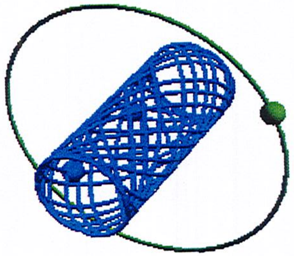
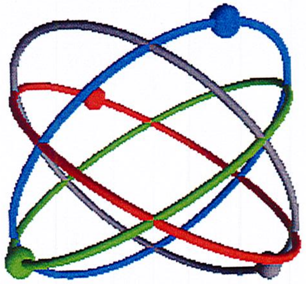
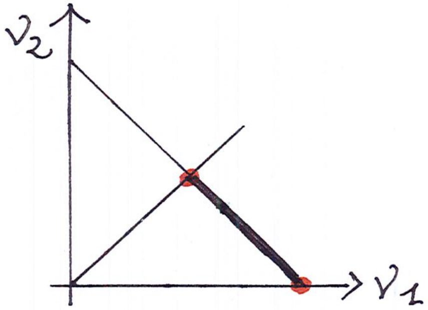
N-BODY RELATIVE EQUILIBRIA IN HIGHER DIMENSIONS

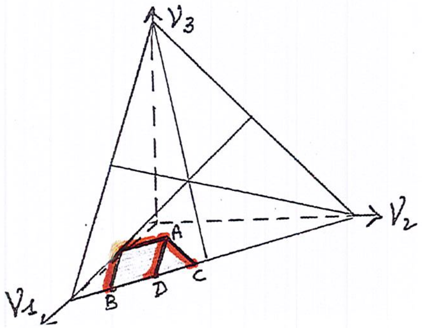


Alain Chenciner

IMCCE & PARIS 7





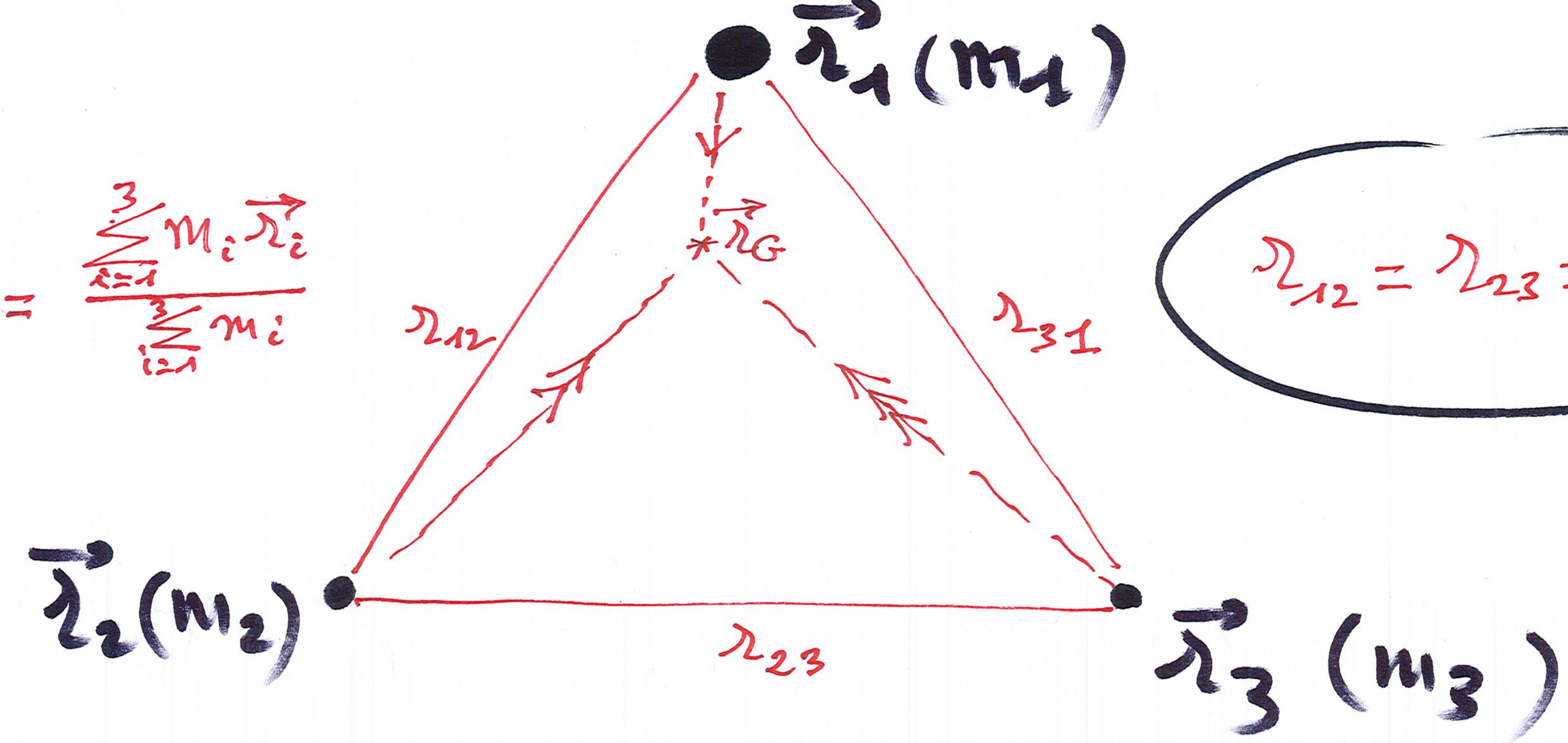


Venetia 18.22 June 2018

LAGRANGE 1772

$\exists\, 1!$ non colinear central configuration

$$\vec{r}_G = \frac{\sum_{i=1}^{3} m_i \vec{r}_i}{\sum_{i=1}^{3} m_i}$$



$$\boxed{r_{12} = r_{23} = r_{31}}$$

$\exists$ also 3 colinear C.C. (Euler)

PROOF

$$\forall i, \quad \ddot{\vec{r}}_i = \sum_{j \neq i} \frac{m_j (\vec{r}_j - \vec{r}_i)}{r_{ij}^3} \Longrightarrow \ddot{\vec{r}}_G = 0$$

$$\text{C.C.} \Longleftrightarrow \exists \lambda, \forall i, \quad \ddot{\overline{\vec{r}_i - \vec{r}_G}} = -\lambda (\vec{r}_i - \vec{r}_G)$$

$$\Updownarrow$$

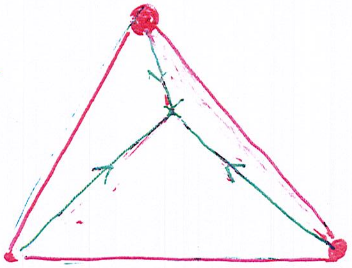
$$\forall i, \quad \sum_{j \neq i} m_j \left(\frac{1}{r_{ij}^3} - \frac{\lambda}{\sum_{j=1}^{3} m_j} \right) \underbrace{(\vec{r}_j - \vec{r}_i)}_{\underline{\text{independents}}} = 0$$

Each C.C. admits HOMOGRAPHIC MOTIONS

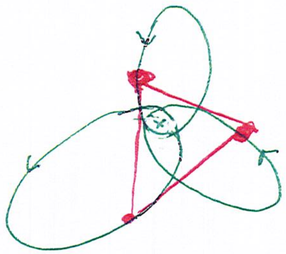
necessarily planar if in $\mathbb{R}^3$ (Lagrange)

in $(\mathbb{R}^2)^3 = (\mathbb{C})^3$, $$\boxed{x(t) = \zeta(t)\,x(0), \quad \ddot{\zeta}(t) = \underbrace{-\frac{k\,\zeta(t)}{|\zeta(t)|^3}}_{\text{Kepler}} \in \mathbb{C}}$$

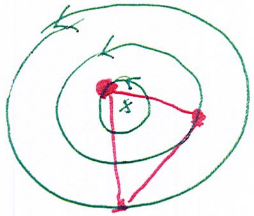
$$\begin{array}{c} x(t) \\ \| \\ \left(\vec{r}_1(t)-\vec{r}_G(t),\ \vec{r}_2(t)-\vec{r}_G(t),\ \vec{r}_3(t)-\vec{r}_G(t)\right) \end{array}$$



$e = 1$



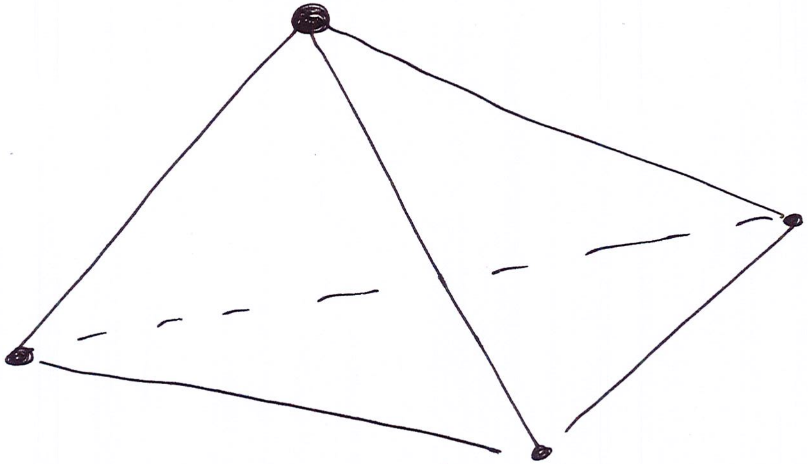
$0 < e < 1$



$e = 0$ (RELATIVE EQUILIBRIUM) ⟺ RIGID

EVERYTHING THE SAME FOR 4 BODIES

$\exists\, 1!$ NON COPLANAR C.C. ,



THE REGULAR TETRAHEDRON

EXCEPT THAT $\nexists$ HOMOGRAPHIC NON HOMOTHETIC MOTIONS IN $\mathbb{R}^3$

SUCH MOTIONS ARE ONLY POSSIBLE IN $\mathbb{R}^4$ or $\mathbb{R}^6$

N-BODY CONFIGURATIONS MOD. TRANSLATIONS IN AN EUCLIDEAN SPACE (E,ε)

A. Albouy & A.C.

$\mathcal{D}$ dispositions $:= \mathbb{R}^n / \mathbb{R}(1,\dots,1)$

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\cong"', "\mu"] \\ \mathcal{D}^* \end{tikzcd}$$

$\mathcal{D}^*$ codispositions $= \{\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \ \sum_{i=1}^{n} \xi_i = 0\}$

$$\mu(x_1,\dots,x_n) = \big(m_1(x_1 - x_G), \dots, m_n(x_n - x_G)\big)$$

$$\boxed{\begin{array}{l} X \in \mathcal{D} \underset{\mu\otimes\varepsilon}{\otimes} E \equiv \mathrm{Hom}(\mathcal{D}^*, E) \\ \qquad\qquad\qquad\qquad \xi \longmapsto \sum_{i=1}^{n} \xi_i \vec{r}_i \end{array}}$$

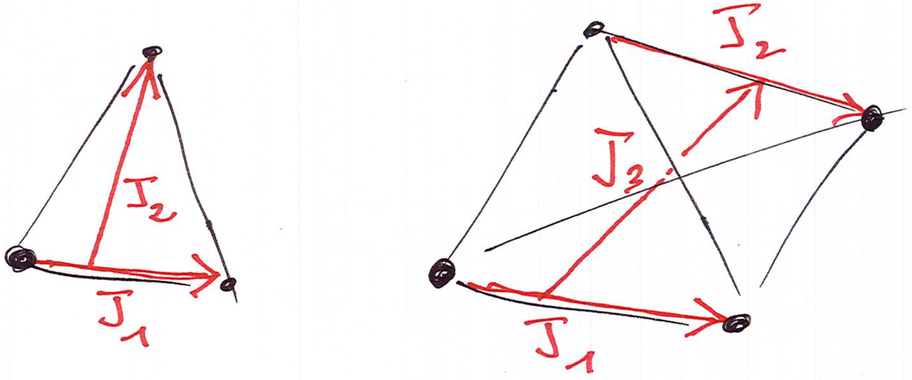
$\|$

mass scalar product: $(\vec{r}'_1,\dots,\vec{r}'_n)\cdot(\vec{r}''_1,\dots,\vec{r}''_n) = \sum_{i=1}^{n} m_i \langle \vec{r}'_i - \vec{r}'_G, \vec{r}''_i - \vec{r}''_G \rangle_E$

in orthogonal bases of $(\mathcal{D}^*, \mu^{-1})$ and (E, ε) :

$$X = \begin{array}{c} \begin{matrix} J_1 & \cdots & J_{n-1} \end{matrix} \\ \left(\begin{matrix} | & & | \\ | & \cdots & | \\ | & & | \end{matrix}\right) \end{array}$$

J_i Jacobi vectors



N.BODY CONFIGURATIONS MOD. ISOMETRIES

SIDE OF BODIES

$$\begin{tikzcd}
\mathcal{D}^* \arrow[r, "X"] \arrow[d] & E \arrow[r, phantom, "\overset{\varepsilon}{\equiv}"] & E^* \arrow[r, "X^{tr}"] & \mathcal{D} \arrow[r, phantom, "\overset{\mu}{\equiv}"] & \mathcal{D}^* \arrow[d] \\
\mathbb{R}^n \arrow[rrrr, "1! \text{ extension s.t.}"] & & & & \mathbb{R}^n
\end{tikzcd}$$

$$(m_1, \dots, m_n) \longmapsto (0, \dots, 0)$$

$$B = X^{tr} X \qquad \mu^{-1}\text{-symmetric}$$

$$\begin{pmatrix}
m_1 |\vec{r}_1 - \vec{r}_G|^2_{\varepsilon} & \cdots & m_1 \langle \vec{r}_1 - \vec{r}_G, \vec{r}_n - \vec{r}_G \rangle_{\varepsilon} \\
\cdots & \cdots & \cdots \\
m_n \langle \vec{r}_n - \vec{r}_G, \vec{r}_1 - \vec{r}_G \rangle_{\varepsilon} & \cdots & m_n |\vec{r}_n - \vec{r}_G|^2_{\varepsilon}
\end{pmatrix}$$

intrinsic inertia matrix

$O(E)$ - invariant

SIDE OF AMBIENT SPACE

$$\begin{tikzcd}
E \arrow[r, phantom, "\overset{\varepsilon}{\equiv}"] & E^* \arrow[r, "X^{tr}"] & \mathcal{D} \arrow[r, phantom, "\overset{\mu}{\equiv}"] & \mathcal{D}^* \arrow[r, "X"] & E
\end{tikzcd}$$

(dim. d)

$$\vec{r}_k - \vec{r}_G = (X_{1k}, \dots, X_{dk})$$

$$S = X X^{tr} \qquad \varepsilon\text{-symmetric}$$

$$\begin{pmatrix}
\sum_{k=1}^{n} m_k X_{1k}^2 & \cdots & \sum_{k=1}^{n} m_k X_{1k} X_{dk} \\
\cdots & \cdots & \cdots \\
\sum_{k=1}^{n} m_k X_{dk} X_{1k} & \cdots & \sum_{k=1}^{n} m_k X_{dk}^2
\end{pmatrix}$$

inertia matrix

$\underbrace{O(\mathcal{D}^*)}_{\text{democracy group}}$ - invariant

FORCES

$\forall i,\ \ddot{\vec{r}}_i = \sum_{j\neq i} m_j \frac{\vec{r}_j - \vec{r}_i}{r_{ij}^3}$ can be written as follows:

$$\begin{pmatrix} \ddot{\vec{r}}_1 & & \ddot{\vec{r}}_n \\ | & \cdots & | \end{pmatrix} = \begin{pmatrix} \vec{r}_1 & \cdots & \vec{r}_n \\ | & \cdots & | \end{pmatrix} \underbrace{\begin{pmatrix} -\sum_{l\neq 1}\frac{m_l}{r_{1l}^3} & \frac{m_1}{r_{12}^3} & \cdots\cdots & \frac{m_1}{r_{1n}^3} \\ - & - & - & - \\ \frac{m_n}{r_{n1}^3} & \frac{m_n}{r_{n2}^3} & \cdots\cdots & -\sum_{l\neq n}\frac{m_l}{r_{nl}^3} \end{pmatrix}}$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "1!\ \text{extension s.t.}"] & \mathbb{R}^n \\ (m_1,\dots,m_n) \arrow[r, mapsto] & (0,\dots,0) \\ \mathcal{D}^* \arrow[uu, hook] \arrow[r, "2A"] & \mathcal{D}^* \arrow[uu, hook] \end{tikzcd}$$

WINTNER-CONLEY ENDOMORPHISM

(μ^{-1} symmetric)

$\Downarrow$ mod. translations

$$\boxed{\ddot{X} = 2XA}$$

what is A ?

If $\hat{U}(B) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$,

$$\boxed{d\hat{U}(B)\Delta B = \operatorname{trace}(A\Delta B)}$$

RELATIVE EQUILIBRIA (= RIGID MOTIONS)

E = space where motion really takes place

Theorem (A. Albouy & A.C.) dim $E = d = 2p$ and

$\exists\, \Omega : E \to E$ ε-antisymmetric non degenerate independent of t

s.t.

$$\boxed{X(t) = e^{\Omega t} X(0)}$$

$$\Downarrow$$

$$\ddot{X} = \Omega^2 X = 2XA$$

SIDE OF BODIES

$$X^{tr}\Omega^2 X = 2BA$$

$$\Downarrow$$

BALANCED CONFIGURATIONS $\boxed{[A, B] = 0}$

$$\Updownarrow$$

B critical pt of $\hat{U}\big|_{\text{Isospectral}(B)}$

(recall C.C. $\Longleftrightarrow$ B critical pt of $\hat{U}\Big|_{\substack{I = \text{trace } B = \text{cste} \\ \text{rank } B = \text{cste}}}$

$$\Updownarrow$$

$$A|_{\text{Im}B} = \lambda \text{Id} \Rightarrow \underline{\Omega = \omega J,\ J^2 = -\text{Id}}$$

SIDE OF AMBIENT SPACE

$$\Omega^2 S = 2XAX^{tr}$$

$$\Downarrow$$

$$\boxed{[\Omega^2, S] = 0}$$

HOW MANY FREQUENCIES ?

B balanced $\Rightarrow$ $\exists$ orthogonal bases resp. of $(\mathcal{D}^*, \mu^{-1})$ and $(E, \mathcal{E})$

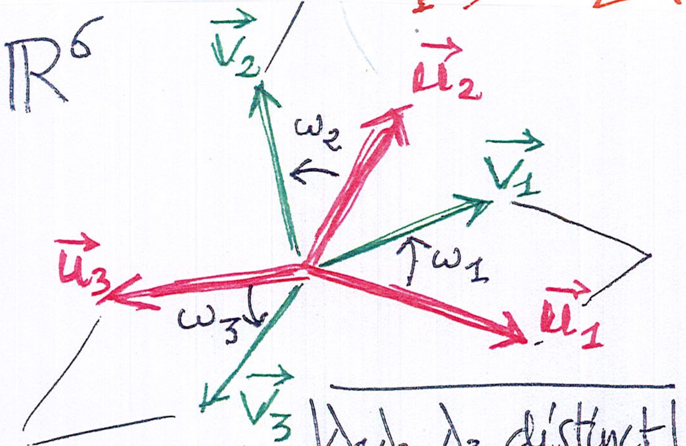
s.t. $2A = \begin{pmatrix} -\lambda_1 & & 0 \\ & \ddots & \\ 0 & & -\lambda_{n-1} \end{pmatrix}$, $B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_{n-1} \end{pmatrix}$, $\Omega^2 = \begin{pmatrix} -\omega_1^2 & & 0 \\ & \ddots & \\ 0 & & -\omega_d^2 \end{pmatrix}$, $S = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_d \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} \overset{\mathrm{Im}B}{V} & 0 \\ 0 & 0 \end{pmatrix} \mathrm{Im}S$, V invertible

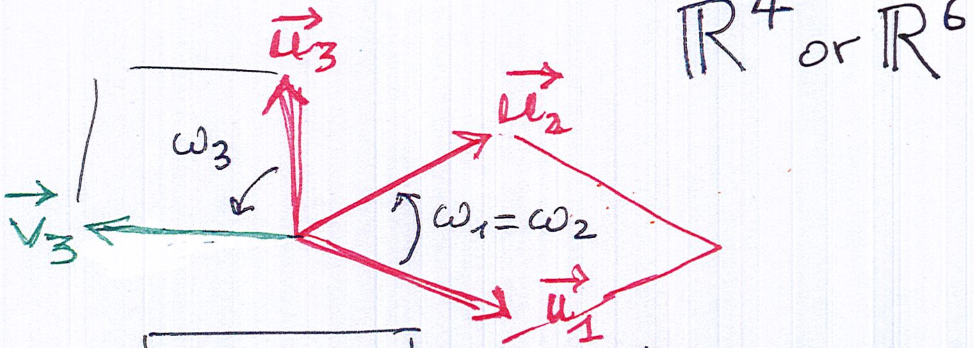
$\Rightarrow$ (after possible permutation of the basis of $\mathcal{D}^*$) $\boxed{\omega_i^2 = \underbrace{\lambda_i}_{\text{spectrum of } A|_{\mathrm{Im}B}}, \quad i = 1, \dots, \operatorname{rank} B}$

Case of a generic set of 4 masses forming a 3dim. configuration ($\Longleftrightarrow$ not 3 masses equal)

$\sigma_1 > \sigma_2 > \sigma_3$



$\boxed{\lambda_1, \lambda_2, \lambda_3 \text{ distinct}}$ (6-BALANCED)



$\boxed{\lambda_1 = \lambda_2}$ (4-BALANCED)

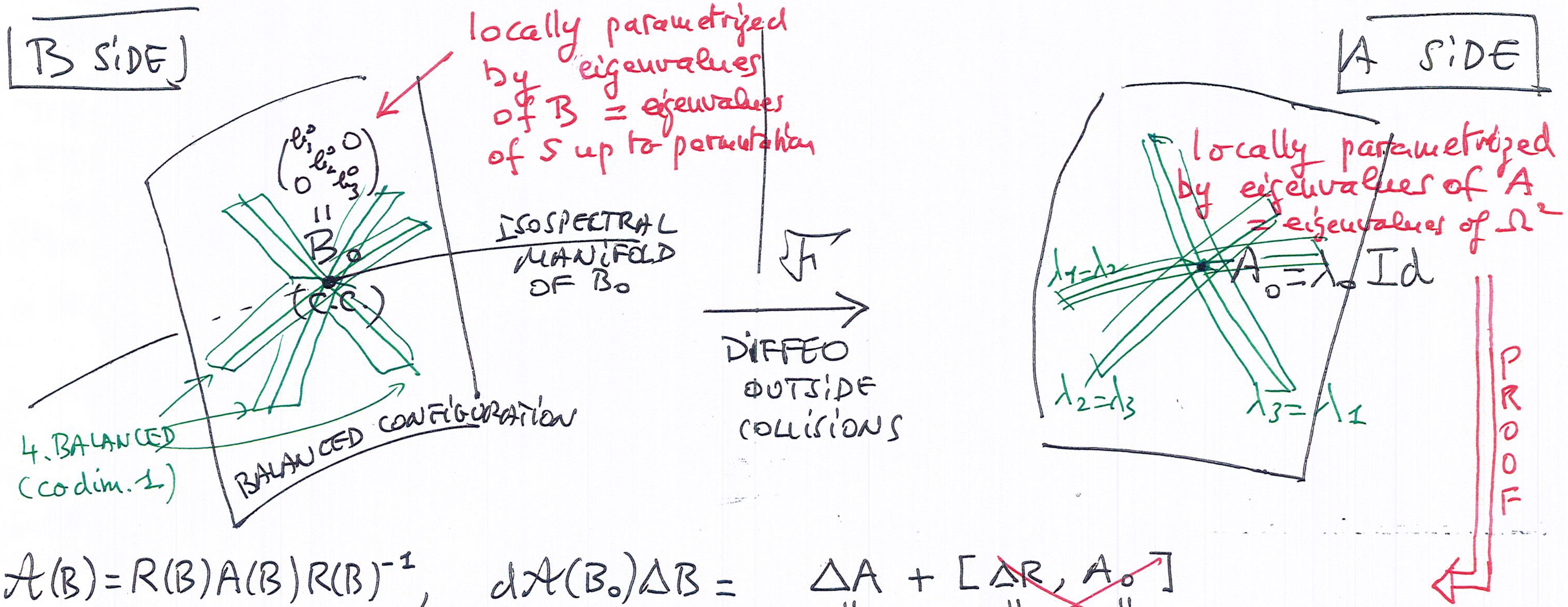
$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ common eigenbasis of S and Ω^2

c.c.
$\Updownarrow$
$\lambda_1 = \lambda_2 = \lambda_3$

BIFURCATION OF QUASI-PERIODIC RELATIVE EQUIL. FROM THE RELATIVE EQUIL. OF A REGULAR  WITH GENERIC MASSES (1) SHAPES

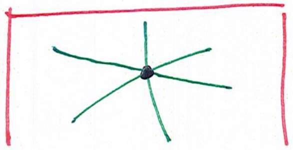
- IN THE UNIQUE ORTHONORMAL EIGENBASIS OF B_0, BOTH B_0 AND A_0 ARE DIAGONAL
- IF B BALANCED CLOSE TO B_0, ∃ 1! $R = R(B)$ SUCH THAT

$$RBR^{-1} = \mathrm{diag}(b_1, b_2, b_3)\ , \quad RAR^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$



IF $\mathcal{A}(B) = R(B)A(B)R(B)^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$,

$$d\mathcal{A}(B_0)\Delta B = \underset{\underset{\text{ISO}}{d\sqrt{F}(B_0)\Delta B}}{\underset{\shortparallel}{\Delta A}} + \cancel{\left[\underset{\underset{dR(B_0)\Delta B}{\shortparallel}}{\Delta R}, \underset{\underset{\lambda_0 \mathrm{Id}}{\shortparallel}}{A_0}\right]}$$

HENCE, AFTER SCALING 

BIFURCATION OF Q.P.R.E (2) ANGULAR MOMENTUM

$X(0)$ c.c. , $X(t) = e^{\omega J t} X(0)$ RELATIVE EQUILIBRIUM IN $E^{d=2p}$

$$\boxed{\mathcal{C} = -X\dot{X}^{tr} + \dot{X}X^{tr} = \omega\underbrace{(S_0 J + J S_0)}_{\text{J-skew hermitian}} = \omega J\underbrace{(J^{-1}S_0 J + S_0)}_{\text{J-hermitian}}}$$

FREQUENCY MAP $\boxed{\mathcal{F} : J \longmapsto \{\nu_1 \geq \dots \geq \nu_p\} = \text{spectrum}(J^{-1}S_0 J + S_0)}$

Theorem (A.C. & H. Jimenez-Perez) Im $\mathcal{F}$ is a convex polytope

$$\left\{(\nu_1, \dots, \nu_p),\ \sum_{i=1}^{p} \nu_i = I(X) = \text{trace } B\right\}$$

- More precisely: if $\left(\begin{array}{l}\text{Spectrum } S_0 = \{\sigma_1 \geq \dots \geq \sigma_{2p}\} \\ \vec{u}_1, \dots, \vec{u}_{2p} \text{ eigenbasis of } S_0\end{array}\right.$

$$\text{Im}\,\mathcal{F} = \text{Im}\left\{J \mid \{\vec{u}_1, \vec{u}_3, \dots, \vec{u}_{2p-1}\} \xrightarrow{J} \{\vec{u}_2, \vec{u}_4, \dots, \vec{u}_{2p}\}\right\}$$

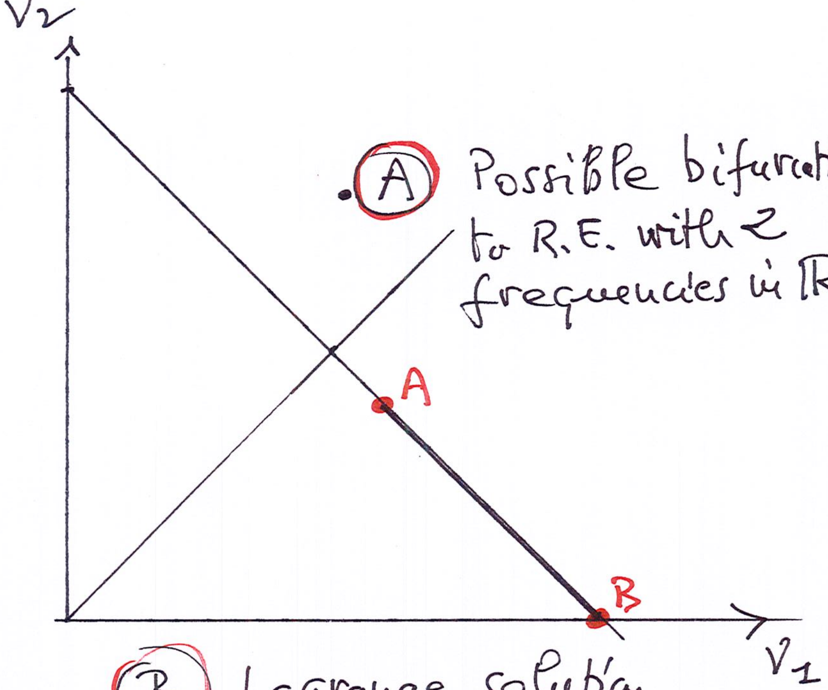
$$= \text{Horn polytope} \left\{\begin{array}{l}\text{ordered spectra} \\ \text{of } \alpha + \beta\end{array} \middle| \begin{array}{l}\text{spectrum } \alpha = \{\sigma_1, \sigma_3, \dots, \sigma_{2p-1}\} \\ \text{spectrum } \beta = \{\sigma_2, \sigma_4, \dots, \sigma_{2p}\}\end{array}\right\}$$

- Other partition into two halves of Spectrum S_0 give rise to subpolytopes of Im $\mathcal{F}$.

BIFURCATIONS TO QUASI-PERIODIC RELATIVE EQUILIBRIA (3)

FROM R.E. OF EQUILATERAL Δ

$$S_0 = \mathrm{diag}(\sigma_1, \sigma_2, 0, 0)$$

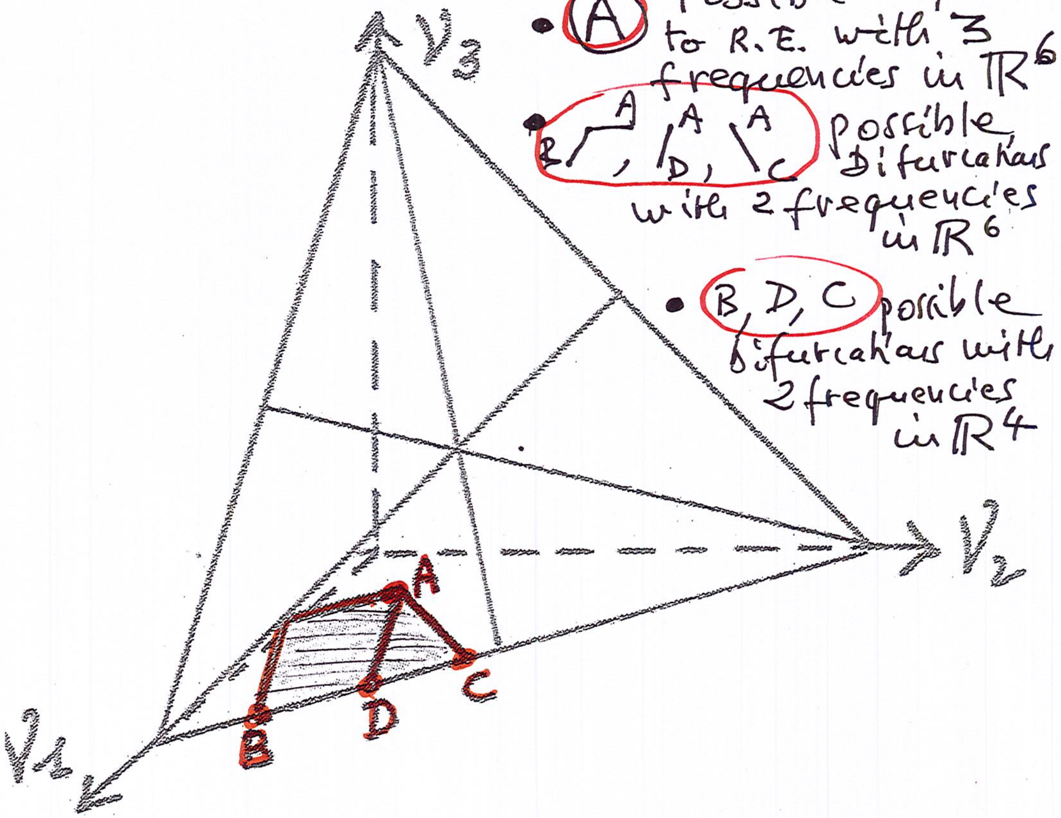


- Ⓐ Possible bifurcation to R.E. with 2 frequencies in $\mathbb{R}^4$

- Ⓑ Lagrange solution in $\mathbb{R}^2$. No possible bifurcation.

FROM R.E. OF REGULAR

$$S_0 = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, 0, 0, 0)$$



- Ⓐ Possible bifurcations to R.E. with 3 frequencies in $\mathbb{R}^6$
- $\overline{BA}$, $\overline{AD}$, $\overline{AC}$ possible bifurcations with 2 frequencies in $\mathbb{R}^6$
- B, D, C possible bifurcations with 2 frequencies in $\mathbb{R}^4$

(case $\sigma_1 > \sigma_2 + \sigma_3 > \sigma_2 > \sigma_3 > 0$)

TWO OPEN QUESTIONS

① B_0 Central configuration

Is the map

$$\left\{ \begin{matrix} \text{Balanced} \\ \text{configurations} \end{matrix} \right\} \ni B \longmapsto A|_{\operatorname{Im} B}$$

a local diffeomorphism near B_0 ?

② Does there exist balanced configurations of 4 equal masses without any symmetry ?


ZATTERE
AGLI INCURABILI

Brodskij Iosif (1940-1996)
grande poeta russo.
premio Nobel.
amò e cantò
questo luogo.

Иосиф Бродский.
великий русский поэт.
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