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4 lectures on the N-body problem

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Abstract

In the first two lectures, hamiltonian techniques are applied to avatars of the N-body problem of interest to astronomers : the first one introduces one of the simplest non integrable equations, the planar circular restricted problem in the lunar case, where most degeneracies of the general (non restricted) problem are not present ; the second one is a quick introduction to Arnold's theorem on the stability of the planetary problem where degeneracies are dealt with thanks to Herman's normal form theorem. The last two lectures address the general (non perturbative) N-body problem : in the third one, a sketch of proof is given of Marchal's theorem on the absence of collisions in paths of N-body configurations with given endpoints which are local action minimizers ; in the last one, this theorem is used to prove the existence of various families of periodic and quasi-periodic solutions with prescribed symmetries and in particular to extend globally Liapunov families bifurcating from polygonal relative equilibria. Celestial mechanics is famous for demanding extensive computations which hardly appear here : these notes only describe the skeleton on which these computations live.

Contents

1 The Poincaré-Birkhoff-Conley twist map of the annulus for the planar circular restricted 3-body problem

1.1 The Kepler problem as an oscillator

The (normalized) motions in a plane of a particle submitted to the Newtonian attraction of a fixed center – the so called *Kepler problem* – are the solutions of the equation

$$\ddot{x} = -x/|x|^3,$$

where $x \in \mathbb{R}^2 = \mathbb{C}$ is identified with a complex number and the dot denotes the time derivative. These equations are the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial \bar{y}},\ \dot{y} = -\frac{\partial H}{\partial \bar{x}}$$

associated to the Hamiltonian $H : (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \to \mathbb{R}$ and the symplectic form ω respectively defined by

$$H(x, y) = |y|^2 - 2/|x|, \quad \omega = dx \wedge d\bar{y} + d\bar{x} \wedge dy.$$

The Levi-Civita mapping $(z, w) \mapsto \big(x = 2z^2,\ y = w/\epsilon\bar{z}\big)$ defines a two-fold covering

$$(L.C.) \qquad K^{-1}(0) \setminus \{z = 0\} \to \Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$$

from the complement of the plane $z = 0$ in the 0-energy 3-sphere $K^{-1}(0)$ of the harmonic oscillator

$$K(z, w) = |z|^2 + |w|^2 - \epsilon^2 = \epsilon^2|z|^2 \left[H\left(2z^2, w/\epsilon\bar{z}\right) + 1/\epsilon^2\right],$$

to the energy hypersurface $\Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$ of the Kepler problem (both diffeomorphic to $S^1 \times \mathbb{R}^2$). It is conformally symplectic and sends integral curves of the harmonic oscillator with energy ϵ^2 to those of the Kepler problem with energy $-1/\epsilon^2$ after the change of time $dt = 2\epsilon|x|dt'$ which prevents the velocity to become infinite at collision. In the coordinates $u_1 = w + iz$, $u_2 = \bar{w} + i\bar{z}$ these integral curves are $u_1(t) = c_1e^{it},\ u_2(t) = c_2e^{it},\ |c_1|^2 + |c_2|^2 = 2\epsilon^2$, that is the intersections of the 3-sphere with the complex lines $u_1/u_2 = cste$, or in other words the fibers of the *Hopf fibration* $(u_1, u_2) \mapsto u_1/u_2 : S^3 \to P_1(\mathbb{C})$. The closest approximation to a section of the Hopf map, the annulus

$$\arg u_1 + \arg u_2 = 0 \mod 2\pi$$

is a global surface of section of the flow of the Harmonic oscillator in a sphere of constant energy : with the exception of the two fibers which form its boundary, all the fibers cut this annulus transversally in two points ; hence, the second return map is the identity. Thus perturbations of the Kepler problem with negative energy are essentially perturbations of the identity map. This is one of the main sources of degeneracies in celestial mechanics.

1.2 The restricted problem in the lunar case

The equations of the n-body problem

$$\ddot{\vec{r}}_i = g\sum_{j\neq i}\frac{m_j(\vec{r}_j-\vec{r}_i)}{||\vec{r}_i-\vec{r}_j||^3}$$

make sense even if some of the masses vanish. Such masses are influenced by the non-zero masses but do not influence them. We shall consider two primaries, say the Sun (mass μ) and the Earth (mass ν) which have a uniform circular motion around their center of mass and a 0-mass third body, say the Moon, which stays close to the Earth. We shall use the normalization $g = 1$ and $\mu+\nu = 1$. We identify the inertial plane with $\mathbb{C}$ (coordinate $X = X_1 + iX_2$ centered on the center of mass of the couple Sun-Earth) and introduce a rotating complex coordinate $x = x_1 + ix_2 = Xe^{-i\omega t} - \mu$ centered on the Earth. Setting $y = \dot{x} + i\omega x$ (up to a translation, this is the velocity in the inertial frame), the equations of motion of the Moon take the Hamiltonian form

$$\dot{x} = \frac{\partial H}{\partial \bar{y}},\ \dot{y} = -\frac{\partial H}{\partial \bar{x}},$$

where H is the *Jacobi integral* (the constant 2μ is added for convenience)

$$H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \frac{2\nu}{|x|} - \frac{2\mu}{|x+1|} - \mu(x+\bar{x}) + 2\mu.$$

More precisely, the vector field is the symplectic gradient of the symplectic form

$$\omega = dx\wedge d\bar{y} + d\bar{x}\wedge dy = 2(dx_1\wedge dy_1 + dx_2\wedge dy_2).$$

As in the first section, we consider the energy hypersurface $H^{-1}(1/\epsilon^2)$, with ϵ a small parameter. Its projection on the x plane is made of three connected components: a neighborhood of the Sun, a neighborhood of the Earth and a neighborhood of infinity (the so-called Hill's regions, which imply Hill's stability result, praised by Poincaré). We shall be interested in the connected component of $H^{-1}(1/\epsilon^2)$ where $|x|$ stays small. Then

$$H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \frac{2\nu}{|x|} - 2\mu\left[\frac{1}{4}|x|^2 + \frac{3}{8}(x^2+\bar{x}^2) + O_3(x)\right].$$

We see that the influence of the Sun on the Moon becomes negligible with respect to the one of the Earth and that at the collision limit, it disappears and one is left with a Kepler problem. To make this apparent, we again apply the Levi-Civita transformation. We get

$$K(z,w) = \epsilon^2|z|^2\left[H\left(2z^2, \frac{w}{\epsilon\bar{z}}\right) + \frac{1}{\epsilon^2}\right] = f^2(z,w)|z|^2 + |w|^2 - \nu\epsilon^2 - \epsilon^2\mu g(z),$$

where

$$f(z,w) = \sqrt{1 + 2i\epsilon(\bar{z}w - z\bar{w})},\quad g(z) = 2|z|^2\left(\frac{1}{|2z^2+1|} - 1 + z^2 + \bar{z}^2\right).$$

As in the Kepler case, the direct image of the restriction to $K^{-1}(0) \setminus \{z=0\}$ of the Hamiltonian flow $\dot{z} = \frac{\partial K}{\partial \bar{w}}$, $\dot{w} = -\frac{\partial K}{\partial \bar{z}}$ becomes the flow of the restricted problem with Jacobi constant $-1/\epsilon^2$ after the change of time $dt = 2\epsilon|x|dt'$.

Each truncation of the Taylor expansion of $K(z,w)$ at the origin,

$$K(z,w) = -\nu\epsilon^2 + |z|^2 + |w|^2 + 2i\epsilon|z|^2(\bar{z}w - \bar{w}z) - \epsilon^2\mu(2|z|^6 + 3|z|^2(z^4 + \bar{z}^4) + 0_8(z)),$$

makes sense dynamically when restricted to $K^{-1}(0)$: we get

at order 2, the harmonic oscillator, which regularizes the Kepler problem ;

at order 4, the regularization of the Kepler problem in a rotating frame ;

at order 6, *Hill's problem.* This is the highest order of interest to us.

1.3 Hill's solutions

The truncation $\hat{K}(z,w) = -\nu\epsilon^2 + f^2(z,w)|z|^2 + w^2$ of K at fourth order is a completely integrable Hamiltonian, a first integral being the angular momentum or, what is equivalent, the function $f^2(z,w)$. This is not surprising as we already knew that the restriction to $K^{-1}(0)$ corresponds to the completely integrable Kepler problem in a rotating frame. The intersection of level hypersurfaces of K and f^2 defines in general a two-dimensional torus, except when the two hypersurfaces are tangent, that is when $w = \pm if(z,w)z$. In this case the intersection degenerates to a circle ; in $K^{-1}(0)$, this defines two solutions which project (by a 2-1 map) onto the two circular solutions (one direct, one retrograde) of the rotating Kepler problem with the given value $-1/\epsilon^2$ of the Jacobi constant.

¿From now on, two roads may be followed : one can, along with Kummer [Ku], stick to symplectic coordinates or one can, as did Conley, use the simpler but not symplectic coordinates

$$\xi_1 = w + if(z,w)z, \quad \xi_2 = \bar{w} + if(z,w)\bar{z}.$$

We shall follow Conley. The equations $\dot{z} = \frac{\partial K}{\partial \bar{w}}$, $\dot{w} = -\frac{\partial K}{\partial \bar{z}}$ take the form

$$\dot{\xi}_1 = i\xi_1\left(1 - \frac{\epsilon}{2}|\xi_1 - \bar{\xi}_2|^2\right) + \epsilon^2 O_5(\xi_1,\xi_2),\ \dot{\xi}_2 = i\xi_2\left(1 + \frac{\epsilon}{2}|\xi_1 - \bar{\xi}_2|^2\right) + \epsilon^2 O_5(\xi_1,\xi_2).$$

For this section, we do not need the exact expression of the terms of order 5.

We shall show that the energy hypersurface $K^{-1}(0)$ contains two periodic solutions of minimal periods close to 2π, corresponding to the so-called *Hill's lunar orbits*, direct and retrograde, which are almost circular periodic motions of the Moon around the Earth in the rotating frame. The value 0 of the energy does not play a special role and it is in fact possible to prove the existence of two "Lyapunov" families of periodic solutions stemming from the origin and foliating two smooth (even analytical) germs of invariant surfaces in the (z,w) four dimensional phase space. This is a degenerate version of Liapunov' theorem, the degeneracy being the double eigenvalues $\pm i$ of the linearization $\dot{\xi}_1 = i\xi_1$, $\dot{\xi}_2 = i\xi_2$, of the vector-field at $\xi_1 = \xi_2 = 0$. Recall that this degeneracy comes from the fact that all solutions of the Kepler problem with a given energy are periodic with the same period. Here are the main steps of the proof.

i) Putting the vector-field into normal form at order 3: the idea, which goes back to Poincaré's thesis and was much developped by Birkhoff is to simplify as much as possible a finite part of the vector-field's Taylor expansion at the origin by means of local change of variables tangent to Identity. It relies on the fact that replacing $X = (x_1, \cdots, x_n)$ by $Y = X + h(X)$, where the components of $h(X)$ start with terms homogeneous in X of degree r, transforms the equation $\dot{X} = AX + F(X)$ into the equation $\dot{Y} = AY + [A,h](Y) + O_{r+1}$, where $[,]$ is the Lie bracket of the two vector-fields. If $A = diag(\lambda_1, \cdots, \lambda_n)$ and $h = (h_1, \ldots, h_n)$ with $h_s(Y) = y_1^{i_1} \cdots y_n^{i_n}$ and $h_j = 0$ if $j \neq s$, one checks that $[A,h] = k$ with $k_s(Y) = (i_1\lambda_1 + \cdots + i_n\lambda_n - \lambda_s) y_1^{i_1} \cdots y_n^{i_n}$ and $k_j = 0$ if $j \neq s$. It follows that one can suppress only *non-resonant* terms, i.e. those for which no *resonance relation* $i_1\lambda_1 + \cdots + i_n\lambda_n - \lambda_s =$ is satisfied.

In our case, this allows to replace the equations by the following (we kept the same name for the variables):

$$\dot{\xi}_1 = i\xi_1\left(1 + \alpha|\xi_1|^2 + \beta|\xi_2|^2\right) + \epsilon^2\varphi_1(\xi_1, \xi_2),$$

$$\dot{\xi}_2 = i\xi_2\left(1 + a|\xi_1|^2 + b|\xi_2|^2)\right) + \epsilon^2\varphi_2(\xi_1, \xi_2),$$

with $\alpha = \beta = -\frac{\epsilon}{2}$, $a = b = +\frac{\epsilon}{2}$, φ_1 and φ_2 of order 5 in $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$. In the neighborhood of the origin, the flow $\Phi_t(\xi_1, \xi_2) = (\xi_1(t), \xi_2(t))$ can be written

$$\xi_1(t) = e^{it}\left[\xi_1\big(1 + i(\alpha|\xi_1|^2 + \beta|\xi_2|^2)t\big) + \epsilon^2\alpha_1(\xi_1, \xi_2, t)\right],$$

$$\xi_2(t) = e^{it}\left[\xi_2\big(1 + i(a|\xi_1|^2 + b|\xi_2|^2)t\big) + \epsilon^2\alpha_2(\xi_1, \xi_2, t)\right],$$

with α_1, α_2 of order 5 in $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$ uniformly in t belonging to a compact.

ii) Regularizing the equations for a periodic solution by means of a blow-up: We look for a periodic solution whose period T is close to the period 2π of the solution $\xi_2 = 0$ of the rotating Kepler problem approximation (an analogous reasoning can be made for a solution close to $\xi_1 = 0$). Because of the existence of the energy first integral, the equations which define a periodic solution of period T, that is $\xi_1(T) = \xi_1$, $\xi_2(T) = \xi_2$, are consequence of the equations

$$Arg\,\xi(T) - Arg\,\xi_1 = 2\pi, \quad \xi_2(T) - \xi_2 = 0.$$

Writing down directly these equations would lead to possibly non differentiable terms like $\alpha_1(\xi_1, \xi_2)/\xi_1$. Indeed, they read

$$2\pi = T + \arg\left[1 + i(\alpha|\xi_1|^2 + \beta|\xi_2|^2)T + \epsilon^2\frac{\alpha_1(\xi_1, \xi_2, T)}{\xi_1}\right],$$

$$\left[e^{iT}\left(1 + i(a|\xi_1|^2 + b|\xi_2|^2)T\right) - 1\right]\xi_2 + \epsilon^2 e^{iT}\alpha_2(\xi_1, \xi_2, T) = 0.$$

We solve this problem by a further localization in a domain of the form $|\xi_2| \leq |\xi_1|$ by means of a complex blow-up

$$\xi_1 = z_1, \quad \xi_2 = z_1 z_2$$

which replaces such a term by $\alpha_1(z_1, z_1 z_2)/z_1$ which is now differentiable. The first equation determines T as a C^3 function of $z_1, \bar{z}_1, z_2, \bar{z}_2$,

$$T = 2\pi - 2\pi|z_1|^2(\alpha + \beta|z_2|^2) + o_3,$$

where o_3 vanishes at order 3 along $z_1 = 0$. The second one becomes

$$2\pi i|z_1|^2\big(a - \alpha + (b - \beta)|z_2|^2\big)z_2 + o_3 = 0_3$$

As $a - \alpha = \epsilon \neq 0$, solving this equation leads to a C^1 surface tangent to the plane $z_2 = 0$, that is in the (ξ_1, ξ_2) space to a C^2 surface N_1 tangent at order 2 to the plane $\xi_2 = 0$. Intersecting with the energy hypersurface $K = 0$ gives the seeked for periodic solution. In the same way, one proves the existence of N_2 tangent to $\xi_1 = 0$.

iii) Proving the analyticity of N_1 and N_2: this is done in Conley's thesis by closely following the proof given in the non resonant case by Siegel and Moser. To understand the formulas, one suppresses the resonant terms of any order by means of a formal (not convergent !) transformation. One gets new (formal coordinates) ζ_1, ζ_2 such that $\dot{\zeta}_1$ and $\dot{\zeta}_2$ become formal series in the resonant terms $\zeta_i|\zeta_j|^2$ and $\zeta_i(\zeta_j\bar{\zeta}_k)$. Rewriting the computation of periodic solutions as above leads to formal surfaces N_1 and N_2 where, for example, N_1 is defined by a (formal) equation of the form $\zeta_2 = \gamma(|\zeta_1|^2)\zeta_1$, the restriction of the vector-field being of the form $\dot{\zeta}_1 = \alpha(|\zeta_1|^2)\zeta_1$ where α has purely imaginary values (this corresponds to the fact that N_1 is foliated by periodic solutions surrounding the origin). One proves the convergence of γ and α by writing down majorant series.

1.4 The annulus twist map

Replacing the boundaries $\xi_1 = 0$ and $\xi_2 = 0$ of the Kepler annulus by the two Hill orbits, one can now construct a global annulus of section of the flow in the 3-sphere $K^{-1}(0)$ and analyze the first return map. Such an annulus is of course not unique and it will be convenient to chose it so as to contain the "collision circle" of equation $z = 0$.

In order to get precise enough information on the first return map, one must analyze the equations up to the 5th order where the influence of the Sun comes into play. Writing down a normal form up to this order implies first computing the effect on terms of order five of the change of variables leading to a normal form at order 3. In fact, one can dispense with this: it is enough to suppress only the non resonant terms of order 5, keeping the terms of order 3 as they stood initially. Moreover, the above analysis of the submanifolds N_1 and N_2 whose intersection with $K = 0$ defines Hill's orbits, shows that there exists an analytic change of variables which transforms them into coordinate planes. A finer analysis shows that such a straightening change of variables differs from Id only by terms $\epsilon A + \epsilon^2 B$, where A is resonant of order 5 and B is of order 7. One deduces that such a straightening of N_1 and N_2 does not bring any new change to the differential equation up to order 5. Finally, we get new coordinates (ζ_1, ζ_2) such that N_1 and N_2 are respectively defined by $\zeta_1 = 0$ and $\zeta_2 = 0$, and the energy hypersurface $K^{-1}(0)$ and the collision circle $z = 0$ by

$$\frac{1}{2}\big(|\zeta_1|^2 + |\zeta_2|^2\big) - \nu\epsilon^2 + \epsilon O_6(\zeta) = 0, \quad \text{and} \quad \zeta_1 - \bar{\zeta}_2 + \epsilon O_5(\zeta) = 0.$$

It follows that an annulus of section in $K^{-1}(0)$ containing the collision circle and bounded by the Hill orbits can be defined by the equation

$$Arg\,\zeta_1 + Arg\,\zeta_2 + \epsilon O_4(\zeta) = 0(mod 2\pi).$$

Computing a little more, one can find coordinates (φ, ρ) on this annulus, such that the two boundaries are close to $\rho = \pm 1$ and the first return map takes the form

$$P_\epsilon(\varphi, \rho) = \left(\varphi + \frac{1}{2} - \frac{\nu}{2}\epsilon^3 - \frac{3\nu^2}{2}(1 - \frac{\mu}{4})\epsilon^6\rho + 0(\epsilon^7),\, \rho + O(\epsilon^7)\right).$$

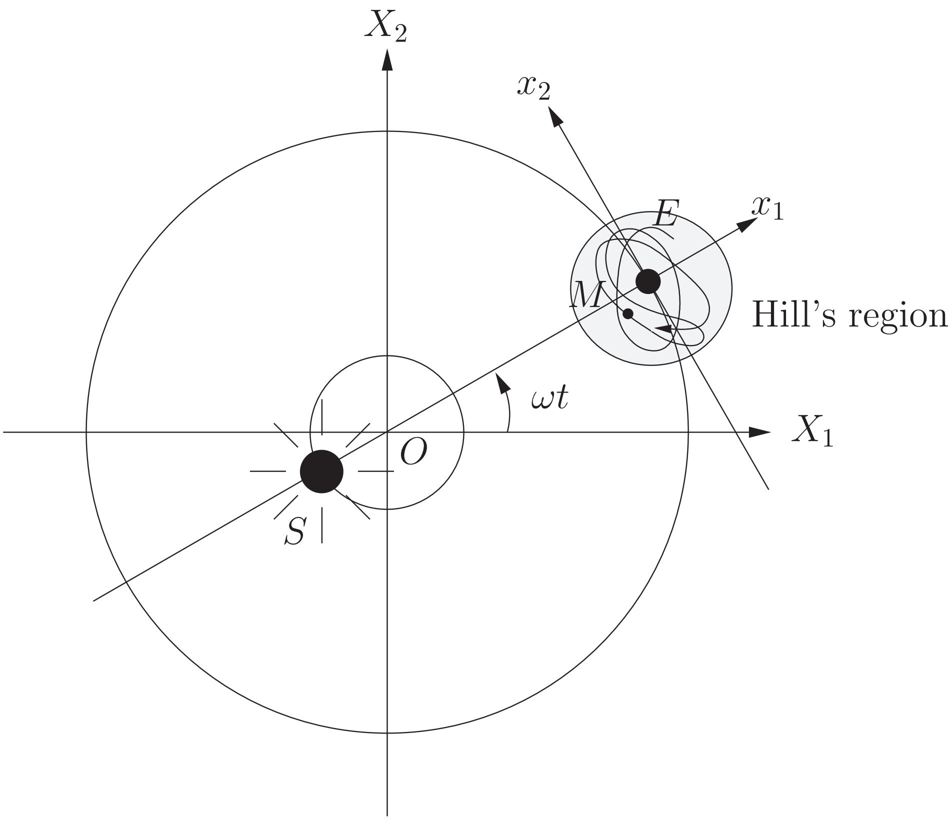
Coming back to the definition of this annulus, one checks that the return map corresponds essentially to the passages of the orbit of the Moon through aphelium in the rotating frame. Originating from a Hamiltonian system, this map necessarily preserves a measure defined by a smooth density. Moreover, it is a $O(\epsilon^7)$ perturbation of an integrable twist map whose twist is of size ϵ^6. This is a perfect ground for applying the main results of the general theory of conservative twist maps, a particular case of the theory of Hamiltonian systems with two degrees of freedom:
1) Applied to the iterates of the return map, the *Birkhoff fixed point theorem* yelds an infinite number of periodic orbits of higher and higher periods to which correspond periodic orbits of long period of the Moon around the Earth in the rotating frame;
2) The *Moser invariant curve theorem* implies the existence of a positive measure Cantor set of invariant curves on which the map is conjugated to a diophantine irrational rotation and to which correspond quasi periodic orbits of the Moon;
3) To the Liouville rotation numbers, the Aubry-Mather theory associates invariant Cantor sets to which correspond orbits of the Moon with a Cantor caustic
4) Finally, it is possible to prove that the image of the collision circle intersects itself transversally at eight points [CL]; in particular, it is not contained in an invariant curve. Varying the value of ϵ moves the invariant curve of a given rotation number across the annulus which forces intersection with the collision curve. This proves the existence of invariant "punctured" tori which correspond to orbits of the Moon which persistently change their direction of rotation around the Earth in the rotating frame (generalization of the punctured tori to the full planar 3-body problem were given by Féjoz in his thesis [Fe1]).

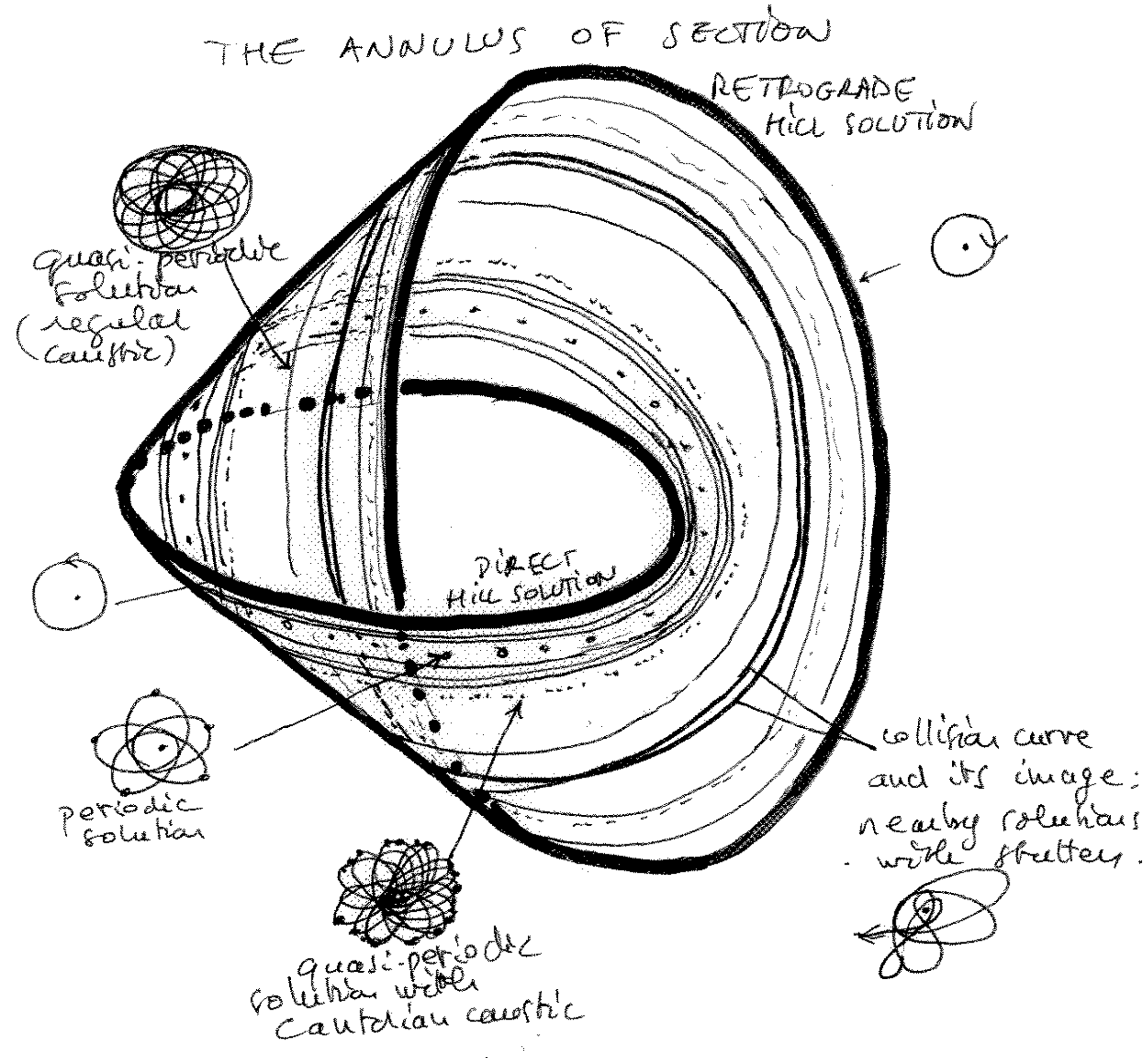
Remark. For writing down formulas, working in the 2-fold covering $K^{-1}(0)$ of the energy hypersurface diffeomorphic to S^3 is convenient but one can prefer to state the results downstairs in the compactification (regularization), diffeomorphic to $SO(3)$ (that is to the real projective space of dimension 3), of the original energy hypersurface $H^{-1}(-\frac{1}{\epsilon^2})$. The first return map then becomes a perturbation of the Identity (the Kepler case) of the form

$$\mathcal{P}_\epsilon(\tilde{\varphi},\rho) = \Big(\tilde{\varphi} - \nu\epsilon^3 - 3\nu^2(1-\frac{\mu}{4})\epsilon^6\rho + 0(\epsilon^7),\ \rho + O(\epsilon^7)\Big)\,.$$

and the collision curve intersects its image only 4 times.

A problem. When the collision curve intersects the set of invariant curves, the closure of the union of its iterates, being the set of intersected curves, is in general of positive measure. What if the collision curve is contained in a Birkhoff region of instability ?




THE ANNULUS OF SECTION
RETROGRADE HILL SOLUTION
quasi-periodic solution (regular caustic)
DIRECT HILL SOLUTION
periodic solution
quasi-periodic solution with Cartesian caustic
collision curve and its image: nearby solutions

2 The Arnold-Herman stability theorem for the spatial (1+n)-body problem

In the so-called *planetary problem*, one mass m_0 is dominant (the Sun) and the others, the planets are of the form $\epsilon m_1, \dots, \epsilon m_n$, where ϵ is small (around 10^{-3} for the "real" solar system). If $x_0 = (x_0^1, x_0^2, x_0^3), x_1, \dots, x_n \in \mathbb{R}^3$ are the positions and $||.||$ the euclidean norm, Newton's equations read

$$\ddot{x}_j = m_0 \frac{x_j - x_0}{||x_j - x_0||^3} + \epsilon \sum_{k \neq j} m_k \frac{x_k - x_j}{||x_k - x_j||^3},\ j = 1, \dots, n.$$

The solutions are the projections on the configuration space of the integral curves of the Hamiltonian vector field defined in the phase space, whose coordinates are denoted by $(x_0, \dots, x_n, y_0, y_1, \dots, y_n)$ and symplectic form is $\sum_{1 \leq k \leq 3} dx_0^k \wedge dy_0^k + \epsilon \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq 3} dx_j^k \wedge dy_j^k$, by the Hamiltonian

$$\frac{1}{2}\frac{||y_0||^2}{m_0} + \epsilon \left(\frac{1}{2} \sum_{1 \leq j \leq n} \frac{||y_j||^2}{m_j} - \sum_{1 \leq j \leq n} \frac{m_o m_j}{||x_j - x_0||} \right) - \epsilon^2 \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{||x_j - x_k||}.$$

One reduces the translation symmetry by restricting to the value $Y_0 = 0$ the total linear momentum and going to the quotient by translations in the so-called Poincaré heliocentric canonical coordinates

$$X_0 = x_0, Y_0 = y_0 + \epsilon y_1 + \dots + \epsilon y_n, \quad X_j = x_j - x_0, \quad Y_j = y_j, \quad j = 1, \dots, n.$$

After dividing the new Hamiltonian and symplectic form by ϵ one obtains a Hamiltonian defined on $T^*\mathbb{R}^{3n}$ (coordinates $(X_1, \dots, X_n, Y_1, \dots, Y_n)$) deprived of the collision set ($X_j = 0$ or $X_j = X_k$) with its canonical symplectic structure :

$$F_\epsilon = \sum_{1 \leq j \leq n} \left(\frac{||Y_j||^2}{2\mu_j} - \frac{\mu_j M_j}{||X_j||} \right) + \epsilon \sum_{1 \leq j < k \leq n} \left(-\frac{m_j m_k}{||X_j - X_k||} + \frac{Y_j \cdot Y_k}{m_0} \right).$$

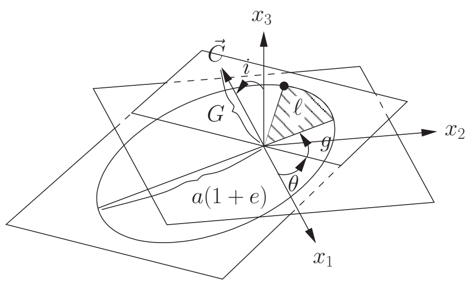
It describes an ϵ perturbation of n uncoupled *Kepler problems* with fictitious masses defined by $M_j = m_0 + \epsilon m_j$ and $\mu_j M_j = m_0 m_j$. Whe shall be interested in solutions which stay close to solutions of F_0 where the planets describes around the sun circular coplanar motions with the same orientation.

Theorem 1 *Given $m_0, \dots, m_n, a_0, \dots, a_n$, there exists $\epsilon_0 > 0$ with the following property : if $\epsilon < \epsilon_0$, there exists in the phase space of the spatial (1+n)-body problem, in the neighborhood of the circular coplanar direct Keplerian motions with semi major axes $a_1, \dots, a_n$, a set of positive Lebesgue measure of initial conditions which lead to quasi-periodic motions with $3n - 1$ frequencies (resp. $2n$ frequencies for the planar problem)*

These solutions are slow (*secular*) modulations of the quasi-periodic motions with n frequencies corresponding to n independent elliptic motions (case $\epsilon = 0$), the new secular frequencies being associated to a slow precession of the perihelia and the nodes. The proof of this theorem was given by Arnold in 1963 under strong non-degeneracy hypotheses which at the time were proved only in the case of the (1+2)-planar problem where no reduction is necessary. What follows is a guide to Herman's proof as written by Féjoz in [Fe2].

2.1 The secular hamiltonian

We make again a symplectic change of coordinates, using the so-called (once more) *Poincaré coordinates* $(\lambda_j, \Lambda_j, \xi_j, \eta_j, p_j, q_j)_{i=1,\dots,n} \in (\mathbb{T}^1 \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)^n$, well adapted to the description of elliptic Keplerian motions which are nearly circular and horizontal. They are defined by the following formulas where the unnamed letters are defined on the figure : $\lambda_j = l_j + g_j + \theta_j$ is the mean longitude and $\Lambda_j = \mu_j\sqrt{M_j a_j}$ its conjugate variable, $r_j = \xi_j + i\eta_j = \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-\epsilon_j^2}}e^{i(g_j+\theta_j)}$ and $z_j = p_j + iq_j = \sqrt{2G_j}\sqrt{1-\sqrt{1-\cos\iota_j}}e^{i\theta_j}$ describe each a symplectic plane. The modules $|r_j| = \sqrt{\Lambda_j/2}\epsilon_j(1+O(\epsilon_j^2))$ and $z_j = \sqrt{\Lambda_j/2}\iota_j(1+O(\epsilon_j^2)+O(\iota_j^2))$ describe respectively the eccentricity and inclination of a Keplerian ellipse; the horizontal circular motions we are interested in correspond to $|r_j| = |z_j| = 0$ for all j. We shall abbreviate the Poincaré coordinates by $(\lambda, \Lambda, Z) \in \mathbb{T}^n \times (\mathbb{R}_+)^n \times \mathbb{C}^{2n}$, with $Z = (r_1, \dots, r_n, z_1, \dots, z_n)$.



In these (analytical, Poincaré proved it) coordinates, the Hamiltonian H becomes an ϵ-perturbation of a sum of n uncoupled Keplerian Hamiltonians

$$H^0(\Lambda) = \sum_{1\le j\le n} -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2}.$$

This is a very degenerate situation indeed, as H^0 depends only on n action variables instead of $3n$. The averaging method tells us to write down H in the form

$$H(\lambda, \Lambda, Z) = H^0(\Lambda) + \epsilon H_\epsilon^1(\Lambda, Z) + \epsilon H_\epsilon^2(\lambda, \Lambda, Z),$$

where $\epsilon H_\epsilon^1(\Lambda, Z)$ is the average of the perturbation $H - H^0$ over the so-called *fast* angles $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n$ (the only ones which move if $\epsilon = 0$) and H_ϵ^2

has zero average over these angles. The hamiltonian H^1_ϵ defines the *first order secular system*. As it does not depend on the mean longitudes λ_j, the conjugate variables Λ_j remain constant under its flow (they are supposed to be such that the (not too excentric) ellipses remain far enough from each other so that the perturbation function deserves its name). Hence, for given values of the Λ_j, i.e. of the semi-major axes a_j, H^1_ϵ defines a flow

$$\frac{dZ_k}{dt} = i\frac{\partial H^1_\epsilon}{\partial \bar{Z}_k},\ k = 1, \dots, 2n,$$

on an open set, diffeomorphic to $\mathbb{R}^{4n} = \mathbb{C}^{2n}$ of the space, diffeomorphic to $(S^2 \times S^2)^n$, of n-tuples of normalized ellipses in $\mathbb{R}^3$. The detailed study of the secular hamiltonian is a sequence of long computations, started by Laplace and Lagrange in the 18th century, of which we only summarize the results :

1) each of the terms $Y_j \cdot Y_k$ is readily seen to have zero average, which implies

$$H^1_\epsilon(\Lambda, Z) = -\sum_{1\le j<k\le n} \int_{\mathbb{T}^n} \frac{m_j m_k}{||X_j - X_k||} d\lambda_1 \dots d\lambda_n,$$

which is the Newtonian potentiel of a set of elliptic rings whose mass repartition would follow Kepler's area law.

2) Being only interested in the neighborhood of the origin, one writes down the expansion up to second order (indeed third because of parity) of H^1_ϵ. This depends on long computations, using the so-called Laplace coefficients, of the Fourier expansion of the inverse distance function of two planets considered as a periodic function of their mean longitudes.

One gets $H^1_\epsilon(\Lambda, Z) = h^0(\Lambda) + Q_\Lambda(Z) + O(|Z|^4)$, with

$$Q_\Lambda(Z) = Q'_\Lambda(\xi_1, \dots, \xi_n) + Q'_\Lambda(\eta_1, \dots, \eta_n) - Q''_\Lambda(p_1, \dots, p_n) - Q''_\Lambda(q_1, \dots, q_n),$$

$$Q'_\Lambda(\xi_1, \dots, \xi_n) = \sum_{1\le j<k\le n} m_j m_k \left(C_1(a_j, a_k)\left(\frac{\xi_j^2}{\Lambda_j} + \frac{\xi_k^2}{\Lambda_k}\right) + 2C_2(a_j, a_k)\frac{\xi_j \xi_k}{\sqrt{\Lambda_j \Lambda_k}}\right),$$

$$Q''_\Lambda(p_1, \dots, p_n) = \sum_{1\le j<k\le n} m_j m_k C_1(a_j, a_k)\left(\frac{p_j}{\sqrt{\Lambda_j}} - \frac{p_k}{\sqrt{\Lambda_k}}\right)^2.$$

The value $h^0(\Lambda)$ of Q_Λ at $Z = 0$ (which is a critical point corresponding to circular horizontal motions) depends on the masses and the semi-major axes while the coefficients $C_1(a_j, a_k)$ and $C_2(a_j, a_k)$ are independent of the masses. All of them have simple expressions in terms of Laplace coefficients. As a good exercise, the reader will show that this form of the quadratic terms is essentially dictated by the symmetries of the problem.

3) If $\rho' \in SO(n)$ and $\rho'' \in SO(n)$ respectively diagonalize Q' and Q'', the linear transformation $\rho = diag(\rho', \rho', \rho'', \rho'') \in SO(4n)$ is symplectic and transforms Q_Λ into a hamiltonian of the form

$$Q_\Lambda \circ \rho(\Lambda, Z) = h^0(\Lambda) + \sum_{1\le j<k\le n} \sigma_j(\xi_j + \eta_j^2) + \sum_{1\le j<k\le n} \zeta_j(p_j^2 + q_j^2) + O(|Z|^4).$$

Finally, applying the above coordinate changes to the full Hamiltonian leads to a Hamiltonian which we shall still write H, defined in a neighborhood of $\mathbb{T}^n \times \mathbb{R}^n_+ \times \{0\}$ in $\mathbb{T}^n \times \mathbb{R}^n_+ \times \mathbb{C}^{2n}$ (symplectic form $d\lambda \wedge d\Lambda + \sum_{1 \le j \le 2n} \frac{1}{2i} d\bar{Z}_j \wedge dZ_j$), of the form

$$H_\epsilon(\lambda, \Lambda, Z) = H^0(\Lambda) + \epsilon \left[h^0(\Lambda) + \sum_{1 \le j \le 2n} \tau_j(\Lambda)|Z_j|^2 + 0(|Z|^4) + H^2_\epsilon(\lambda, \Lambda, Z) \right],$$

where $\tau_j = \sigma_j$ if $1 \le j \le n$, $\tau_j = \zeta_j$ if $n+1 \le j \le 2n$, the term $O(|Z|^4)$ does not depend of λ and H^2_ϵ has zero average with respect to $\lambda \in \mathbb{T}^n$.

The degeneracy of the integrable approximation $H_\epsilon - \epsilon H^2_\epsilon$ appears clearly: for $\epsilon = 0$ or for $Z = 0$, the dimension of the invariant tori drops down to n. We shall later encounter other degeneracies which affect the spatial problem but we first turn to Herman's way of proving an appropriate KAM theorem.

2.2 Herman's normal form theorem and how to use it

Herman's powerful idea is to separate a normal form theorem for Hamiltonians close to what could be called a "Kolmogorov Hamiltonian" – one for which $\mathbb{T}^m \times \{0\}$ is a diophantine invariant torus – from the actual verification of a non-degeneracy hypothesis which allows a tuning of the available parameters which turns such a normal form into a conjugacy to some Kolmogorov Hamiltonian.

The following theorem is a far reaching generalization of the Arnold-Moser theorem on vector-fields on the torus which states that, *among all C^∞ vector-fields on $\mathbb{T}^2$ close enough to a constant vector-field (noted $\omega = (\omega_1, \omega_2)$) whose frequencies ω satisfy a diophantine condition $HD_{\gamma,\tau}$ (defined below), the ones which are C^∞-conjugated to it form a submanifold of codimension 2*; more precisely, that the mapping

$$\Phi_\omega : \mathrm{Diff}^\infty(\mathbb{T}^2, 0) \times \mathbb{R}^2 \to \mathcal{X}^\infty(\mathbb{T}^2)$$

defined by $\Phi_\omega(h, \lambda) = h_*\omega + \lambda$ (where $h_*\omega$ is the direct image by h of the constant vector-field ω) is a C^∞ (i.e. "tame" in the sense of Hamilton) diffeomorphism of a neighborhood of $(Id, 0)$ onto a neighborhood of ω in $\mathcal{X}^\infty(\mathbb{T}^2)$.

We study hamiltonians $H(r, \theta)$ on $T^*\mathbb{T}^m \equiv \mathbb{T}^m \times \mathbb{R}^m$ (in our case, $m = 3n, r = (\Lambda - \Lambda_0, |Z| - |Z|_0), \theta = (\lambda, ArgZ)$). The role of the constant vector field of frequencies ω on the torus is now held by the set $\mathcal{N}_\omega$ of *Kolmogorov Hamiltonians* $N(r, \theta) = N_\omega(r) + O(r^2)$, where $N_\omega(r) = \omega \cdot r$. This is the set of Hamiltonians whose Hamiltonian vector-field leaves invariant the torus $r = 0$ and induces on it the constant vector-field with frequency vector ω. Let also $\mathcal{G}$ be some space (which we will not describe, see [Fe2]) of Hamiltonian diffeomorphisms close to Identity, defined on a neighborhood of $\mathbb{T}^m \times \{0\}$ in $\mathbb{T}^m \times \mathbb{R}^m$. Let $C^\infty_+(\mathbb{T}^m \times \mathbb{R}^m)$ be the quotient of the space of Hamiltonians by the real constants. We note

$$HD_{\gamma,\tau} = \left\{ \omega \in \mathbb{R}^m, \forall k \in \mathbb{Z}^m \setminus 0, |l \cdot \omega| \ge \gamma ||k||^{-\tau} \right\}.$$

Theorem 2 (Herman's normal form) *For every* $\omega \in HD_{\gamma,\tau}$ *and for every* $N^o \in \mathcal{N}_\omega$*, the map*

$$\begin{array}{lccl} \Phi_\omega : & \mathcal{N}_\omega \times \mathcal{G} \times \mathbb{R}^m & \to & C^\infty_+(\mathbb{T}^m \times \mathbb{R}^m) \\ & (N, G, \Delta\omega) & \mapsto & H = N \circ G + N_{\Delta\omega}, \end{array}$$

is a local C^∞*-diffeomorphism in a neighborhood of* $(N^o, id, 0)$*. Moreover, the inverse map* Φ_ω^{-1} *depends smoothly in the sense of Whitney of* $\omega \in HD_{\gamma,\tau}$*.*

As in the Arnold-Moser theorem, this theorem asserts that the set of Hamiltonians which are conjugated to a normal form with a diophantine frequency vector (i.e. those of the form $H = N \circ G$ with $N = N_\omega + O(r^2)$) form a submanifold of codimension m of the set of Hamiltonians modulo constants. Herman's theorem is in fact more general (see [Fe2]) in that it works also with normal forms which leave invariant tori of dimension lower than n. Following Herman, the proof given in [Fe2] uses a "hard" implicit function theorem, that is one valid in a scale of Fréchet spaces. The key feature of such theorems is the necessity of inverting (or inverting approximately) the differential of the mapping Φ_ω on a whole neighborhood of $(N^0, Id, 0)$ (invertibility is not an open property in Fréchet spaces).

Of course, it is only when the frequency correction $\Delta\omega$ vanishes that Herman's normal form implies the existence of an invariant torus. The beautiful idea of Herman was to use the Whitney extension theorem and the usual implicit function theorem to draw the following corollary (I use the name given by Féjoz): let $\mathcal{N} = \cup_{\omega \in \mathbb{R}^m} N_\omega = \{\omega \cdot r + O(r^2)\}_{\omega \in \mathbb{R}^m}$ be the set of all normal forms.

Corollary 3 (hypothetical conjugacy) *For every* $N^0 \in \mathcal{N}$*, there is a (non unique) germ of* C^∞*-diffeomorphism*

$$C^\infty_+(\mathbb{T}^m \times \mathbb{R}^m) \ni H \mapsto \Theta(H) = (N_H = \omega_H \cdot r + O(r^2), G_H) \in \mathcal{N} \times \mathcal{G}$$

at $N^0 \mapsto (N^0, Id)$ *such that* $H = N_H \circ G_H$ *for each* H *verifying* $\omega_H \in HD_{\gamma,\tau}$*.*

The proof is in two steps : first, the Whitney extension theorem allows to extend (non uniquely) from $C^\infty_+(\mathbb{T}^m \times \mathbb{R}^m) \times HD_{\gamma,\tau}$ to $C^\infty_+(\mathbb{T}^m \times \mathbb{R}^m) \times \mathbb{R}^m$ the map $(H, \omega) \mapsto \Phi_\omega^{-1}(H) = (N, G, \Delta\omega)$; then, one deduces from the identity $N^0 = (N^0 + N_{\omega-\omega^0}) \circ Id + N_{\omega^0-\omega}$ that, at (N^0, Id), one has $\frac{\partial\Delta\omega}{\partial\omega} = -Id$. Hence, from the usual implicit function theorem, it is possible to define a function $\omega \mapsto \omega_H$ by locally solving the equation $\Delta\omega(\omega) = 0$.

We are now left with a serious problem : how to check that ω_H which we do not know satisfies a diophantine condition ? The magic word here is "parameters". If we were in the non-degenerate case of Kolmogorov where the *frequency map* from the actions to the frequencies of the corresponding invariant torus is a local diffeomorphism the existence of a positive measure set of "good" values of the actions would follow immediately from the fact that $HD_{\gamma,\tau}$ has positive measure. But in our case, the frequency map $H \mapsto \omega_H$ is of the form

$$(\Lambda, \rho) \mapsto \left[\nu(\Lambda) + O(\epsilon), \epsilon\left(\tau(\Lambda) + O(\rho^2)\right)\right].$$

Going back to Arnold and first used by Rüssmann, the key idea is that in the analytic case, the non-degeneracy hypothesis implying a positive measure set of good actions can be much weakened; thanks to the following result, it is enough that the image of the mapping $s \mapsto \omega_s^0$ lies in no proper vector subspace of $\mathbb{R}^m$:

Theorem 4 (Arnold, Margulis, Pyartli) *If some real-analytic map $s \mapsto \omega_s^o$ from a domain of $\mathbb{R}^p$ to $\mathbb{R}^m$ is non-planar in the sense that its image is nowhere locally contained in some proper vector space of $\mathbb{R}^m$, the Lebesgue measure of $\{s,\ \omega_s^o \in HD_{\gamma,\tau}\}$ is positive provided that γ is small enough and τ large enough.*

2.3 A stability theorem

We come back to Hamiltonians on $\mathbb{T}^n \times (\mathbb{R}_+)^n \times \mathbb{R}^{2p}$ of the form obtained at the end of section 2.1 (for the spatial (resp. planar) secular system $p = 2n$ (resp. $p = n$)).

$$H_\epsilon(\lambda, \Lambda, Z) = H^0(\Lambda) + \epsilon H_\epsilon^1(\Lambda, Z) + \epsilon H_\epsilon^2(\lambda, \Lambda, Z),$$

with $H_\epsilon^1(\Lambda, Z) = h^0(\Lambda) + \sum_{1\le j\le 2n} \tau_j(\Lambda)|Z_j|^2 + 0(|Z|^4)$, and H_ϵ^2 has zero average with respect to $\lambda \in \mathbb{T}^n$. We note as before $\nu_i = \frac{\partial H^0}{\partial \Lambda_i}(\Lambda)$.

Theorem 5 (Herman's stability theorem) *If, for Λ near Λ_0, the frequency map $\alpha : \Lambda \mapsto (\nu_1, \ldots, \nu_n, \tau_1, \ldots, \tau_{2p})$ is non planar, there is a positive measure set of Lagrangian invariant tori close to $\mathbb{T}^n \times \{\Lambda_0\} \times \{0\} \in \mathbb{T}^n \times (\mathbb{R}_+)^n \times \mathbb{R}^{2p}$.*

One starts by changing coordinates in order that H_ϵ appears as a close enough approximation of an integrable Hamiltonian in the neighborhood of a Lagrangian invariant torus. There are standard ways of simplifying such a Hamiltonian by symplectic transformations defined by polynomial generating functions; the non planarity hypothesis implies that the set A_2 of Λ's on which this is possible has positive measure and moreover that it intersects any neighborhood of Λ_0. In the case of the $(1+n)$-body problem, the assertion on the bigger set A_1 defined below is directly insured by the non degeneracy of the map $\Lambda \mapsto \nu(\Lambda) = \big(\nu_1(\Lambda), \cdots \nu_n(\Lambda)\big)$.

1) *Elimination "à la Lindstedt" of the dependence on the fast angles λ_j at a sufficiently high order N_1.* This is possible if Λ belongs to the set A_1 on which $\nu(\Lambda) \in HD_{\gamma,\tau}$. Moreover, Whitney regularity allows to extend this to a (non unique) symplectic transformation L such that $H_\epsilon \circ L$ keeps the same form with $H_\epsilon^2(\lambda, \Lambda, Z))$ replaced by $R_1(\epsilon, \lambda, \Lambda, Z) + O(\epsilon^{N_1})$, where R_1 vanishes at infinite order along $\{(\epsilon, \lambda, \Lambda, Z) | \Lambda \in A_1\}$.

2) *Transformation to Birkhoff normal form up to order N_2.* This is possible if Λ belongs to the subset A_2 of A_1 defined by diophantine conditions on the set $(\nu_1, \ldots, \nu_n, \tau_1, \ldots, \tau_p)$ of all frequencies. As above, one can get a symplectic transformation B such that

$$H_\epsilon \circ L \circ B(\lambda, \Lambda, Z) = H^0(\Lambda) + \epsilon \tilde{H}^1(\epsilon, \Lambda, Z) + \epsilon R_2(\epsilon, \lambda, \Lambda, Z) + 0(\epsilon^{N_1}),$$

$$\tilde{H}^1(\epsilon, \Lambda, Z) = h^0(\Lambda) + \sum_{1\le j\le p} \tau_j(\Lambda)|Z_j|^2 + K(\Lambda, |Z^2|) + O(|Z|^{2N_2}),$$

where K is a polynomial in the $|Z_j|^2$ with terms of degree between 2 and $N_2 - 1$ and R_2 vanishes at infinite order along $\{(\epsilon, \lambda, \Lambda, Z)|\Lambda \in A_2\}$. On this subset, H_ϵ appears now as a $O(\epsilon^{N_1}, |Z|^{N_2})$-perturbation of the completely integrable system with Hamiltonian $H^0(\Lambda)+\epsilon\left[h^0(\Lambda) + \sum_{1\leq j\leq p}\tau_j(\Lambda)|Z_j|^2 + K(\Lambda, |Z_1|^2, \dots, |Z_p|^2)\right]$. To focus the attention on the Lagrangian invariant tori $\Lambda = \Lambda_0$, $|Z| = |Z|_0$ of this integrable approximation, one moves to symplectic polar coordinates $Z_k = \sqrt{\rho_k}e^{i\theta_k}$, which leads to

$$\mathcal{H}_\epsilon = H^0(\Lambda) + \epsilon\left[h^0(\Lambda) + \mathcal{K}(\Lambda, \rho)\right] + \epsilon R_3 + O(\epsilon^{N^1}, \rho^{N_2}),$$

where R_3 vanishes at infinite order along $\{(\epsilon, \lambda, \Lambda, Z)|\Lambda \in A_2\}$. In order to show that enough of these tori do survive the perturbation, one considers the $(m = n+p)$-parameter family $H_{(\Lambda,\rho)}$ of Hamiltonians H obtained by translating the origin of the actions at (Λ, ρ). If $\Lambda^0 \in A_2$, $\rho^0 > 0$ and if (Λ, ρ) is close to (Λ^0, ρ^0), the flow of $H_{(\Lambda,\rho)}$ is close to the flow of $H^0(\Lambda) + \epsilon\left[h^0(\Lambda) + \mathcal{K}(\Lambda, \rho)\right]$ in the neighborhood of the Lagrangian torus $\mathcal{T}_{(\Lambda,\rho)} = \mathbb{T}^n \times \{\Lambda\} \times \{|Z|^2 = \rho\}$. The non planarity being an open condition, it will be verified at Λ and the conclusion follows from the hypothetical conjugacy theorem.

2.4 Herman's degeneracy

For the planar $1+n$-body problem, a thorough study of the Laplace coefficients after complexification of the semi-major axes, allows proving by induction on the number of planets (letting one semi-major axis go to zero) that the frequency map is non planar. For the spatial problem, this map presents an expected degeneracy, say $\zeta_n = 0$, due to the invariance under rotation of the problem, as well as an unexpected one: the trace $\sum_{1\leq j\leq n}\sigma_j + \sum_{1\leq j\leq n}\zeta_j$ of Q_Λ is always zero. In the study of the motion of the Moon, this resonance is responsible for the well-known fact that "at the first order of the theory of perturbations" the retrograde motion of the node is exactly opposite to the mean motion of the apogee. Nevertheless, it is only Herman who noticed it in its generality. An induction similar to the one done in the planar case shows that these are the only degeneracies. The first resonance is well known to disappear when the direction of the (non-zero) angular momentum is fixed (here, vertical), which corresponds to restricting the system to a codimension 2 symplectic submanifold; the second one disappears when completing the reduction by fixing the angular momentum and quotienting by the rotations around its axis. This comes from the fact that in the Poincaré coordinates, the vertical component of the angular momentum becomes the quadratic form $\mathcal{C}_z = \Sigma_{1\leq j\leq n}\left(\Lambda_j - \frac{1}{2}(|r_j|^2 + |z_j|^2)\right)$
whose trace is different from zero. Hence, after reduction, the frequency map becomes non planar and the stability theorem yields diophantine Lagrangian invariant tori of dimension $3n-2$. To these tori correspond, for the non reduced system, invariant tori of dimension $3n-1$ whose number of independent frequencies is $3n-2$ or $3n-1$.

3 Minimal action and Marchal's theorem

3.1 Central configurations and their homographic motions

The equations of the n-body problem in an euclidean space E can be given the particularly simple form

$$\ddot{x} = \nabla U(x), \tag{$*$}$$

where $x = (\vec{r}_1, \ldots, \vec{r}_n) \in E^n$ and $U(x) = \sum_{i<j} m_i m_j ||\vec{r}_i - \vec{r}_j||^{-1} \in \mathbb{R}$ are respectively an n-body configuration and its *potential function*, and where the gradient is relative to the *mass scalar product* (or *kinetic energy scalar product*), defined by

$$x' \cdot x'' = (\vec{r}'_1, \ldots, \vec{r}'_n) \cdot (\vec{r}''_1, \ldots, \vec{r}''_n) = \sum_{i=1}^{n} m_i \, \langle \vec{r}'_i - \vec{r}'_G, \vec{r}''_i - \vec{r}''_G \rangle_E \, .$$

The presence of the centers of mass $\vec{r}_G = \frac{1}{\sum m_i} m_i \vec{r}_i$ makes the formula translation invariant; one may as well consider only configurations x such that $\vec{r}_G = 0$.

In addition to being invariant under translation, equation $(*)$ is invariant under isometries of E and it inherits from the homogeneity of U the following scaling property : if $x(t)$ is a solution, so is $\lambda^{-\frac{2}{3}} x(\lambda t)$ for any positive real number λ. When $n = 2$, any change in the configuration is necessarily a similarity (a segment has no shape !); when n is at least 3, the simplest motions (called *homographic*) are such that the similarity class of their configuration does not change. If $\dim E \leq 3$, such motions are necessarily of Keplerian type: if for example, the total energy $\frac{1}{2}||\dot{x}||^2 - U(x)$ is negative, the solution is periodic, each body following an ellipse of the same excentricity according to Kepler law. Such solutions were first discovered for $n = 3$ by Euler and Lagrange at the end of 18th century. The configurations x which admit homographic motions are called *central configurations* and their determination for $n \geq 4$ is a very difficult problem. They are characterized by the existence of a Keplerian motion with excentricity 1, which means that they collapse on their center of mass when released with 0 initial velocity. In other words, $\nabla U(x)$ is proportional to x. But $x = \frac{1}{2} \nabla I(x)$, where $I(x) = ||x||^2$ is the *moment of inertia* of the configuration with respect to its center of mass. Hence central configurations are the critical points of the restrictions of the potential function U to the spheres $I =$ constant. As an exercise, the reader will use (squared) mutual distances as coordinates on the space of "triangles mod isometries" and prove Lagrange's result that, whatever be the masses, the only non colinear central configuration of 3 masses is the equilateral triangle.
Another important fact, already proved by Lagrange for $n = 3$, is that a homographic solution with excentricity $e < 1$ is necessarily planar. Note that only the case of a *relative equilibrium* (that is $e = 0$) is "physically" obvious.

3.2 Variational characterizations of Lagrange's equilateral solutions

Equations of the type $\ddot{x} = \nabla U(x)$ are known, since Lagrange, to be the so-called Euler-Lagrange equations of an action functional, the *Lagrangian action*

$$\int L\big(x(t), \dot{x}(t)\big)dt, \quad L(x, \dot{x})) = \frac{1}{2}||\dot{x}||^2 + U(x),$$

where the Lagrangian $L(x, \dot{x})$ is the difference between the kinetic energy $\frac{1}{2}||\dot{x}||^2$ and the potential energy $-U(x)$. This means that the solutions of $(*)$ are exactly the set of "extremal" curves of the action functional. It is the mathematical formulation of the so-called *principle of least action*. Poincaré was the first to try to obtain new solutions of an n-body problem using minimization. In a short note written in 1896, he looks for quasi-periodic (periodic in a rotating frame) solutions of the 3-body problem in $\mathbb{R}^2$ as functions $x(t)$ defined on $[0, T]$, with value in 3-body configurations, which minimize the Lagrangian action $\int_0^T L\left(x(t), \dot{x}(t)\right)$ among those with the following property: after the "period" T, the new triangle $x(T)$ is the image of the initial one $x(0)$ by a rigid rotation and the three sides have respectively turned by the real (not mod 2π) angles $\alpha, \alpha + k_1, \alpha + k_2$ where k_1 and k_2 are fixed integers. This amounts to fixing a 1-dimensional homology class in the space of triangles up to rotation (this space has the topology of $\mathbb{R}^3$ deprived of three half-lines from the origin). Assuming existence (this is a consequence of Tonelli's theorem proved around 1930 because $k_1 \neq 0$ and $k_2 \neq 0$ garantee *coercivity*, that is the impossibility of minimizer "at infinity"), he was blocked by the collision problem caused by the weakness of the Newtonian attraction. Indeed, around 1913 Sundman proved that in any solution of the n-body problem which ends in a collision (partial or total) at time t_0, two bodies i, j involved in the collision satisfy the estimates

$$||\vec{r}_i(t) - \vec{r}_j(t)|| = O(|t - t_0|^{\frac{2}{3}}), \quad ||\dot{\vec{r}}_i(t) - \dot{\vec{r}}_j(t)|| = O(|t - t_0|^{-\frac{1}{3}}).$$

For the 2-body problem, these estimates are an easy exercise which was enough to convince Poincaré that the action of a solution ending in collision might (in fact always does) converge, hence that a minimizer could a priori be the mere concatenation through collisions of segments of solutions. He simply eliminated the problem by assuming a "strong force" potential (proportional to the inverse squared distance).

Poincaré's retreat was in a sense wise because very often such homology constraints indeed lead to minimizers with collisions. The simplest example is given by the Kepler problem of attraction by a fixed center in the plane (the 2-body problem can be reduced to this). Let us look for periodic solutions of the equation $\ddot{x} = -\frac{x}{|x|^3}$ in $\mathbb{R}^2 \setminus 0$. The action is $\int_0^T \left(|\dot{x}(t)|^2 - \frac{1}{|x(t)|}\right)dt$ and one seeks for minimizers in the space of loops $x(t)$ of period T going k times around the origin (i.e. loops belonging to a fixed homology class). Coercivity is insured as soon as the integer k is different from 0. It was proved by Gordon that for

$k = \pm 1$, minimizers are exactly the elliptic solutions of the given period T, with any excentricity (along a curve of critical points, a function stays constant !) while, if $k \neq 0, \pm 1$, minimizers are only collision-ejection solutions (ellipses with excentricity 1). The main point was to notice that, by convexity of the action, a sequence of ejection collisions in a given time T has a higher action than a single ejection collision solution during the same time.
A partial generalization of this result was given by Venturelli (and also Zhang...) for the three body problem : action minimizers among loops of configurations $x(t)$ of a given period T such that, during time T the three sides of the triangle make respectively k_1, k_2, k_3 complete turns, where the k_i are fixed integers, are the equilateral elliptic homographic solutions of the given period and any excentricity if $(k_1, k_2, k_3) = \pm(1, 1, 1)$, a collision ejection of the given period if this is not the case and all k_i are different from 0, unknown if one of the k_i is 0.
Let us give a sketch of the case $(1, 1, 1)$. In a frame fixing the center of mass, a classical identity going back to Leibniz allows to write the action as the sum of three Keplerian actions:

$$\sum_{i<j} \frac{m_i m_j}{M} \int_0^T \left[\frac{||\dot{\vec{r}}_{ij}(t)||^2}{2} + \frac{M}{||\vec{r}_{ij}(t)||} \right] dt,$$

where $M = \sum m_i$ and $\vec{r}_{ij}(t) = \vec{r}_j(t) - \vec{r}_i(t)$. By the result of Gordon, an a priori lower boundof the action is obtained by replacing each term by its minimum, obtained if each $\vec{r}_{ij}(t)$ is a Kepler elliptic solution of period T. The end of the proof consists in showing that the Lagrange equilateral solution is the only one which achieves this lower bound: from $\sum \vec{r}_{ij}(t) \equiv 0$ it follows that $\sum \ddot{\vec{r}}_{ij}(t) \equiv 0$ that is $\sum \frac{\vec{r}_{ij}(t)}{||\vec{r}_{ij}(t)||^3} \equiv 0$ from which it follows that the $\vec{r}_{ij}(t)$ cannot be colinear and the three mutual distances $|\vec{r}_{ij}(t)|$ must be equal at each instant of time.

Notice that in all the cases considered above, collision solutions exist among minimizers. This will not be the case anymore if we minimize the action among loops $x(t)$ of configurations of period T satisfying the *italian symmetry*

$$x(t - T/2) = -x(t).$$

This symmetry selects the relative equilibria (excentricity 0) among all Keplerian motions and indeed, minimizers for the 2-body and 3-body problem are exactly the circular solutions (with equilateral configuration in the latter case). The proof is even simpler as above, the reason for the selection of the equilateral triangle among central configurations being more clearly seen to originate from the fact that it is the unique configuration which realizes the minimum of the restriction of U to $I =$ constant or, what amounts to the same, the minimum U_0 of the normalized potential function $\tilde{U}(x) = I(x)^{\frac{1}{2}} U(x)$. On the other hand, the Fourier series of a symmetric loop has no constant term and this implies the inequality

$$\int_0^T ||\dot{x}(t)||^2 dt \geq \frac{4\pi^2}{T^2} \int_0^T ||x(t)||^2 dt.$$

Hence, the action A of a symmetric loop satisfies

$$A \geq A_0 = \int_0^T \left[\frac{2\pi^2}{T^2} I(x(t)) + U_0 I(x(t))^{-\frac{1}{2}} \right] dt \geq T \inf_I \left(\frac{2\pi^2}{T^2} I + U_0 I^{-\frac{1}{2}} \right),$$

with equality if and only if there exist two configurations α and β such that $x(t) = \alpha \cos \frac{2\pi}{T} + \beta \sin \frac{2\pi}{T}$ (no harmonics of order higher than 1), and the fonction $\frac{2\pi^2}{T^2} I(x(t)) + U_0 I(x(t))^{-\frac{1}{2}}$ is constant and equals its absolute minimum. Hence $I(x(t)$ is constant, from which it follows that the two configurations α and β are orthogonal and have the same norm. Finally, $x(t)$ is a rigid circle in the configuration space. One concludes that the motion is a relative equilibrium by using the fact that the similitude classes of 3-bodies central configurations are isolated.

The two proofs above are misleading. As soon as the constraints select more complicated (non a priori known) solutions, one needs proving the existence of collision-free minimizers. In the next paragraph, an idea is given of the proof of Marchal's theorem which is the basic tool explaining why action minimizers under symmetry constraints are very often collision-free.

3.3 Marchal's theorem

Theorem 6 *Let $x' = (\vec{r}_1', \vec{r}_2', \cdots, \vec{r}_n')$ and $x'' = (\vec{r}_1'', \vec{r}_2'', \cdots, \vec{r}_n'')$ be two arbitrary configurations, possibly with collisions, of n material points with positive masses $m_1, m_2, \cdots, m_n$ in the plane or in space. For any $T > 0$, any local minimizer of the action among paths $x(t) = (\vec{r}_1(t), \vec{r}_2(t), \cdots, \vec{r}_n(t))$ in the configuration space which start at $x(0) = x'$ and end at $x(T) = x''$ is collision-free and hence a true solution of Newton's equations in the open interval $]0, T[$.*

Already in the case of two bodies, this theorem is non trivial. Translated in terms of the Kepler problem, it asserts that given two points $x', x'' \in \mathbb{R}^2 \setminus 0$ and $T > 0$, a minimizing path $x(t) \in \mathbb{R}^2 \setminus 0$ $x(0) = x'$, $x(T) = x''$, is a collision-free solution of the equation $\ddot{x}(t) = -x/||x(t)||^3$. Many proofs can be given of this special case but Marchal's one is still among the simplest.

In what follows, I give the main idea of the proof of Marchal's theorem (see [Ma3, C3, FT]) . Suppose that the minimum of the action is attained by a path $x(t)$ which has a collision at time t_0. In order to get a contradiction, we try to slightly modifiy the path in such a way as to decrease the action. The problem which was faced in the early attempts to prove that minimizers of some kind are collision-free is that, except in the case of three bodies, not much is known about the configuration taken by the bodies entering the collision. There is Sundman's theory, which says that the normalized configuration tends to the set of central configurations, but the latter ones are so poorly understood that it is of no use (for 5 bodies and more one even does not know if the number of similitude classes is finite !). Marchal proposes to chose any one of the bodies, say $\vec{r}_i$ involved in the collision and to shift slightly its position at time t_0, replacing $r_i(t)$ by $r_i(t) + \epsilon\varphi(t)\vec{v}_i$, where $\vec{v}_i$ is a unit vector and $\varphi(t)$ is a smooth function such that $\varphi(t_0)=1$, supported by a small interval $[t_0-\eta, t_0+\eta]$. Controling the modification brought to the action by this single modification is impossible but Marchal makes the striking observation that replacing the original action by the average of the modified action when $\vec{v}_i$ takes every possible direction amounts to replacing the perturbed body i by a uniform repartition of its mass over a sphere in the spatial case (resp. a circle in the planar case). But, in the spatial case, the potential generated by a homogeneous sphere is constant inside the ball bounded by the sphere and equal to the potential of a point mass at the center with the same total mass outside. This is a strong hint that the averaged action is strictly smaller than the original one.

Let us prove that it is indeed the case in the simplest possible situation, to which it is indeed possible to reduce the general case. We suppose that the minimizer $x(t)$ is a parabolic homothetic collision-ejection solution of the n-body problem in $\mathbb{R}^3$, that is:

$$x(t)=|t|^{\frac{2}{3}}x_0,\ t\in[-T,T]$$

where x_0 is some central configuration. Thanks to the linearity of the mean, we may treat separately ejection and collision, hence we can restrict the attention to the time interval $[0,T]$. We study deformations of $x(t)$ of the form

$$x^k_{\vec{s}}(t)=\big(\vec{r}_1(t),\dots,\vec{r}_k(t)+R(t)\vec{s},\dots,\vec{r}_n(t)\big),$$

where $1\le k\le n$ and $R(t)=(1-\frac{t}{T})\rho$ with ρ a small positive real number and $\vec{s}$ belongs to the unit sphere. Taking the mean of the actions over $\vec{s}$ and exchanging the order of integration amounts to truncating the potential of the (k,j)-interactions to $m_jm_k/R(t)$ for t belonging to the interval $[0,t_j]$, where t_j is the characteristic time after which this potential is the same as the one for the original path, that is

$$R(t_j)=r_{jk}(t_j)=r^0_{jk}t_j^{\frac{2}{3}},$$

which implies

$$\rho=r^0_{jk}t_j^{\frac{2}{3}}(1+O(t_j)).$$

Hence

$$\mathcal{A}^k_m-\mathcal{A}\le\frac{m_k}{2}\frac{\rho^2}{T}+\sum_{j\neq k}m_jm_k\int_0^{t_j}\left[\frac{1}{R(t)}-\frac{1}{r_{jk}(t)}\right]dt,$$

(the inequality sign comes from the fact that the deformations do not keep the center of mass fixed)

In other words, the last term is the integral over the whole interval $[0,T]$ of the function $\left[\frac{1}{R(t)}-\frac{1}{r_{jk}(t)}\right]^-$, where for any $f:[0,T]\to\mathbb{R}$, we have denoted by $f(t)^-$ the function which is equal to $f(t)$ when $f(t)\le 0$ and to 0 otherwise.

Hence

$$\mathcal{A}^k_m-\mathcal{A}\le\frac{m_k}{2T}\rho^2-\sum_{j\neq k,\,j\le p}m_jm_k\Delta_j\,,$$

where

$$\Delta_j = \frac{T}{\rho}\log(1-\frac{t_j}{T}) + \int_0^{t_j}\frac{1}{r_{jk}(t)}dt.$$

Hence

$$\mathcal{A}_m^k - \mathcal{A} \leq \frac{m_k}{2T}(r_{jk}^0)^2 t_j^{\frac{4}{3}} + O(t_j^{\frac{7}{3}}) - \sum_{j\neq k,\, j\leq p} m_j m_k \left(\frac{1}{r_{jk}^0}t_j^{\frac{1}{3}} + o(t_j^{\frac{1}{3}})\right),$$

and we conclude that $\mathcal{A}_m^k - \mathcal{A} < 0$.

The proof that one can reduce the general problem to this special case is given in [ICM]. It uses the ideas of R. Montgomery, S. Terracini and A. Venturelli; the two main steps in this proof are 1) the existence of an *isolated* collision in any local minimizer $x(t)$ and 2) the reduction, via blow-up, of the case of an arbitrary isolated collision to the case of a parabolic homothetic collision-ejection solution. In [FT] an important generalization is given,with detailed proofs, to some equivariant cases, to other exponents of the potential and any space dimension greater than 1. The main remark is that in many cases (the ones possessing the "rotating circle property"), averaging over a well-chosen circle is sufficient.

3.4 Minimization under symmetry constraints

The simplest case where Marchal's theorem applies directly is the already mentioned *italian symmetry* $x(t-T/2) = -x(t)$, which corresponds to an action of the group $\mathbb{Z}/2\mathbb{Z}$ on the space of T-periodic loops in the configuration space of the n-body problem in $\mathbb{R}^p$. Indeed, let $[t_0, t_0+T/2] \subset [0,T]$ be a fundamental domain of this action: the restriction of x to $[t_0, t_1]$ must be an unrestricted local minimizer of the action $\mathcal{A}$ among paths with the same endpoints, and as such collision-free in the open interval $]t_0, t_1[$. As the starting point t_0 may be chosen arbitrarily, we deduce that x cannot have a collision.

For the planar problem ($p=2$), this result is somewhat disapointing as one can prove that a relative equilibrium whose configuration minimizes the scaled potential $U_0 = I^{\frac{1}{2}}U$ is always an absolute minimizer and that these are the sole minimizers provided certain technical conditions are satisfied (which are at least satisfied for $n=3$ and $n=4$) . Hence, in order to get interesting minimizers, one must either look at the spatial problem ($p=3$) or impose sronger symmetry constraints. These two routes lead to interesting new families of periodic solutions of the n-body problem, the *Hip-Hops* and the *choreographies.*

1) the Hip-Hops (see [CV, C4]) Combined with known results on central configurations [Mo3] and the above remark that a relative equilibrium solution whose configuration minimizes U_0 is a minimizer for the italian symmetry, a simple analysis of Hessian of the action along such a relative equilibrium solution shows that a minimizer for the spatial problem cannot be a planar solution as soon as the number n of bodies is at least 4. The simplest case is the one of 4 equal masses for which a minimizer should be (this is not proved) the original Hip-Hop with its $D_4 \times \mathbb{Z}_2$ symmetry. In this solution, to the relative equilibrium of the square is added a vertical oscillation of the two diagonals; twice per period, the shape is the one of a regular tetrahedron. It is a remarkable compromise between the relative equilibrium of the square and the relative equilibrium of the regular tetrahedron which should have been the minimizer if it existed (it does in $\mathbb{R}^4$). More generally, whatever be the masses, the corresponding minimizers are likely to be among the "simplest" non planar solutions of the corresponding n-body problem.

2) the choreographies (see [CM, Si, CGMS])
In this case, one imposes equal masses and a symmetry constraint which implies that after time T/n, the bodies occupy the same positions save for a circular permutation (i.e. the symmetry group G contains as a subgroup a copy of $\mathbb{Z}/n\mathbb{Z}$ which acts in the indicated way). This implies the existence of a curve along which the bodies move, separated by equal time lags. It is likely that the equality of the masses is a necessary condition for such a solution to exist but up to now this is proved only when $n \leq 5$ [C6]. The simplest choreographies are the relative equilibria of n equal masses which are the vertices of a regular n-gon. Surprizingly we shall see in the next section that they are related through families of relatively periodic solutions to more complicated choreographies (in particular the figure eight solution when $n = 3$) and to Hip-Hops. An extensive search for choreographies was done by Carles Simó (see his website for animations).

4 Global continuation via minimization

We study the 3-dimensional dynamics in the neighborhood of the equilateral relative equilibrium of the regular n-gon with equal masses ($\forall i, m_i = 1$).

The fact that, when perturbed in an orthogonal direction, the length of a straightline segment stays constant at the first order of approximation, implies a splitting of the variational equation of the n-body problem along any planar solution into a part (HVE) describing the "horizontal variations" (along the plane of motion) and one (VVE) describing the vertical ones (orthogonal to the plane of motion). When the planar solution is a relative equilibrium, this last equation takes the particularly simple form

$$\ddot{z}_i = \sum_{j \neq i} \frac{m_j}{r_{ij}^3}(z_j - z_i), \qquad (VVE)$$

where the r_{ij} are the (constant) mutual distances of the bodies in the relative equilibrium and $(z_1, z_2, \cdots, z_n) \in \mathbb{R}^n$ are supposed to be such that $\sum_{i=1}^n z_i = 0$, which amounts to fixing the center of mass at the origin. In what follows, we suppose that all the m_i are equal.

After reducing the rotation symmetry by fixing the angular momentum and quotienting by the rotations around its axis, the relative equilibrium becomes an isolated equilibrium. One reads directly from the variational equation the spectrum of the linearized vector-field at this equilibrium: The corresponding $6n-10$ dimensional matrix splits into a $4n-6$ "horizontal" block and a $2n-4$ "vertical" block whose eigenvalues are all purely imaginary because the Newton force is attractive.

In the next sections, we concentrate essentially on the case $n = 3$, giving only hints at the end for the cases $n = 4$ (partially understood) and $n > 4$ more conjectural.

4.1 Bifurcations from the Lagrange equilateral relative equilibrium

When $n = 3$, after reducing the rotation symmetry and restricting to a center manifold one gets into a situation very similar to the one in the lunar problem, with a 1-1 resonant spectrum and energy surfaces diffeomorphic to the 3-sphere. Here also the local existence of two Lyapunov families of (relatively) periodic solutions can be proved: one is already known, it is the homographic family; the other one, when globally continued (see the next section) goes all the way to the reverse equilateral relative equilibrium through the planar figure eight solution. In an energy surface close to the relative equilibrium, the flow admits an annulus of section whose Poincaré return map is a twist map which, because of a resonance which persists all along the homographic family, is the identity on the corresponding boundary. I shall not reproduce the computations of [CF2] but be content with explaining the similarities and the differences with the first chapter.

For the relative equilibrium of an equilateral triangle whose edges have length 1 and vertices have masses m_i, (VVE) reads $\ddot{z}_i = \sum_{j\neq i} m_j(z_j - z_i),\ i = 0, 1, 2$. As $\sum_{i=0}^{2} m_i z_i = 0$, this becomes the following (with $M = \sum_{i=0}^{2} m_i$):

$$\ddot{z}_i = -Mz_i,\ i = 0, 1, 2.$$

We shall choose the masses to be $1/3$ so that the period of the relative equilibrium solution is 2π and the $2n - 4 = 2$ "vertical" eigenvalues are $\pm i$.

On the other hand, the $4n - 6 = 6$ "horizontal" eigenvalues are $\pm i$ and a quadruple $\pm\frac{1}{\sqrt{2}} \pm i$ (see for instance [Mo2]), so that the spectrum is completely resonant. Using Maple, an analogue of the normal form described in the first chapter can be computed. This leads to complex coordinates (u, v, h, k) (I keep the notations of [CF2]) in the tangent space (identified to $\mathbb{C}^4$) of the 8 dimensional reduced phase space such that the linearized vector field becomes free of non resonant terms up to order three. The normal form, which is not unique at a general order, can be chosen so that the vector field is invariant under $\mathcal{T}(u, v, h, k) = (u, -v, h, k)$. This corresponds to the symmetry with respect to the invariant horizontal plane, which is defined by the equation $v = 0$.
The result is of the following form:

$$\begin{aligned}
\dot{u} &= iu[1 + \alpha|u|^2 + \beta|v|^2 + \gamma hk + \bar{\gamma}\bar{h}\bar{k}] + O_5 \\
\dot{v} &= iv[1 + a|u|^2 + b|v|^2 + chk + \bar{c}\bar{h}\bar{k}] + A\bar{v}h\bar{k} + O_5 \\
\dot{h} &= \lambda h[1 + r|u|^2 + s|v|^2 + thk + t'\bar{h}\bar{k}] + Rv^2\bar{h} + O_5 \\
\dot{k} &= -\lambda k[1 + r|u|^2 + s|v|^2 + thk + t'\bar{h}\bar{k}] - R\bar{v}^2\bar{k} + O_5,
\end{aligned}$$

where the coefficients have the following non-zero values:

$$\begin{gathered}
\alpha = -1,\ \beta = -1, \gamma = \tfrac{9}{2} + 6i\sqrt{2}, \\
a = -1,\ b = -\tfrac{21}{19},\ c = \tfrac{186}{19} + \tfrac{126\sqrt{2}}{19}i,\ A = -\tfrac{120}{19}, \\
r = -\tfrac{11}{12} - \tfrac{\sqrt{2}}{12}i,\ s = -\tfrac{73}{57} + \tfrac{10\sqrt{2}}{57}i,\ t = \tfrac{275}{57} + \tfrac{334\sqrt{2}}{57}i, \\
t' = \tfrac{105}{19}(1 - i\sqrt{2}),\ R = \tfrac{5\sqrt{2}}{19}i,
\end{gathered}$$

and where O_5 stands for real analytic functions of order 5 in $u, \bar{u}, v, \bar{v}, h, \bar{h}, k, \bar{k}$.

Even if the situation looks more complicated than fin the restricted problem, it is not really so. This is because one can restrict the attention to a so-called "center manifold" tangent to the invariant space associated to the purely imaginary part of the spectrum, and containing all the local recurrence near the equilibrium. A simple analysis shows that, when lifted up to the non reduced phase space, such a 4 dimensional center manifold at the equilibrium becomes a 6 dimensional manifold tangent to the one obtained from the relative equilibrium solution by making the rotations act independently on positions and momenta. From this description of the tangent space one can deduce that the restriction of the reduced Hamiltonian to a center manifold has the equilibrium as a non degenerate minimum, which implies that its levels close enough to the equilibrium are 3 spheres (and in fact, as noted by Moeckel, that the center manifold is unique). In restriction to the center manifold (coordinates $u, \bar{u}, v, \bar{v}$), the normal form, still invariant under the mapping $\tau : (u,v) \mapsto (u,-v)$, is of the form

$$\begin{aligned}\dot{u} &= iu[1+\alpha|u|^2+\beta|v|^2]+O_5\\ \dot{v} &= iv[1+a|u|^2+b|v|^2]+O_5,\end{aligned}$$

with $v=0$ defining the Lyapunov family of equilateral homographic motions. Moreover, the energy becomes

$$H=-\frac{1}{2}+\frac{|u|^2}{36}+\frac{|v|^2}{6}+O_4.$$

The problem is now similar to the planar circular restricted problem in the Lunar case (see [Co, C0, Ku] or [Du] in a more general situation), where the Lyapunov orbits are Hill's direct and retrograde orbits. The proof of existence and local uniqueness of the vertical Lyapunov family (the one tangent to $u=0$) follows exactly as in the first chapter because $b \neq \beta$; moreover, if we knew that our center manifold is analytic, we would get also analyticity of the family. On the contrary, the higher order resonance $a=\alpha$ would prevent us from applying the same proof to the horizontal homographic family tangent to $v=0$ if we did not know that it exists. A simple analysis of the vertical variational equation along the homographic family shows that this resonance must persist in normal forms of any order: the coefficients of the monomials $u|u|^{2k}$ in $\dot{u}$ and $v|u|^{2k}$ in $\dot{v}$ are necessarily equal. One can nevertheless prove that no other Lyapunov family bifurcates from the relative equilibrium by showing that the Poincaré return map in an annulus of section, whose one boundary belongs to the homographic family and the other one to the P_{12} family, is a monotone twist map.

4.2 From the equilateral triangle to the Eight

The vertical Lyapunov family has a high symmetry. Indeed, after choosing a phase, it is tangent to the "linear" family

$$q_j(t)=\left(\frac{1}{\sqrt{3}}\zeta^j e^{i2\pi t}, A\mathrm{Re}(\bar{\zeta}^j e^{i2\pi t})\right)\in\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3,\quad j\in\mathbb{Z}/3\mathbb{Z}, \qquad (S_1)$$

where $\zeta=e^{\frac{2\pi}{3}}$ and the amplitude A is a real parameter. The discrete symmetry group of such a motion is seeked as a subgroup of

$$G_0=O(\mathbb{R}/\mathbb{Z})\times\Sigma(3)\times O(\mathbb{R}^3),$$

where $g=(\tau,\sigma,\rho)\in G_0$ acts naturally on the space of 1-periodic loops:

$$\begin{array}{rccccc} q: & \mathbb{R}/\mathbb{Z} & \times & \{1,2,3\} & \to & \mathbb{R}^3\\ & \tau\downarrow & & \sigma\downarrow & & \rho\downarrow\\ gq: & \mathbb{R}/\mathbb{Z} & \times & \{1,2,3\} & \to & \mathbb{R}^3.\end{array}$$

If q is a loop in the configuration space, the transformed loop by the (left) action of $g = (\tau, \sigma, \rho)$ is

$$gq_j(t) = \rho q_{\sigma^{-1}(j)}(\tau^{-1}(t)).$$

Lemma 7 *The stabilizer $G_1 \subset G_0$ of S_1 is isomorphic to the dihedral group D_6 with 12 elements.*

The proof is an easy exercise. One finds that the elements (τ, σ, ρ) of G_1 act as follows (vectors in $\mathbb{R}^3$ are decomposed into a horizontal part h and a vertical part v):

$$\tau^{-1}(t) = \xi(t-\theta),\ \sigma^{-1}(j) = \xi(j+\delta),\ \rho(h,v) = (e^{i2\pi\alpha}\bar{h}^{\xi},\ e^{i\pi\beta}v),$$

with $\xi = \pm 1$ (and $\bar{h}^{\xi} = h$ or $\bar{h}$ according to whether $\xi = +1$ or -1) and $\alpha \in \mathbb{R}/\mathbb{Z}$, $\beta \in \mathbb{Z}/2\mathbb{Z}$, $\delta \in \mathbb{Z}/3\mathbb{Z}$, $\theta \in \mathbb{R}/\mathbb{Z}$ satisfying

$$\alpha = \theta - \frac{\delta}{3} \pmod 1, \quad \theta = \frac{\beta}{2} - \frac{\delta}{3} \pmod 1.$$

The choices of $(\xi = 1, \beta = 1, \delta = 1)$ and $(\xi = -1, \beta = 0, \delta = 0)$ define generators g_1 and g_2 of G_1 which satisfy the relations $g_1^6 = g_2^2 = 1, g_1 g_2 = g_2 g_1^{-1}$, which is a presentation of D_6.

In a frame which rotates uniformly in the opposite direction with the same frequency as the relative equilibrium, (S_1) becomes

$$\hat{q}_j(t) = \left(\zeta^j e^{i4\pi t}, \mathrm{Re}(\bar{\zeta}^j e^{i2\pi t}\right) \in \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3, \quad j \in \mathbb{Z}/3\mathbb{Z}, \qquad (S_1).$$

The symmetry group does not change but its action does: the formula defining α is changed to $\alpha = 2\theta - \frac{\delta}{3} = \beta - \delta \pmod 1 = 0 \pmod 1$. The resulting curve in rotating frame is now a *choreography.* Indeed, the group element defined by $\xi = 1, \beta = 0, \delta = 1$, transforms $(h_j(t), v_j(t))$ into $\left(h_{j+1}(t - \frac{1}{3}), v_{j+1}(t - \frac{1}{3})\right)$: all bodies lie on one and the same spatial curve (see figure).

Global continuation of the family is based on the following remark (see[CF2]): let us consider the following family (parametrized by ϖ) of paths in the configuration space:

$$q_j^{\varpi}(t) = \left(\frac{1}{\sqrt{3}}\left(\frac{4\pi+\varpi}{2\pi}\right)^{-\frac{2}{3}} \zeta^j e^{i(4\pi+\varpi)t}, 0\right), \quad j \in \mathbb{Z}/3\mathbb{Z}. \qquad (L)$$

In a frame which rotates uniformly with frequency ϖ, it becomes a loop making two complete rotations during the period 1. Hence, for all values of ϖ, the corresponding path has the G_1 symmetry in the rotating frame. Its action during the period 1 is readily computed to be proportional to $\left(\frac{4\pi+\varpi}{2\pi}\right)^{\frac{2}{3}}$. In particular, it tends to its absolute minimum zero as ϖ tends to -4π, the limit situation corresponding in the inertial frame to bodies at rest at infinity. When ϖ varies from -4π to 0, the action increases. It can stop being the absolute minimum among paths which, in the rotating frame become loops with the G_1

symmetry only when it appears a 1-periodic Jacobi field, that is a solution of the variational equation which, in the rotating frame, is 1-periodic and possesses the required G_1 symmetry. This is the case only when $\varpi = -2\pi$. For values of ϖ closer to 0, the minimum is no more the (L) family but an appropriate lift of the vertical Lyapunov family. The global continuation is obtained by looking, for each value of ϖ between -2π and 0 to such a minimizer among paths which are G_1-symmetric in the rotating frame. The end of the family is the figure Eight solution for which the D_6 symmetry can be interpreted as the symmetry of the space of similitude classes of plane oriented triangles (the so-called *shape sphere* (see[CM, Mo1]). It is the maximal discrete symmetry that a solution of the 3-body problem may possess in the case of equal masses (see [Ma1]).

Technically, one is faced with the problem of showing that, for each value of ϖ, a (local) minimizer has no collision. This is not a direct consequence of Marchal's theorem because of the time reversal symmetry which implies that the boundaries of a fundamental domain of the τ action on the time circle cannot be chosen arbitrarily. Nevertheless, this can be proved by a direct estimation of a lower bound of the action of paths with collision with the given symmetry : this lower bound happens to be exactly the value of the action of the last member of (L) corresponding to $\varpi = 0$.

Remarks. 1) Using obvious symmetries, the P_{12} family can be continued into a loop of quasi-periodic solutions containing the horizontal equilateral relative equilibria rotating in both directions (figure). Applying isometries and scaling, this defines in the 12 dimensional (after reduction of translations) phase space a compact invariant 6 dimensional submanifold entirely foliated by relatively periodic solutions. Topologically, this manifold is a fibre space over the lens space $L(4, 1)$.

2) It is interesting to recall a remark made by C. Marchal at page 257 of his book[Ma1]: after having determined the expansion ot the vertical Lyapunov family up to order 6 in a small parameter c_1 corresponding to the vertical extension of the solution (opening of the mouth of the oyster described in the rotating frame), he asks for their continuation, mentioning as an example of surprising continuation the family of retrograde Hill solutions up to the colinear "Schubart" solution (see [He]).

4.3 From the square to the Hip-Hop

In the case of the square relative equilibrium of 4 equal masses, there are two Lyapunov families in addition to the homographic family; one of these leads by continuation to the Hip-Hop, which is the simplest non-planar solution of the 4 body problem. The possibility of obtaining this family by minimization of the action is related to the fact that, in $\mathbb{R}^3$, a relative equilibrium must be planar (this is indeed true of any homographic solution of the n-body problem but it is an easy result only in the case of relative equilibria).

The case of the Hip-Hop corresponds to a frequency which is not in resonance with the frequency of the relative equilibrium; the local study is done in [Ba] (compare also with [MS] for the case of an additional central mass).

The global continuation of the Hip-Hop family is done in [TV]. Here also, the proof that there are no collisions for minimizers in this family cannot appeal to Machal's theorem (except in the case of a non rotating frame, that is for the original Hip-Hop where, if one insists on the full $D_4 \times \mathbb{Z}_2$ symmetry, the strengthening given in [FT] is needed). The problem is the topological constraint attached to the rotating frame: one has to minimize among loops whose starting point and end point make a fixed real (not mod 2π) angle between 0 and 2π, in way similar to what one can be done for the planar Kepler problem. The method is a nice idea of introducing obstacles. The end of the family should be a simultaneous double collision but this is not proved.

Remark. The fate of the first vertical Lyapunov family, associated to the frequency of the relative equilibrium is complicated, probably leading through a secondary bifurcation to a planar solution proved to exist at first numerically by J. Gerver and then with a computer assisted proof by Kapela and Zglyczinski (this solution lies in the horizontal plane and not the vertical one because its angular momentum, contrarily to the figure eight solution, is not zero).

4.4 The avatars of the regular n-gon relative equilibrium: eights, chains and generalized Hip-Hops

Symmetries of the solutions of VVE along the regular n-gon relative equilibrium are easily analyzed [CF3] and may lead to Lyapunov families with interesting continuation [CF1]. Possible problems connected to minimization under the corresponding symmetry constraints could appear for $n \geq 6$ because of the appearance of new imaginary eigenvalues of the Horizontal Variational Equation [Mo2] which could lead to different types of bifurcations with the given symmetries.

Remark. It is easy to prove that when, looked in the inertial frame, the members of the vertical Lyapunov families attached to the regular n-gon relative equilibrium are choreographies for a dense set of values of the frequency ϖ of the rotating frame in which they have the symmetries of the linearized equation.

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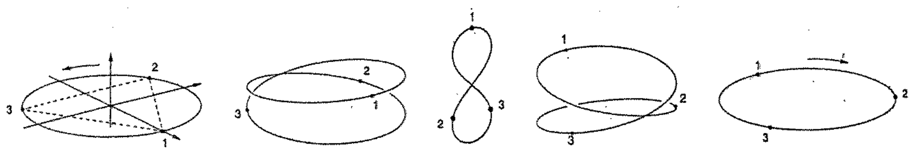
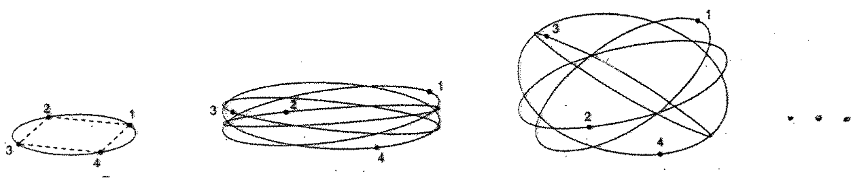
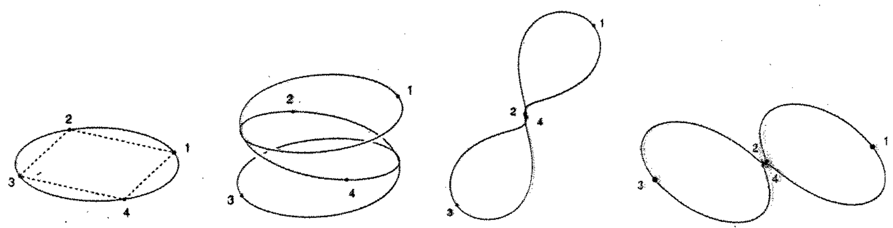


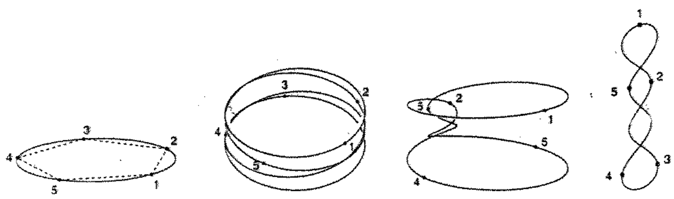
FROM EQUILATERAL TO EIGHT : THE P12 FAMILY



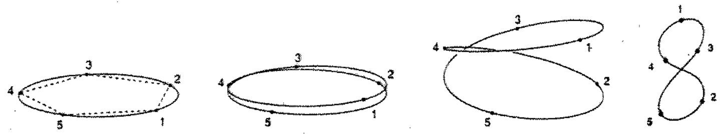
THE HIP-HOP FAMILY



4 BODIES : THE FIRST VERTICAL FREQUENCY



FROM PENTAGON TO 4.CHAIN



FROM PENTAGON TO EIGHT

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