Are there perverse choreographies ?

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Let $\mathcal{C}(t) = \big(q(t+1), q(t+2), \ldots, q(t+n) = q(t)\big)$ be a *planar choreography* of period n of the n punctual masses $m_1, m_2, \ldots, m_n$, that is a planar n-periodic solution of the n-body problem where all n bodies follow one and the same curve $q(t)$ with equal time spacing (see [CGMS]). In the sequel, we shall identify the planar curve $q(t)$ with a mapping $q : \mathbb{R}/n\mathbb{Z} \to \mathbb{C}$ (for convenience of notation, we have chosen the period to be n; well chosen homotheties on configuration and velocities reduce the general case to this one).

Question. *Does there exist planar choreographies (with equal time spacing) whose masses are not all equal ?*

The following proposition says that it is enough to study planar choreographies with equal masses.

Proposition 1. *The curve $\mathcal{C}(t)$ is still a planar choreography, with the same center of mass (and the same sum of the masses), when each mass m_j is replaced by the arithmetic mean $m = \sum m_i/n$.*

Notations. We shall denote respectively by ρ_j and a_j the complex numbers denoted by z_{jn} and a_{jn} in [C], that is

$$\rho_j(t) = q(t+j) - q(t), \quad a_j(t) = \frac{\rho_j(t)}{|\rho_j(t)|^3} \text{ if } j \neq n,\ a_n(t) = 0.$$

We can suppose that the center of mass is at 0, that is

$$\sum_{i=1}^{n} m_i q(t+i) = 0.$$

In particular, translating the time by integers, we get that for any t we have

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_n & m_1 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ m_2 & m_3 & \cdots & m_1 \end{pmatrix} \begin{pmatrix} q(t+1) \\ q(t+2) \\ \cdots \\ q(t+n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}.$$

In the same way, Newton's equations

$$\forall s,\ 1 \leq s \leq n,\ \ddot{q}(t+s) = \sum_{1 \leq k \leq n,\, k \neq s} m_k \frac{q(t+k) - q(t+s)}{|q(t+k) - q(t+s)|^3},$$

may be written

$$\forall s,\ \ddot{q}(t) = \sum_{j=1}^{n-1} m_{j+s} \frac{q(t+j) - q(t)}{|q(t+j) - q(t)|^3},$$

that is

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_n & m_1 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ m_2 & m_3 & \cdots & m_1 \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ \cdots \\ a_n(t) \end{pmatrix} = \begin{pmatrix} \ddot{q}(t) \\ \ddot{q}(t) \\ . \\ \ddot{q}(t) \end{pmatrix}.$$

Let us denote by $\mathcal{M}$ the above "circulant" $n \times n$ matrix of masses. We shall use the following property of such matrices (see [MM]):

Lemma 1. *The matrix $\mathcal{M}$ is diagonalisable over $\mathbb{C}$. An orthogonal basis of $\mathbb{C}^n$ is defined by the eigenvectors $X_k = (\zeta^k, \zeta^{2k}, \ldots, \zeta^{nk} = 1)$, where $\zeta = e^{2\pi i/n}$. The corresponding eigenvalues are the $\lambda_k = m_1 + m_2\zeta^k + \cdots + m_n\zeta^{(n-1)k}$. As the masses are positive, it follows that the image of $\mathcal{M}$ always contains the line generated by $(1, 1, \ldots, 1)$ and that its kernel is always contained in the hyperplane $\mathcal{H} = \big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n,\ z_1 + z_2 + \cdots + z_n = 0\big\}$.*

Here, $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)$ orthogonal means that $\sum u_i \overline{v}_i = 0$.
It is important to notice that the X_k are independent of the masses m_i. Indeed, they are also the eigenvectors, with eigenvalues ζ^k, of the "circular permutation matrix" (endomorphism of $\mathbb{C}^n$ which acts by a circular permutation of the coordinates)

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As the circulant mass matrix is $\mathcal{M} = m_1\mathrm{Id} + m_2\mathcal{P} + m_3\mathcal{P}^2 + \cdots + m_n\mathcal{P}^{n-1}$, this is indeed the key to the proof of the lemma.

Proof of Poposition 1. One immediately deduces from the fact that $X_0 = (1, 1, \ldots, 1)$ is an eigenvector with associated eigenvalue the sum $M = \sum m_i$ of the masses that, if

$$\tilde{a}_i(t) = a_i(t) - \frac{1}{M}\ddot{q}(t),$$

we have

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_n & m_1 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ m_2 & m_3 & \cdots & m_1 \end{pmatrix} \begin{pmatrix} \tilde{a}_1(t) \\ \tilde{a}_2(t) \\ \cdots \\ \tilde{a}_n(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \end{pmatrix}.$$

¿From Lemma 1 it then follows that for all t,

$$\sum_{j=1}^{n} q(t+j) = 0, \quad \text{and} \quad \sum_{j=1}^{n} \tilde{a}_j(t) = 0.$$

The last identity may also be written

$$\ddot{q}(t) = \frac{\sum m_i}{n} \sum_{j=1}^{n-1} \frac{q(t+j) - q(t)}{|q(t+j) - q(t)|^3},$$

which proves the proposition.

Before stating the next proposition, we recall from [C] some terminology.

Definition 1. *A solution $R(t) = (\vec{r}_1(t), \ldots, \vec{r}_n(t))$ of the n-body problem with masses $m_1, \ldots, m_n$ is called* perverse *if it is also a solution for at least another set of masses. Any set of masses for which $R(t)$ is a solution will be called* admissible.

Corollary 1. *The choreographies whose masses are not all equal are exactly the perverse choreographies.*

Notation.

$$\mu_i = m_i - \frac{M}{n}.$$

One deduces from the definitions and from Proposition 1 that, for all t,

$$(*) \qquad \sum_{i=1}^{n} \mu_i = 0, \quad \sum_{i=1}^{n} \mu_i \rho_i(t) = \sum_{i=1}^{n} \mu_i q(t+i) = 0, \quad \sum_{i=1}^{n} \mu_i a_i(t) = \sum_{i=1}^{n} \mu_i \tilde{a}_i(t) = 0.$$

Definition 2. *A non-trivial $\mathcal{P}$-decomposition of $\mathbb{C}^n$ is an orthogonal decomposition*

$$\mathbb{C}^n = (1, 1, \ldots, 1)\mathbb{C} \oplus K \oplus L$$

into $\mathcal{P}$-invariant subspaces such that: (i) each subspace K, L is invariant under complex conjugation, (ii) neither K nor L is reduced to $\{0\}$. It will be noted (K, L).

Definition 3. *Let $\mathcal{C} = \mathcal{C}(t) = \big(q(t+1), q(t+2), \ldots, q(t+n) = q(t)\big)$ be a planar choreography of period n whose all masses are equal to m and whose center of mass is at 0. A $\mathcal{P}$-decomposition (K, L) of $\mathbb{C}^n$ is said to be* adapted *to $\mathcal{C}$ if it is non-trivial and such that $\big(q(t+1), q(t+2), \ldots, q(t+n)\big)$ and $\big(\tilde{a}_1(t), \tilde{a}_2(t), \ldots, \tilde{a}_n(t)\big)$ belong to K for all t.*

Proposition 2. *Let $\mathcal{C}(t) = \big(q(t+1), q(t+2), \ldots, q(t+n) = q(t)\big)$ be a planar choreography of period n whose all masses are equal to m and whose center of mass is at 0. It is perverse (= really perverse) if and only if there exists an adapted $\mathcal{P}$-decomposition (K, L). If this is the case, the admissible sets of masses $(m_1, \ldots, m_n)$ are exactly the ones of the form $m_i = m + \mu_i > 0$, where $(\mu_1, \mu_2, \ldots, \mu_n)$ is a real vector belonging to L.*

Proof of Proposition 2. Let $(\mu_1, \mu_2, \ldots, \mu_n) \neq (0, 0, \ldots, 0)$, $\sum \mu_i = 0$, be such that $\mathcal{C}(t)$ is still a choreography when the bodies are endowed with the masses $m_i = m + \mu_i$. By the proof of proposition 1, the (complex) vectors $\big(q(t+1), q(t+2), \ldots, q(t+n)\big)$ and $\big(\tilde{a}_1(t), \tilde{a}_2(t), \ldots, \tilde{a}_n(t)\big)$ must be contained in the kernel K of $\mathcal{M}$, that is in the space generated by the eigenvectors X_k such that $\lambda_k = m_1 + m_2\zeta^k + \cdots + m_n\zeta^{(n-1)k} = 0$. As X_k is also an eigenvector of $\mathcal{P}$, K is $\mathcal{P}$-invariant; as $\lambda_{n-k} = \overline{\lambda}_k$ is also equal to zero if λ_k is, K is invariant under complex conjugation. Finally, K is neither reduced to $\{0\}$ (the bodies would be in collision) nor to the orthogonal $\mathcal{H}$ of $(1, 1, \dots, 1)$ (all the masses m_i should be equal). This proves that if L is the space generated by the eigenvectors $X_l \neq (1, 1, \dots, 1)$ not belonging to K, the pair (K, L) is an adapted $\mathcal{P}$-decomposition. The condition $\lambda_k = 0$ may be written $\zeta^k(\mu_1 + \mu_2\zeta^k + \cdots + \mu_n\zeta^{(n-1)k}) = 0$, that is $(\mu_1, \mu_2, \dots, \mu_n)$ orthogonal to X_k. As by hypothesis it is already orthogonal to $(1, 1, \dots, 1)$, it belongs to L.

In the other direction, let us suppose that (K, L) is an adapted $\mathcal{P}$-decomposition, and let $(\mu_1, \mu_2, \dots, \mu_n)$ be any real vector in L. As such, it is orthogonal to $(1, 1, \dots, 1)$ and to any vector in K. In particular $(*)$ is satisfied. But $\mathcal{C}(t)$ is a choreography with all masses equal to m and center of mass at zero, hence

$$\sum_{i=1}^{n} mq(t+i) = 0 \quad \text{and} \quad \ddot{q}(t) = \sum_{j=1}^{n-1} ma_j(t).$$

Comparing to $(*)$, one obtains that $\mathcal{C}(t)$ is also a choreography with masses $m_i = m + \mu_i$ and center of mass at zero.

Equations. As the decomposition $\mathbb{C}^n = (1, 1, \dots, 1)\mathbb{C} \oplus K \oplus L$ is orthogonal, it is easy to write equations of a subspace K belonging to a non-trivial $\mathcal{P}$-decomposition (K, L). As it is invariant under complex conjugation, one can find real equations: it is enough to write that K is the set of vectors orthogonal to $(1, 1, \dots, 1)$ and to the vectors

$$c_l = (1/2)(X_l + \overline{X}_l) = \left(\cos\frac{2l\pi}{n}, \cos\frac{4l\pi}{n}, \dots, \cos\frac{2(n-1)l\pi}{n}, 1\right),$$
$$s_l = (1/2i)(X_l - \overline{X}_l) = \left(\sin\frac{2l\pi}{n}, \sin\frac{4l\pi}{n}, \dots, \sin\frac{2(n-1)l\pi}{n}, 0\right),$$

where the eigenvectors X_l, $l = l_1, \dots, l_s \leq n/2$, span L.

Starting from this set of equations for K, we now get the following criterion. Let us say that a set of integers $1 < l_1 < \cdots < l_s \leq n/2$ is non-trivial if, defining L as the subspace of $\mathbb{C}^n$ generated by $X_{l_1}, \overline{X}_{l_1}, \dots, X_{l_s}, \overline{X}_{l_s}$, the pair (K, L) is non-trivial.

Corollary 2. *Let $\mathcal{C}(t) = \big(q(t+1), q(t+2), \dots, q(t+n) = q(t)\big)$ be a planar choreography of period n whose all masses are equal and whose center of mass is at 0. It is perverse (= really perverse) if and only if there exists a non-trivial set of integers $1 < l_1 < \cdots < l_s \leq n/2$ such that, for all t, the following vectors in $\mathbb{C}^{n-1}$,*

$$R(t) = \big(\rho_1(t), \dots, \rho_{n-1}(t)\big), \quad A(t) = (a_1(t), \dots, a_{n-1}(t)) = \left(\frac{\rho_1(t)}{|\rho_1(t)|^3}, \dots, \frac{\rho_{n-1}(t)}{|\rho_{n-1}(t)|^3}\right),$$

belong to the kernel of the

$$C_{l_1,\dots,l_s} = C_L = \begin{pmatrix} \cos\frac{2l_1\pi}{n} & \cos\frac{4l_1\pi}{n} & \cdots & \cos\frac{2(n-1)l_1\pi}{n} \\ \sin\frac{2l_1\pi}{n} & \sin\frac{4l_1\pi}{n} & \cdots & \sin\frac{2(n-1)l_1\pi}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \cos\frac{2l_s\pi}{n} & \cos\frac{4l_s\pi}{n} & \cdots & \cos\frac{2(n-1)l_s\pi}{n} \\ \sin\frac{2l_s\pi}{n} & \sin\frac{4l_s\pi}{n} & \cdots & \sin\frac{2(n-1)l_s\pi}{n} \end{pmatrix}.$$

Proof of Corollary 2. The orthogonality of the vectors $\big(q(t+1), q(t+2), \dots, q(t+n)\big)$ and $\big(\tilde a_1(t), \tilde a_2(t), \dots, \tilde a_n(t)\big)$ to $(1,1,\dots,1)$ just states that $\mathcal{C}(t)$ is a planar choreography of period n whose all masses are equal to m and whose center of mass is at 0 (compare to Proposition 1). It is equivalent to

$$q(t) = -\frac{1}{n}\sum_{j=1}^{n-1}\rho_j(t) \quad \text{and} \quad \frac{1}{m}\ddot q(t) = \sum_{j=1}^{n-1}\frac{\rho_j(t)}{|\rho_j(t)|^3}.$$

As L is orthogonal to $(1,1,\dots,1)$, the orthogonality of $\big(q(t+1), q(t+2), \dots, q(t+n)\big)$ and $\big(\tilde a_1(t), \tilde a_2(t), \dots, \tilde a_n(t)\big)$ to the c_l and s_l is equivalent to the orthogonality of

$$\begin{aligned}\big(\rho_1(t), \dots, \rho_{n-1}(t), 0\big) &= \big(q(t+1), q(t+2), \dots, q(t+n)\big) - q(t)(1,1,\dots,1),\\ \left(\frac{\rho_1(t)}{|\rho_1(t)|^3}, \dots, \frac{\rho_{n-1}(t)}{|\rho_{n-1}(t)|^3}, 0\right) &= \big(\tilde a_1(t), \tilde a_2(t), \dots, \tilde a_n(t)\big) + \frac{1}{nm}\ddot q(t)(1,1,\dots,1),\end{aligned}$$

to the c_l and s_l. This shows the Corollary.

Remark 1. The rank of C_L equals the dimension of L. This is because the rank of C_L does not change if one adds to C_L the nth column $(1,0,\dots,1,0,\dots,1,0)$, whose sum with the $(n-1)$ columns of C_L is 0. But the lines of the extended matrix are the vectors $c_{l_1}, s_{l_1}, \dots, c_{l_s}, s_{l_s}$ which generate L.

The end of the paper is devoted to the proof of

Proposition 3. *For $n \leq 5$, the planar n-body problem does not possess any perverse choreography.*

This is a very modest result: comparing to Proposition 3 of [C], which asserts that the planar n body problem does not possess any really perverse solution for $n \leq 4$, we gain only the case $n = 5$. To go further, one should take into account dynamical considerations and not only algebraic ones.

Proof of Proposition 3.

We start we some general remarks.

1) Saying that all masses are necessarily equal amounts to saying that the kernel of $\mathcal{M}$ must coincide with the orthogonal $\mathcal{H}$ of $(1,1,\dots,1)$, that is

$$\forall k \in \mathbb{Z},\ 1 \leq k \leq n-1,\ \lambda_k = 0$$

(as the eigenvalues go by conjugate pairs $\lambda_{n-k} = \overline{\lambda}_k$, it is sufficient to take $1 \leq k \leq \frac{n}{2}$).

This is of course the same as saying that every adapted $\mathcal{P}$-decomposition must be trivial.

2) The bigger K is, the less constrained the motion is but the more restricted the masses are (L small).

2-i) n **even:** at the two extremes we have
– a 1-dimensional K generated by $X_{n/2} = (-1,1,-1,1,\ldots,-1,1)$, which is ruled out because it implies collisions,
– a 1-dimensional L generated by $X_{n/2} = (-1,1,\ldots,-1,1)$, that is

$$K = \big\{(z_1,\ldots,z_n) \in \mathbb{C}^n,\ z_1 + z_3 + \cdots + z_{n-1} = 0,\ z_2 + z_4 + \cdots + z_n = 0\big\}.$$

In this case, in fact *as soon as L contains $X_{\frac{n}{2}}$*, each $(n/2)$-tuple $1,3,\ldots,n-1$ and $2,4,\ldots,n$ is a choreography when the bodies are endowed with masses all equal to $2m$. Indeed, one deduces from the fact that $(q(t+1), q(t+2), \ldots, q(t+n))$ and $(\tilde{a}_1(t), \tilde{a}_2(t), \ldots, \tilde{a}_n(t))$ belong to K that:

$$q(t+2) + q(t+4) + \cdots + q(t+n-2) + q(t) = 0,$$
$$\frac{q(t+2)-q(t)}{|q(t+2)-q(t)|^3} + \frac{q(t+4)-q(t)}{|q(t+4)-q(t)|^3} + \cdots + \frac{q(t+n-2)-q(t)}{|q(t+n-2)-q(t)|^3} = \frac{1}{2m}\ddot{q}(t),$$

and permutations.
As $X_{\frac{n}{2}}$ is either orthogonal to L (that is $\lambda_{\frac{n}{2}} = m_1 - m_2 + m_3 - m_4 + \cdots + m_{n-1} - m_n = 0$) or contained in L (that is $\lambda_{\frac{n}{2}} \neq 0$), we can state the

Lemma 2. *For any choreography with an even number n of bodies, either*

$$m_1 + m_3 + \cdots + m_{n-1} = m_2 + m_4 + \cdots + m_n\,,$$

or each $(n/2)$-tuple $1,3,5,\ldots,n-1$ and $2,4,\ldots,n$ is itself a choreography when endowed with masses all equal to $2M/n$. In this last case, one could freely transfer mass from one $(n/2)$-tuple to the other.

2-ii) n **odd:** *at the two extremes we have*

– a 2-dimensional K generated by X_k and $\overline{X}_k$ for some k between 1 and $(n-1)/2$.

– a 2-dimensional L generated by $X_l, \overline{X}_l$, for some l between 1 and $(n-1)/2$.

Lemma 3. *For any non trivial $\mathcal{P}$-decomposition satisfying the conclusions of Proposition 2, the dimension of K must be strictly bigger than 2.*

Remark 2. If K is of dimension 2, either $k > 1$ is not prime to n and any configuration in K must have collisions, or the configuration is at all times the image by a linear transformation of the standard (stellated if $k \neq 1$) regular n-gon. Indeed, if k is not prime to n, at least two coordinates of X_k are equal. Otherwise, the coordinates of X_k are exactly the vertices of a direct regular n-gon (stellated if $k \neq 1$). The conclusion follows from the fact that the linear transformations of $\mathbb{R}^2 = \mathbb{C}$ are exactly those of the form $z \mapsto \alpha z + \beta \overline{z}$.

Note that choreographies whose configuration at all times is the linear image of a fixed regular polygon do exist: relative equilibria with a regular polygon configuration of course, but also any choreography with $n \leq 3$ bodies (because any triangle is the linear image of an equilateral one), and Gerver's supereight for four bodies where the configuration is

always symmetric with respect to the origin (that is a parallelogram), which is equivalent to being a linear image of the square (the existence of this orbit is for the moment known only numerically, see [CGMS]). This does not mean that these solutions possess an adapted $\mathcal{P}$-decomposition with $\dim K = 2$; for this, we should have asserted also that the configurations of the $\tilde{a}_i(t)$ are a linear image of the same fixed regular polygon.
Indeed, none of these choreographies is perverse. In the case of 3 or 4 bodies this is stated in Proposition 3. In the case of relative equilibria with regular polygon configuration, this is implied by [E] or [PW] but we shall give a direct proof (Proposition 5).

Proof of Lemma 3. We have already ruled out the case $\dim K = 1$ because it implies collisions for the configuration, so let us suppose that $\dim K = 2$.
Corollary 2 asserts that $R(t) = \big(\rho_1(t), \ldots, \rho_{n-1}(t)\big)$ and $A(t) = \Big(\frac{\rho_1(t)}{|\rho_1(t)|^3}, \ldots, \frac{\rho_{n-1}(t)}{|\rho_{n-1}(t)|^3}\Big)$ both belong to the kernel $\tilde{K}$ of C_L. By Remark 1, the rank of C_L is equal to the dimension of L, that is to $n-3$ (the number of bodies must be at least 4 for the $\mathcal{P}$-decomposition to be non trivial). Hence the dimension of $\tilde{K}$ is equal to 2. In particular, there exists two indices $1 \leq j \neq k \leq n-1$, such that the projection of $\tilde{K}$ on the plane generated by the j-th and the k-th elements of the canonical basis of $\mathbb{C}^{n-1}$ is injective. In other words, $\tilde{K}$ can be defined by equations of the form $u_i = \alpha_i u_j + \beta_i u_k,\ i \leq n-1,\ i \neq j, k$. Moreover, $\tilde{K}$ being invariant under conjugation, such equations exist with real coefficients α_i, β_i. This implies that for all t and all $1 \leq i \leq n-1,\ i \neq j, k$,

$$\left\{ \begin{array}{c} \rho_i(t) = \alpha_i \rho_j(t) + \beta_i \rho_k(t), \\ \dfrac{\rho_i(t)}{|\rho_i(t)|^3} = \alpha_i \dfrac{\rho_j(t)}{|\rho_j(t)|^3} + \beta_i \dfrac{\rho_k(t)}{|\rho_k(t)|^3}. \end{array} \right\} \qquad (**)$$

The coefficients α_i and β_i are both different from 0. Otherwise, if for example $\alpha_i = 0$, the coefficient β_i satisfies $|\beta_i|^3 = 1$, that is $\beta_i = 1$, which implies collision.
Now, either t is such that $\rho_j(t)$ and $\rho_k(t)$ are independent over the reals, and equations $(**)$ imply the equality of the norms $|\rho_i(t)|, |\rho_j(t)|$ and $|\rho_k(t)|$ for all $i \neq j, k$, or, on the contrary, t is such that $\rho_j(t)$ and $\rho_k(t)$ are dependent over the reals, and equations $(**)$ imply that the n bodies $q(t+1), q(t+2), \ldots, q(t+n)$, are collinear. As one cannot pass from one situation to the other without avoiding a collision, one of them must be realized all the time. But perpetual collinearity for a choreography would imply collision at some time (in fact it was proved in 1904 by Pizzetti that this condition forces the solution to be homographic). Hence the equality of the norms $|\rho_i(t)|, |\rho_j(t)|$ and $|\rho_k(t)|$ for all $i \neq j, k$ is realized at all times. Replacing t by $t+1, t+2, \ldots$, this implies the equality at all times of all the mutual distances between the bodies. But for more than three bodies, this is impossible in the plane.

Remark 3. We showed in Remark 2 that if $\dim K = 2$, a configuration in K either has a collision, or is the linear image of a regular n-gon. In this last case, if n is even, such a configuration is symmetric with respect to the origin. *This is more generally the case if, n being even, K is generated by eigenvectors X_k sharing this symmetry property, that is if all the corresponding indices k are odd.* Then the motion of each pair $(i, i+n/2)$ of symmetric bodies endowed with equal masses must be also a choreography with center of mass at zero:

$$\begin{cases} \quad q(t+n/2)+q(t)=0, \\ \dfrac{q(t+n/2)-q(t)}{|q(t+n/2)-q(t)|^3}=\dfrac{2}{M}\ddot{q}(t) \quad \text{(and permutations).} \end{cases}$$

As each couple must lie on one and the same circle, the configuration would be at all times a regular n-gon with uniform motion. By Proposition 5 this is impossible if not all masses are equal.

End of the proof of Proposition 3.

1) For $n=3$, any $\mathcal{P}$-decomposition is trivial.

2) For $n=4$ or $n=5$, in any non-trivial $\mathcal{P}$-decomposition, the dimension of K cannot exceed 2, and one concludes with Lemma 3.

Choreographies with 6 bodies. There remains three cases to study:

1) K is of dimension 4, generated by $X_1, \overline{X}_1, X_2, \overline{X}_2$,

2) K is of dimension 3, generated by $X_1, \overline{X}_1, X_3$,

3) K is of dimension 3, generated by $X_2, \overline{X}_2, X_3$.

Case 1 The equations of L give that

$$m_1=m_3=m_5, \quad m_2=m_4=m_6.$$

According to Lemma 2, if not all the masses are equal, the motion of the three bodies 2,4,6 (resp. 1,3,5) endowed with equal masses $m_1+m_2=M/3$, is also a choreography with center of mass at zero.

If the three-body choreography is Lagrange relative equilibrium motion (equilateral triangle, the configuration must be at all times a regular hexagon and the motion is uniform. By Proposition 5 this is impossible if not all masses are equal.

If the three-body choreography is the "eight", a collision occurs each time one of the triples is in Euler (=collinear) configuration.

It remains to understand the case of an arbitrary three-body choreography...

Case 2. Corollary 2 gives the relations

$$\begin{aligned} &\rho_3=\rho_1+\rho_4=\rho_5+\rho_2, \\ &\frac{\rho_3}{\rho_3^3}=\frac{\rho_1}{\rho_1^3}+\frac{\rho_4}{\rho_4^3}=\frac{\rho_5}{\rho_5^3}+\frac{\rho_2}{\rho_2^3}. \end{aligned}$$

This implies $|\rho_1|=|\rho_2|=|\rho_3|=|\rho_4|=|\rho_5|$. Translating the time, we get that the mutual distances between the six bodies must all be the same at all times, which is impossible in the plane.

One could also have remarqued that a configuration in K is necessarily symmetric with respect to 0 and conclude as in Remark 3.

Case 3. Corollary 2 gives the relations

$$\rho_3 = \rho_1 - \rho_4 = \rho_5 - \rho_2,$$
$$\frac{\rho_3}{|\rho_3|^3} = \frac{\rho_1}{|\rho_1|^3} - \frac{\rho_4}{|\rho_4|^3} = \frac{\rho_5}{|\rho_5|^3} - \frac{\rho_2}{|\rho_2|^3}.$$

This implies $|\rho_1| = |\rho_2| = |\rho_3| = |\rho_4| = |\rho_5|$ and one concludes as in Case 2.
Finally, we have proved the

Proposition 4. *A perverse choreography of 6 bodies could only be of the following type: $m_1 = m_3 = m_5 \neq m_2 = m_4 = m_6$ and each triple 1,3,5 and 2,4,6 is itself a three-body choreography with the same center of mass when endowed with masses all equal to $m_1 + m_2 = M/3$.*

Polygonal relative equilibria. We extract from [PW] a proof of the

Proposition 5. *The relative equilibrium choreographic solutions of the regular n-gon are not perverse.*

Proof. Let $\mathcal{C}(t) = \big(q(t+1), q(t+2), \ldots, q(t+n)\big)$ be the choreography (with all masses equal to 1) defined by

$$q(t) = r\zeta^t, \quad \zeta = e^{\frac{2\pi i}{n}}, \quad 4\pi^2 r^3 = n^2 \sum_{k=1}^{n-1} \alpha_k, \quad \alpha_k = \frac{1-\zeta^k}{|1-\zeta^k|^3}\,, k = 1, \ldots, n-1.$$

As we have

$$\begin{cases} \big(q(t+1), q(t+2), \ldots, q(t+n)\big) = r\zeta^t(\zeta, \zeta^2, \ldots, \zeta^n), \\ \big(\tilde{a}_1(t), \tilde{a}_2(t), \ldots, \tilde{a}_n(t)\big) = \dfrac{1}{r^2}\zeta^t \left[(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 0) + \dfrac{4\pi^2 r^3}{n^3}(1, 1, \ldots, 1)\right], \end{cases}$$

it is enough, by Proposition 2, to prove that the smallest subspace of $\mathbb{C}^n$ which is invariant under $\mathcal{P}$ and complex conjugation and contains the vectors

$$X = (\zeta, \zeta^2, \ldots, \zeta^n), \quad Y = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 0) \quad \text{and} \quad (1, 1, \ldots, 1),$$

coincides with $\mathbb{C}^n$.
Let us introduce the circulant matrix U whose columns are $\mathcal{P}^{n-1}Y, \mathcal{P}^{n-2}Y, \ldots, \mathcal{P}Y, Y$ (this is the transposed (=complex conjugate) of the matrix $A = C_0$ of [PW]):

$$U = \begin{pmatrix} \alpha_n & \alpha_{n-1} & \ldots & \alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_n & \ldots & \alpha_3 & \alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{n-2} & \alpha_{n-3} & \ldots & \alpha_n & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & \alpha_n \end{pmatrix} \qquad \text{(we have set } \alpha_n = 0).$$

By Lemma 1, the eigenvalues of U are, for $k = 1, \ldots n$,

$$\lambda_k = \alpha_n + \alpha_{n-1}\zeta^k + \ldots + \alpha_1\zeta^{(n-1)k} = \sum_{l=1}^{n-1} \frac{\zeta^{lk} - \zeta^{l(k-1)}}{|\zeta^{lk} - \zeta^{l(k-1)}|^3} = -\frac{1}{4}\sum_{l=1}^{n-1} \frac{\sin[(2k-1)\frac{\pi l}{n}]}{\sin^2 \frac{\pi l}{n}}.$$

(Note that we call λ_k what is called λ_{n-k+1} in [PW]). One sees immediately that

$$\lambda_{n-k+1} + \lambda_k = 0.$$

This implies that, if $n = 2p+1$, one has $\lambda_{p+1} = 0$. Moreover, $\lambda_n = -\lambda_1 = 4\pi^2 r^3/n^2 > 0$.

Lemma 4. *With the exception of λ_{p+1} when $n = 2p + 1$, all the eigenvalues of U are different from 0. In particular, the rank of U is equal to $n - 1$ if $n = 2p + 1$ and to n if $n = 2p$.*

We have already noticed that $\lambda_1 = -\lambda_n = -4\pi^2 r^3/n^2$ is strictly positive. For the remaining eigenvalues $\lambda_2 = -\lambda_{n-1}, \cdots, \lambda_{[n/2]} = -\lambda_{n+1-[n/2]}$, an elementary but clever proof of Lemma 4 is given in [PW] (lemmas 10,11,12).

Corollary 3. *The smallest subspace of $\mathbb{C}^n$ which is invariant under $\mathcal{P}$ and complex conjugation and contains $Y = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n = 0)$ coincides with $\mathbb{C}^n$.*

Proof of Corollary 3. It follows from Lemma 4 that the subspace generated by Y and its images under the iterates of $\mathcal{P}$ is $\mathbb{C}^n$ if n is even and the orthogonal of the complex line generated by the eigenvector $X_{p+1} = (\zeta^{p+1}, \zeta^{2(p+1)}, \ldots, \zeta^{n(p+1)})$ corresponding to λ_{p+1} if $n = 2p+1$ is odd. In this last case, the complex conjugate $\bar{X}_{p+1}$ of X_{p+1} is orthogonal to X_{p+1}:

$$\zeta^{p+1}\zeta^{p+1} + \zeta^{2(p+1)}\zeta^{2(p+1)} + \cdots + \zeta^{n(p+1)}\zeta^{n(p+1)} = \zeta + \zeta^2 + \cdots + \zeta^n = 0.$$

This proves Corollary 3 and hence Proposition 5.

Remark 4. Let us call $\beta_k = 1 - \zeta^k$, V the matrix obtained from U by replacing the alphas by the betas and $\mu_1, \ldots, \mu_n$ the eigenvalues of V:

$$\mu_k = \beta_n + \beta_{n-1}\zeta^k + \cdots + \beta_1\zeta^{(n-1)k} = \sum_{l=1}^{n-1}(\zeta^{lk} - \zeta^{l(k-1)}) = -2\sum_{l=1}^{n-1} \sin\frac{\pi l}{n}\sin[(2k-1)\frac{\pi l}{n}].$$

As

$$(\beta_k, \beta_{k+1}, \ldots, \beta_{k-1}) = (1, 1, \ldots, 1) - \zeta^{k-1}(\zeta, \zeta^2, \ldots, \zeta^n),$$

the image of V is generated by the eigenvectors $(\zeta, \zeta^2, \ldots, \zeta^n)$ and $(1, 1, \ldots, 1)$ whose eigenvalues are respectively $\mu_1 = -n$ and $\mu_n = n$. This implies that

$$\mu_2 = \mu_3 = \cdots = \mu_{n-1} = 0.$$

On one hand, this remark "explains" the nature of Lemma 4: as always in these questions (compare [C]), we had to show that for a certain collection of vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$ such that $\vec{A} + \vec{B} + \vec{C} + \ldots = \vec{0}$, one has

$$\frac{\vec{A}}{|\vec{A}|^3} + \frac{\vec{B}}{|\vec{B}|^3} + \frac{\vec{C}}{|\vec{C}|^3} + \ldots \neq \vec{0}$$

(take $\vec{A} = \sin\frac{\pi l}{n} e^{i(2k-1)\frac{\pi l}{n}}$, etc...)
On the other hand, it is with Lemma 4 the key of the proof in [PW] that the regular n-gon with at least four vertices is a central configuration only when all masses are equal: indeed, the condition is easily seen to be that the mass vector $(m_1, m_2, \ldots, m_n)$ belongs to

the kernel of the matrix $C_{-\frac{\omega^2}{M}}$ of [PW], which is the transposed (=complex conjugate) of $U - \frac{4\pi^2 r^3}{n^3} V$. As the circulant matrices U and V have the same eigenvectors, one deduces from the computation of the n numbers $\lambda_k - \frac{4\pi^2 r^3}{n^3}\mu_k$, $k = 1, \ldots, n$, that the kernel of $U - \frac{4\pi^2 r^3}{n^3} V$ is generated by $X_1 = (\zeta, \zeta^2, \ldots, \zeta^n)$ and $X_n = (1, 1, \ldots, 1)$ if $n = 2p$ and by X_1, X_n and the supplementary vector $X_{p+1} = (\zeta^{p+1}, \zeta^{2(p+1)}, \ldots, \zeta^{n(p+1)})$ if $n = 2p+1$. When $n = 2$, X_1 and X_2 generate $\mathbb{C}^2$, when $n = 3$, X_1, X_2, X_3 generate $\mathbb{C}^3$, but as soon as $n \geq 4$, the kernel of $U - \frac{4\pi^2 r^3}{n^3} V$ does not contain real vectors other than the multiples of $(1, 1, \ldots, 1)$.

Two questions.

1) Choreographies also exist in three-space. The problem of their possible perversity is completely open.

2) It is natural to wonder about the existence of choreographies with unequal masses and unequal time spacings between the bodies. This leads for example to the following unsolved question: is the regular n-gon with equal masses the sole central configuration such that

1) all the bodies lie on a circle,
2) the center of mass coïncides with the center of the circle?

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