Action minimizing periodic solutions of the N-body problem

A. CHENCINER, Paris

I.C.M.S. Edinburgh 25.5.2001

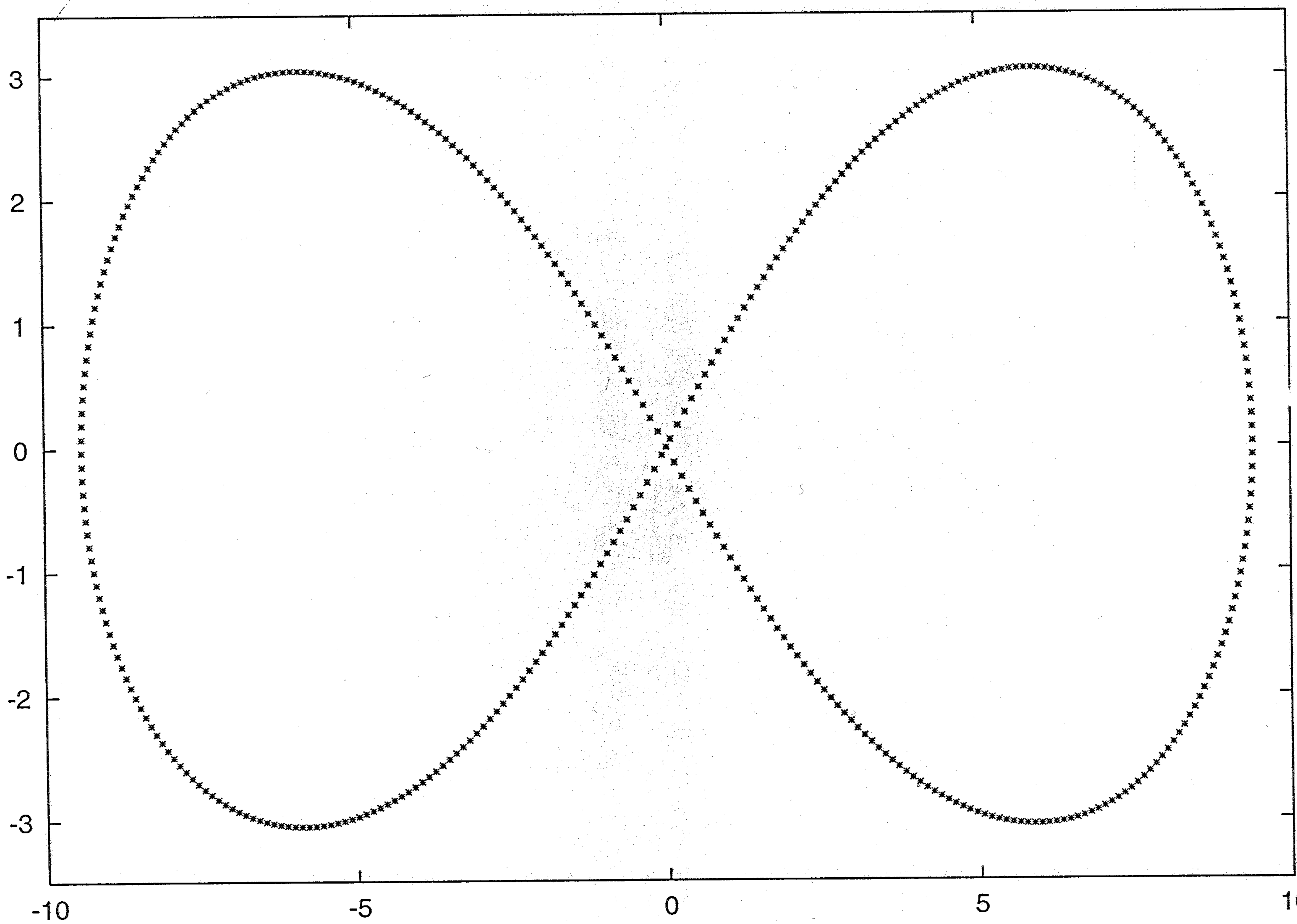
Lorentz center, Leiden 26.5.2001

Oberwolfach, 17.7.2001

Trieste, 1.8.2001

399 bodies of equal masses

399 corps [illegible]



COMPUTED BY C. SIMÓ

Newton equations

$$\cancel{m_i}\,\ddot{\vec{r}}_i = -\sum_{j\neq i} \cancel{m_i}\, m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

$$\Updownarrow$$

$$\ddot{x} = \nabla U(x),$$

where ∇ = grad. for the "mass metric"

$$\|x\|^2 = I$$

N bodies in $E \cong \mathbb{R}^k$

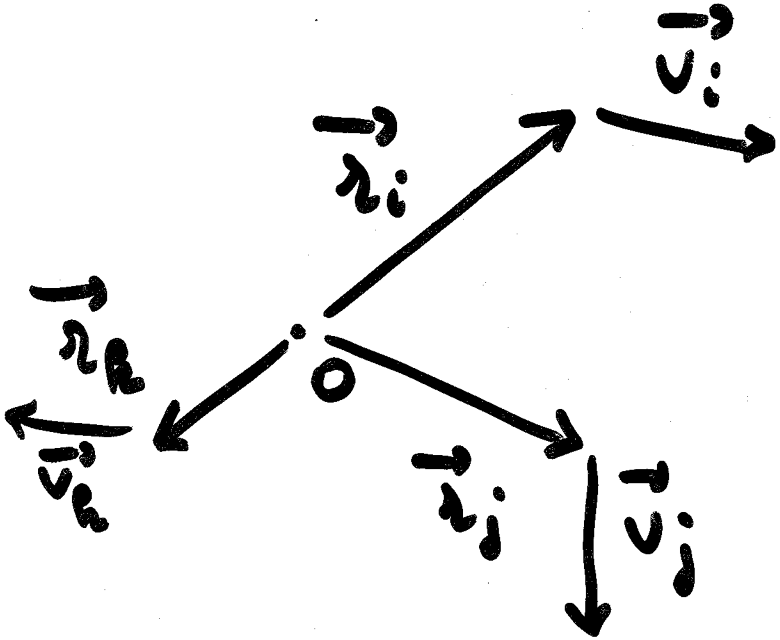
Configuration space $\hat{X} = X \setminus \text{coll.}$

$$X = \left\{ x = (\vec{r}_1, \dots, \vec{r}_N) \in E^N, \sum m_i \vec{r}_i = \vec{0} \right\}$$

$$\text{coll.} = \left\{ x \in X, \exists\, i \neq j, \vec{r}_i = \vec{r}_j \right\}$$

Phase space $T\hat{X} \equiv \hat{X} \times X$

$$(x, \; y)$$



$$x = (\vec{r}_1, \dots, \vec{r}_n)$$

$$y = (\vec{v}_1, \dots, \vec{v}_n)$$

Moment of inertia / 0

$$I(x,y) = \sum_i m_i |\vec{r}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\vec{r}_i - \vec{r}_j|^2$$

2 × Kinetic energy

$$K(x,y) = \sum_i m_i |\vec{v}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\vec{v}_i - \vec{v}_j|^2$$

$$I = \|x\|^2, \quad K = \|y\|^2$$

Newtonian potential

$$U(x,y) = \sum_{i<j} \frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|}$$

Lagrangian

$$L = \frac{K}{2} + U > 0$$

Action

$$S : H^1(\mathbb{R}/T\mathbb{Z}, X) \longrightarrow \mathbb{R} \cup \{+\infty\}$$

$$(t \mapsto x(t)) \longmapsto \int_0^T L(x(t), \dot{x}(t))\,dt$$

<u>Problem</u>: Can one <u>choose</u>

$$\Lambda \subset H^1(\mathbb{R}/T\mathbb{Z}, X)$$

s.t. $S|_\Lambda$ attains its minimum at an interesting T-periodic solution x of Newton equation

$$\ddot{x} = \nabla U(x)$$

?

3 ways one could fail:

① NON COERCIVITY — A minimiser could be "at infinity"

② COLLISION — A minimiser could have collisions

③ TRIVIALITY — A minimiser could be a well-known solution

(1) If $\Lambda = H^2$, $\min S|_\Lambda = 0$,

realized by limit of

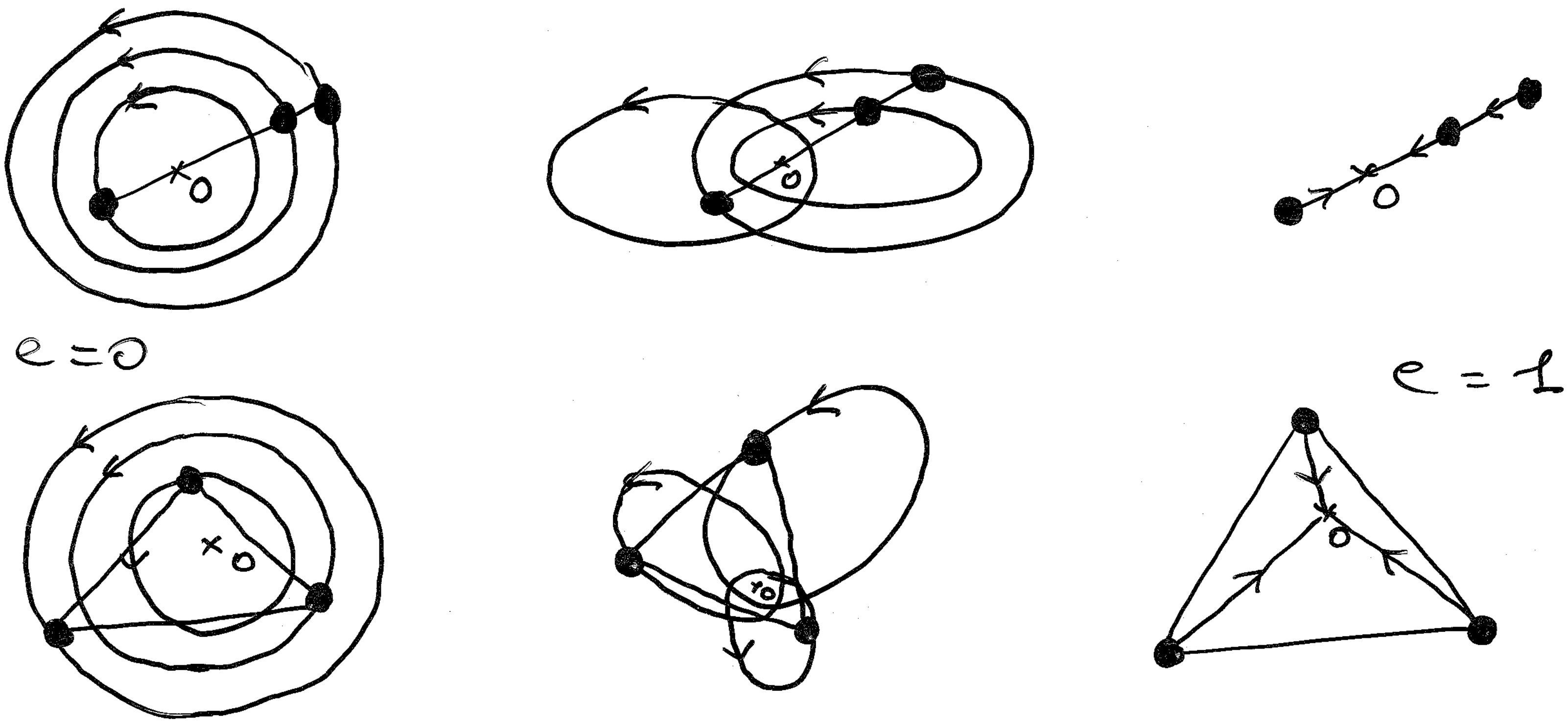
(2) If a solution of Newton's equations is such that $\vec{x}_i(t_0) = \vec{x}_j(t_0)$,

$$\begin{cases} |\vec{x}_i(t) - \vec{x}_j(t)| \sim \text{cste}\, |t-t_0|^{2/3} \\ |\dot{\vec{x}}_i(t) - \dot{\vec{x}}_j(t)| \sim \text{cste}\, |t-t_0|^{-1/3} \end{cases}$$

(explicit for 2 bodies, Sundman 1913 in general)

$\Longrightarrow$ $S(x)$ stays finite on such a solution !!!

(3) "Trivial solutions = HOMOGRAPHIC = Kepler-like
3 bodies (Euler 1763, Lagrange 1772)



Exist only for central configurations x :

$$x \parallel \nabla U(x) \iff x \text{ crit. pt of } U\big|_{I=cste}$$

SUR LES SOLUTIONS PÉRIODIQUES
ET LE PRINCIPE DE MOINDRE ACTION

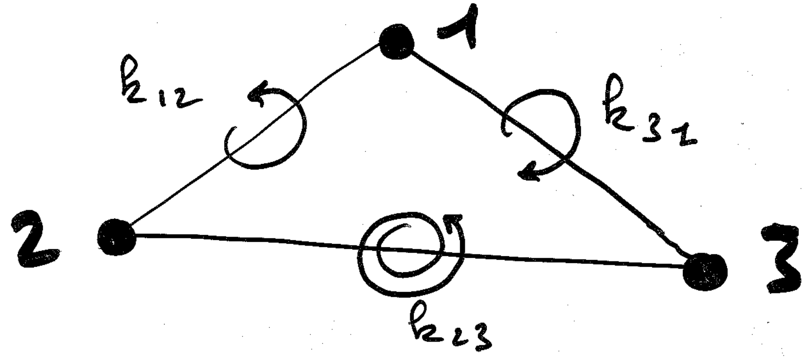
Comptes rendus de l'Académie des Sciences, t. 123, p. 915-918 (30 novembre 1896).

La théorie des solutions périodiques peut, dans certains cas, se rattacher au principe de moindre action.

Supposons trois corps se mouvant dans un plan et s'attirant en raison inverse du cube des distances ou d'une puissance plus élevée de ces distances; j'appelle a, b, c ces trois corps.

Poincaré's way of solving the problem:

① $\Lambda = \{$ loops in a given $\underbrace{\text{homology class}}_{= \; (k_{12}, k_{23}, k_{31}) \in \mathbb{Z}^3} \}$

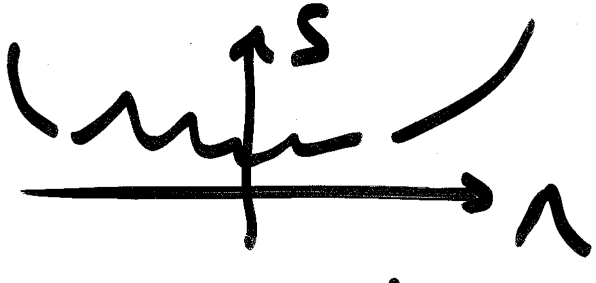


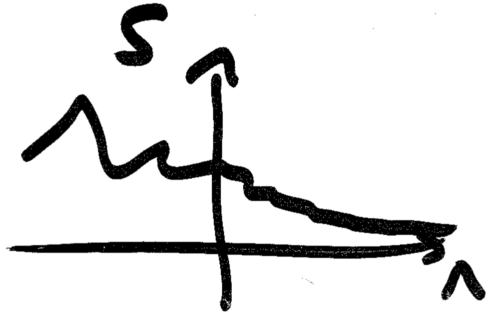
② cheat: replace $\frac{1}{r}$ by $\frac{1}{r^2}$ (strong force)

③ $(k_{12}, k_{23}, k_{31}) \neq \pm(1,1,1)$ excludes Lagrange solutions

Comments: Poincaré addresses the pb mod $SO(2)$ but idem

① If 2 of the k_{ij}'s are $\neq 0$, $\Longrightarrow$ COERCIVITY

i.e. 

and not 

because if some body goes far away

$\Longrightarrow$ the length of the loop is big

$\Longrightarrow$ S is big

Then Tonelli 1920 + Gordon 1977 (S l.s.c. on H^1_{weak}) $\Longrightarrow$ $\exists$ min. on Λ

② $\frac{1}{r^2}$ pot $\Rightarrow$ $S(\longrightarrow \text{collision}) = +\infty$

SYMMETRY

$$\begin{tikzcd} \mathbb{R}/T\mathbb{Z} \arrow[r, "x"] \arrow[d, "\beta"', "\text{isometry}"'] & X \arrow[d, "\alpha", "\text{isometry}"'] \\ \mathbb{R}/T\mathbb{Z} \arrow[r, "(\alpha,\beta).x"] & X \end{tikzcd}$$

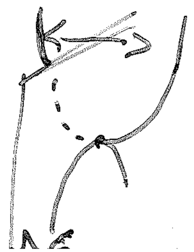
$$(\alpha, \beta) \in G$$

Principle of symmetric criticality (Palais)

if S is G-invariant,

a crit. pt of $S|_{\Lambda_G}$ is a crit. pt of S

Λ_G: G-invariant loops



2 bodies in $\mathbb{R}^2$ $\Longleftrightarrow$ Kepler pb

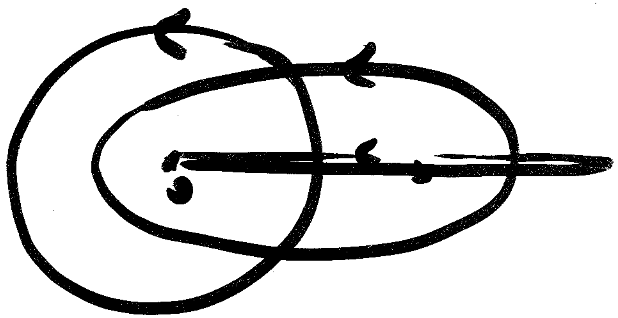
Homology constraints (Gordon 1977)

$$\Lambda_{\neq 0},\ \Lambda_k \subset H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$$

$$\shortparallel$$

Weak closure of $\left\{ x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2 \setminus \{0\}),\ \text{s.t. index } x \neq 0,\ \text{or} = k \right\}$

x min. of $S|_{\Lambda_{\neq 0}}$ $\Longleftrightarrow$



$\Uparrow$

x min of $S|_{\Lambda_{\pm 1}}$

x min of $S|_{\Lambda_{k \neq -1,0,1}}$ $\Longleftrightarrow$

collision!

TOOL: convexity of $S = \text{cste}\, T^{1/3}$

2 bodies in $\mathbb{R}^k$

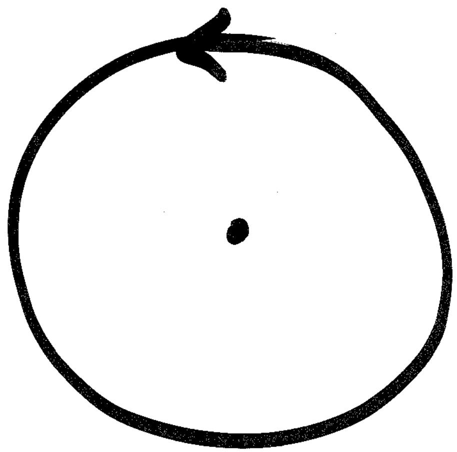
Symmetry constraints

$\mathbb{Z}/2\mathbb{Z}$ $\quad \alpha(x) = -x$
$\quad \beta(t) = t + T/2$

(De Giovanni, Giannoni, Marino 1987
v. Coti Zelati 1990)

Invariant loops $\Lambda_a = \{x,\ x(t - \frac{T}{2}) = -x(t)\}$

x min of $\Lambda_a \iff$



3 bodies in $\mathbb{R}^2$

Homology constraints (A. Venturelli 2001)

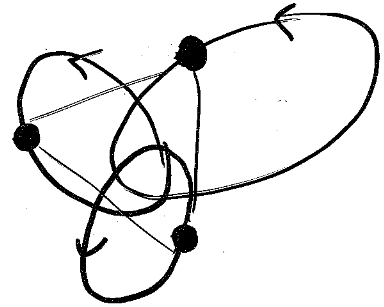
$\Lambda_{\neq 0}$: each $k_{ij} \neq 0$

Λ_{k_1, k_2, k_3} : each k_{ij} fixed

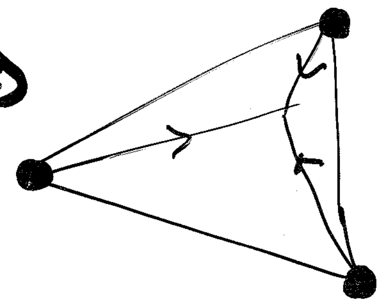
x min. of $S|_{\Lambda_{\neq 0}}$

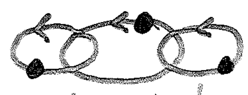
$\Updownarrow$

x min of $S|_{\Lambda_{\pm(1,1,1)}}$ $\Longleftrightarrow$



x min of $S|_{\Lambda_{k_1,k_2 k_3}}$ $\neq \left| \begin{array}{l} -(1,1,1) \\ (0, k_2, k_3) \\ (k_1, 0, k_3) \\ (k_1, k_2, 0) \\ (1,1,1) \end{array} \right.$ $\Longleftrightarrow$

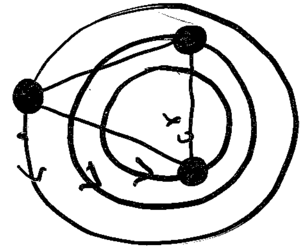


? $(1,0,1)$ 

Broucke, Hénon

Symmetry constraints

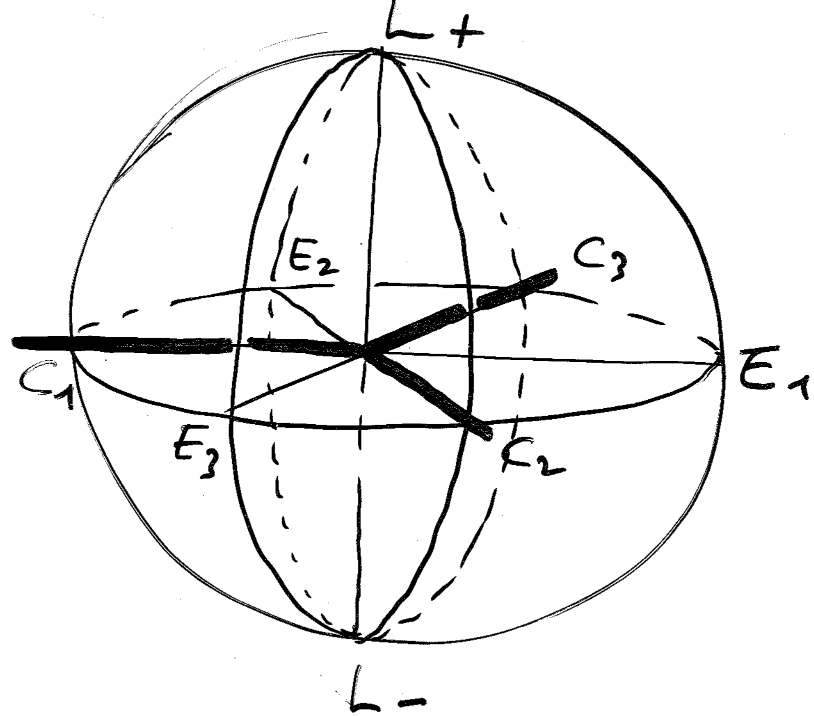
$\mathbb{Z}/2\mathbb{Z}$ A.C. & N. Desolneux 1998

x min. of $\mathcal{S}|_{\Lambda_a} \Longleftrightarrow$ 

D_6 **EQUAL MASSES**

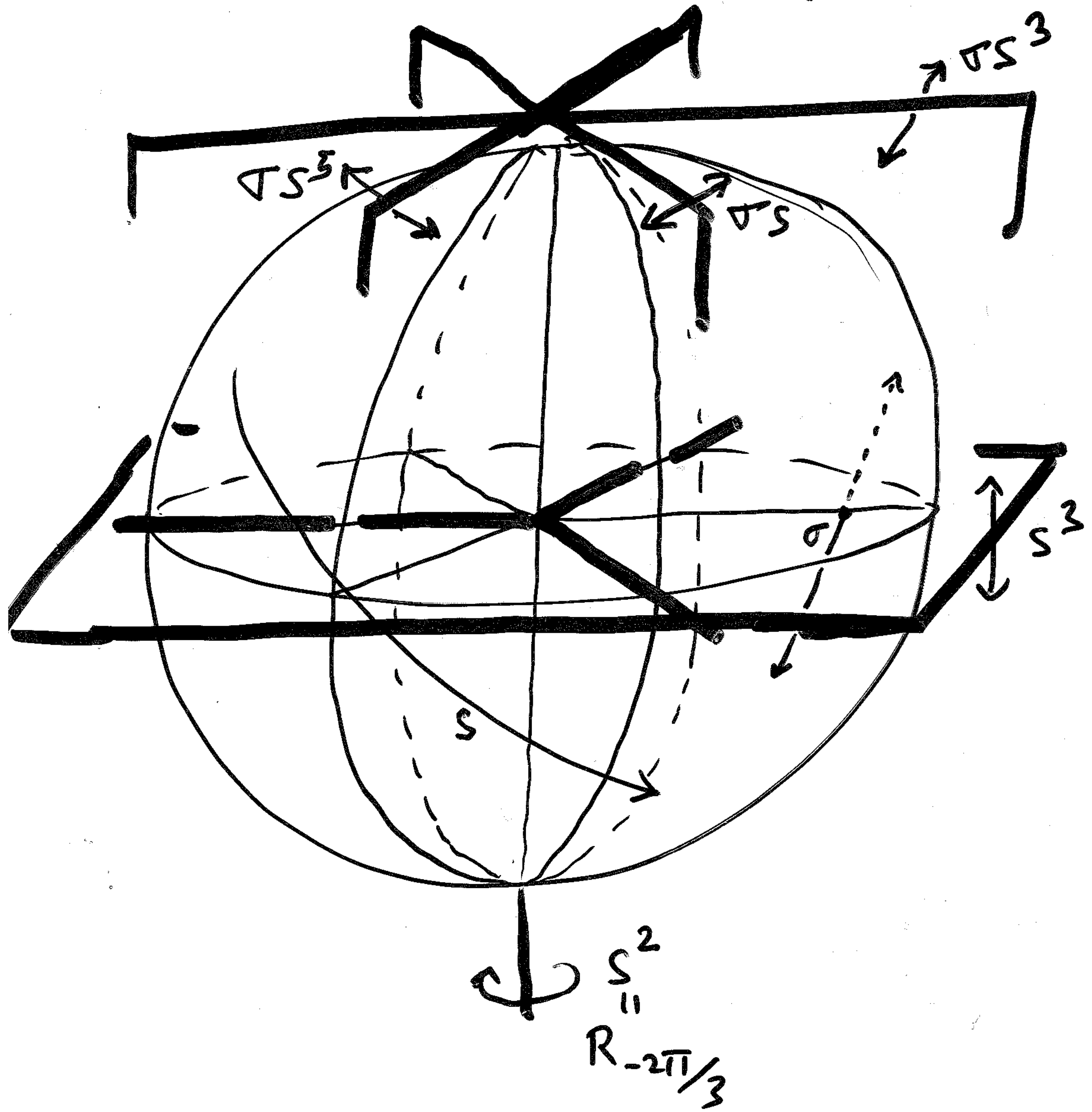
D_6 = sym. group of the space of similitude classes of oriented triangles : $\to S^2$

$$X \simeq \mathbb{R}^4 \xrightarrow[\text{Hopf}]{/SO(2)} \mathbb{R}^3 \overset{}{\underset{I=1}{\hookleftarrow}} S^2$$



The $\mathcal{E}$-action of D_6 on $H^1(\mathbb{R}/T\mathbb{Z}, X)$

$$D_6 = \{ s, \sigma ; \; s^6 = 1, \; \sigma^2 = 1, \; s\sigma = \sigma s^{-1} \}$$



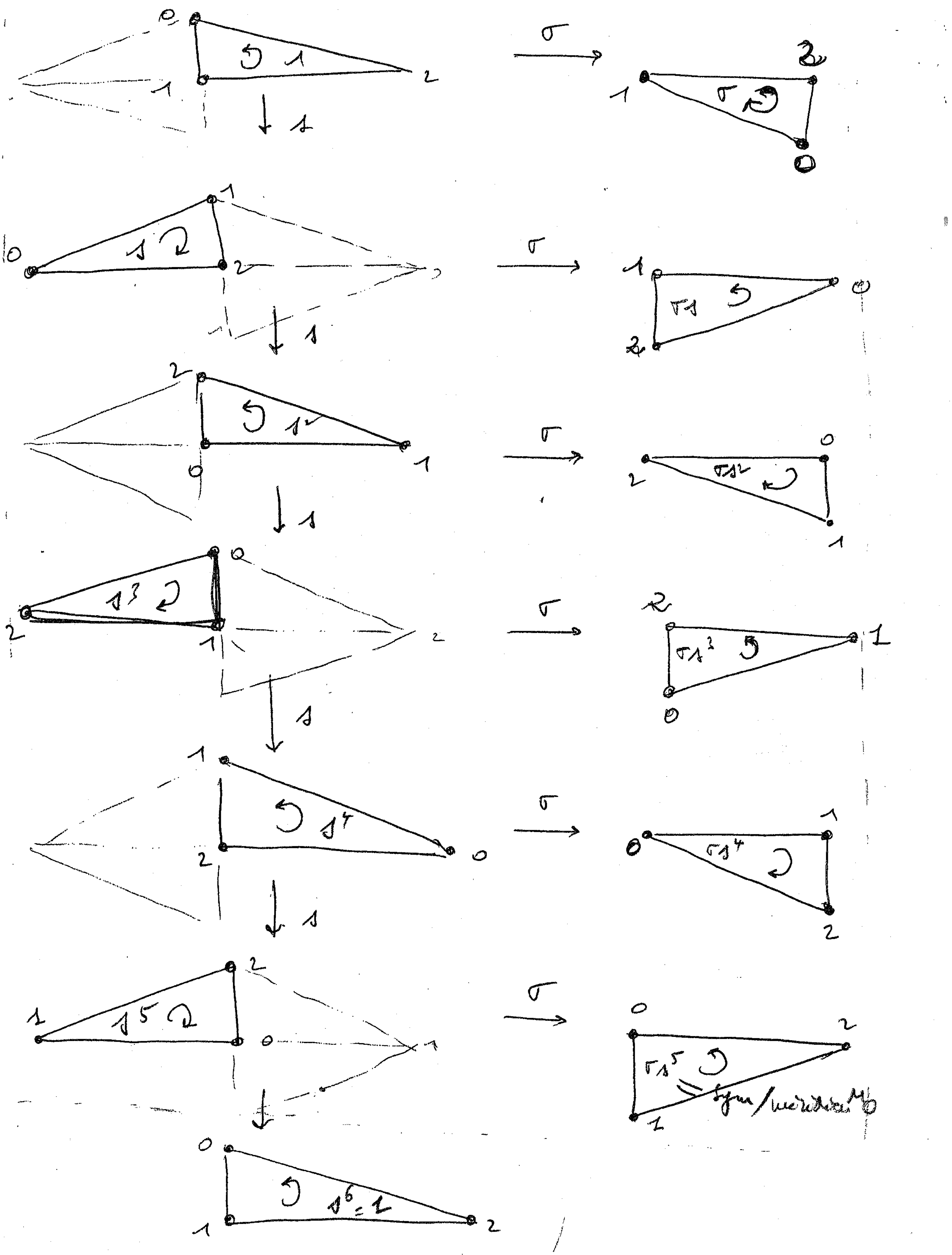
$$\alpha(s)(x_0, x_1, x_2) = (-\overline{x}_2, -\overline{x}_0, -\overline{x}_1)$$

$$\alpha(\sigma)(\text{——}) = (-x_0, -x_2, -x_1)$$

$$\beta(s)(t) = t + T/6$$

$$\beta(\sigma)(t) = -t$$

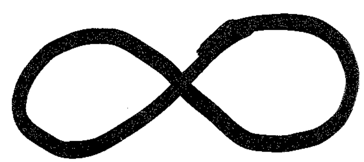
$$\begin{tikzcd}[column sep=small]
 & D_6 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
D_3 \arrow[dr, dash] & \mathbb{Z}_6 \arrow[d, dash] & D_3 \arrow[dl, dash] \\
 & \mathbb{Z}_3 &
\end{tikzcd}$$


σ
s
s²
s³
s⁴
s⁵
s⁶ = 1
σs
σs²
σs³
σs⁴
σs⁵

Thm (A.C. – R. Montgomery)

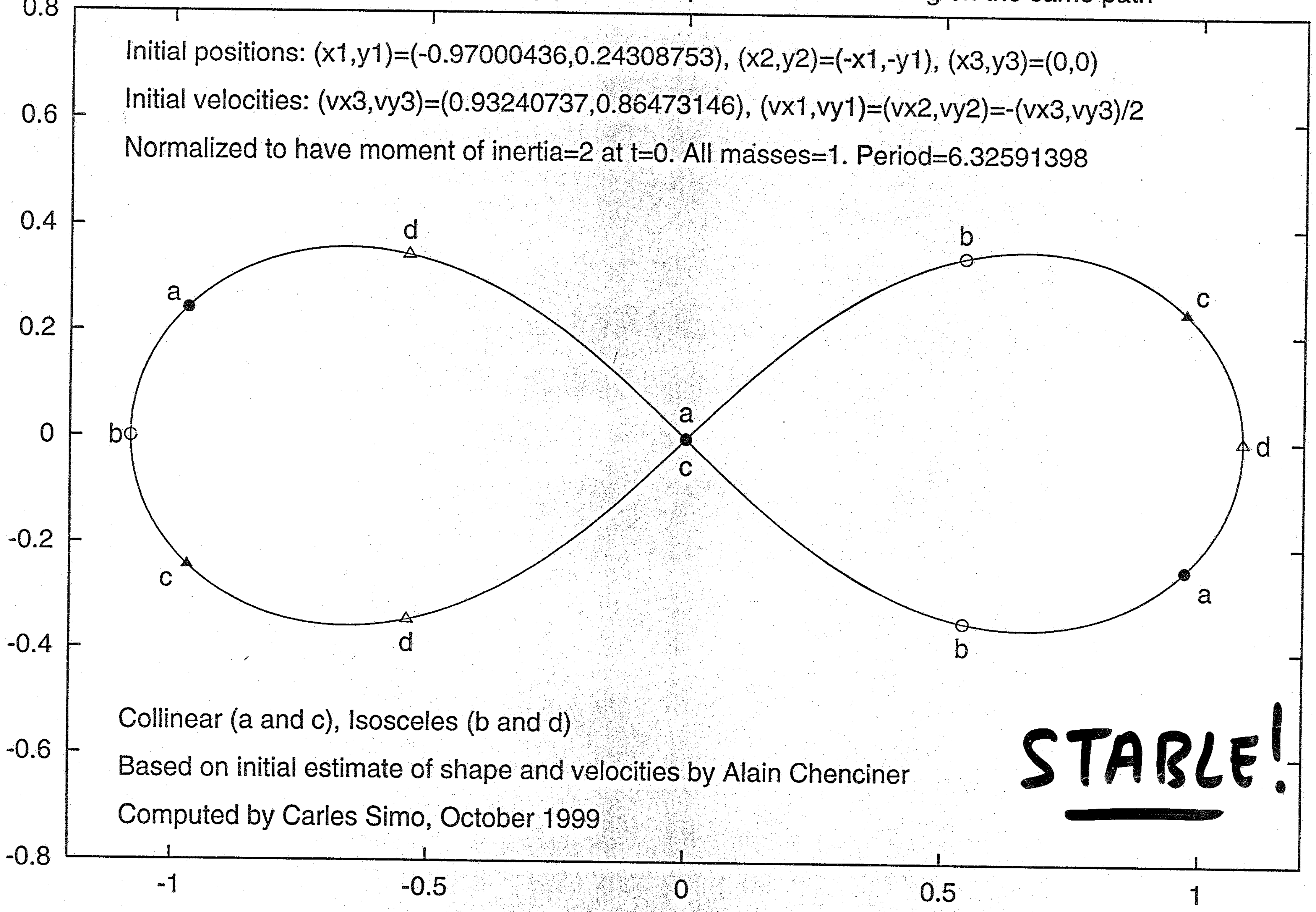
EQUAL MASSES

If $x(t)$ minimizes $S\big|_{\Lambda_{D_6}}$

- x has no collision
- $\mathbb{Z}/3 \Rightarrow x(t) = \left(q(t), q\left(t+\frac{T}{3}\right), q\left(t+\frac{2T}{3}\right)\right)$
  $=\{s^3\}$
- $q(t)$ is eight-shaped 

$D_6 \supset \mathbb{Z}/3\mathbb{Z} \Rightarrow x(t) = (q(t), q(t+T/3), q(t+2T/3))$

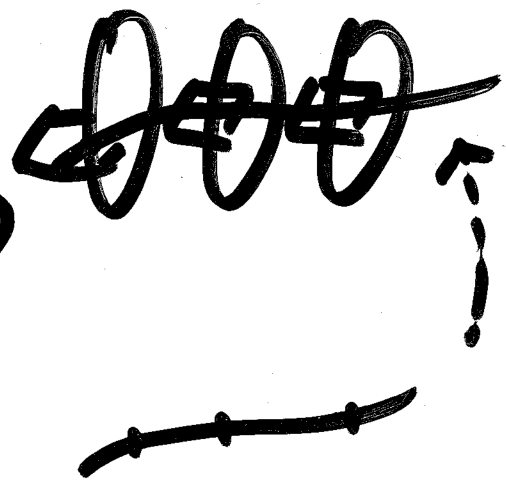
Periodic orbit of the 3-body problem, equal masses travelling on the same path



Proof :

- Coercivity : obvious
- no collision : equipotential model
- non trivial : D_6 - sym. excludes 

$$\begin{array}{c} X = \mathbb{R}^4 \\ \big\downarrow /SO(2) \\ \mathbb{R}^3 \end{array}$$

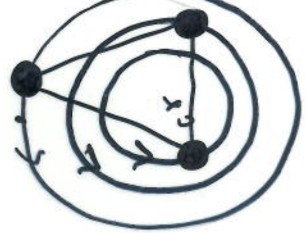


horiz $\iff$ $\boxed{\mathcal{C} = 0}$

Symmetry constraints

$\mathbb{Z}/2\mathbb{Z}$ A.C. & N. Desolneux 1998

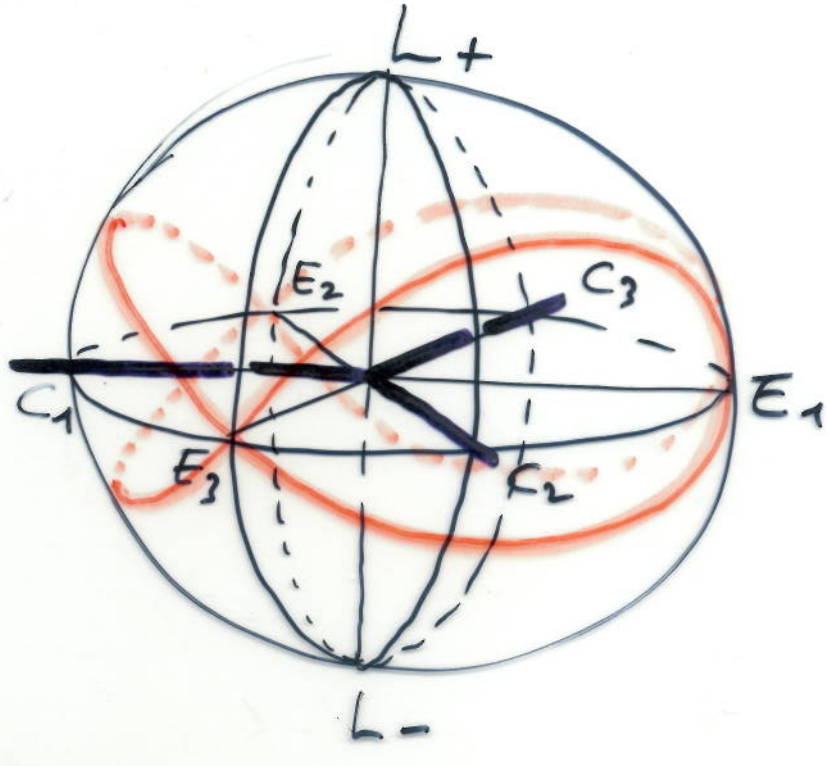
x min. of $S|_{\Lambda_a} \Longleftrightarrow$



D_6 = sym. group of the space of similitude classes of oriented triangles:

EQUAL MASSES

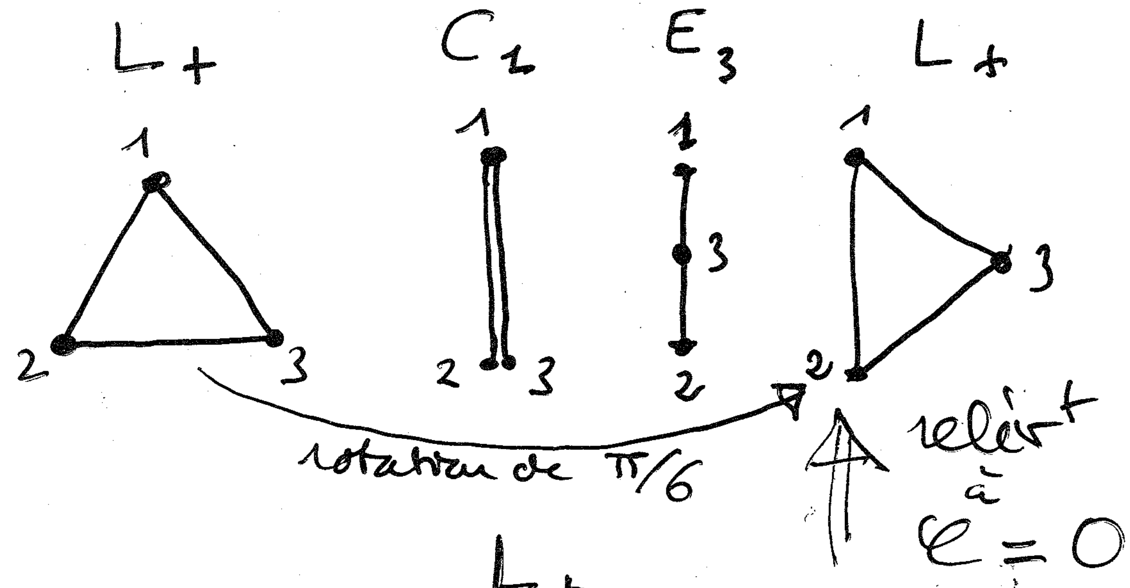
$$X \cong \mathbb{R}^4 \xrightarrow[\text{Hopf}]{/SO(2)} \mathbb{R}^3 \underset{I=1}{\hookleftarrow} S^2$$



Close to C-E. to Saari's conjecture

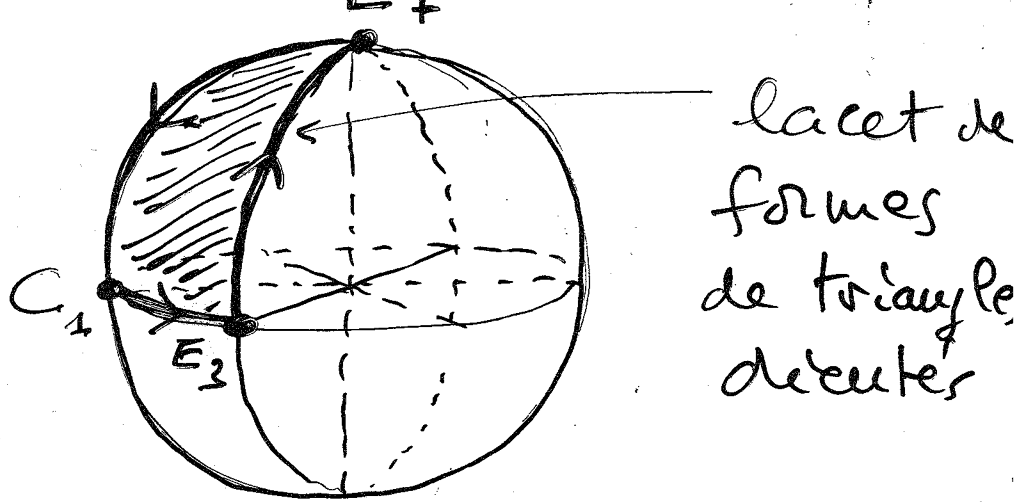
Tourner sans rotation, ou comment un chat retombe sur ses pattes :

$$\left(\begin{array}{l}\text{Triangles } {}_{2}\overset{1}{\Delta}{}_{3} \\ \text{dans } \mathbb{R}^2\end{array}\right) = \mathbb{R}^4$$



$\big\downarrow$ quotient par les rotations

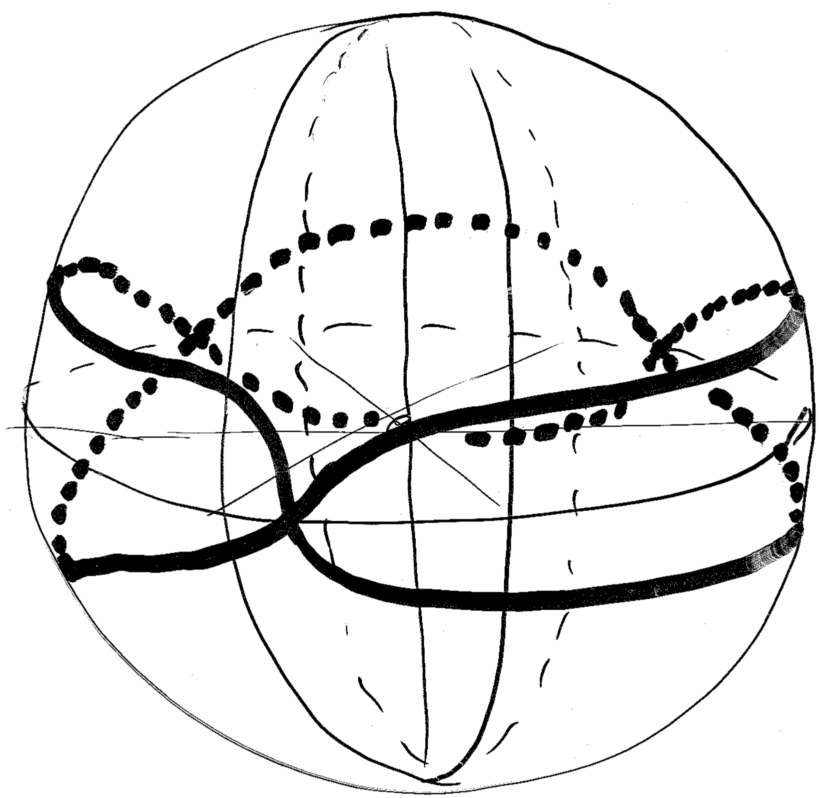
$$\left(\begin{array}{l}\text{Formes de} \\ \text{triangles orientés}\end{array}\right) = \mathbb{R}^3 \supset$$

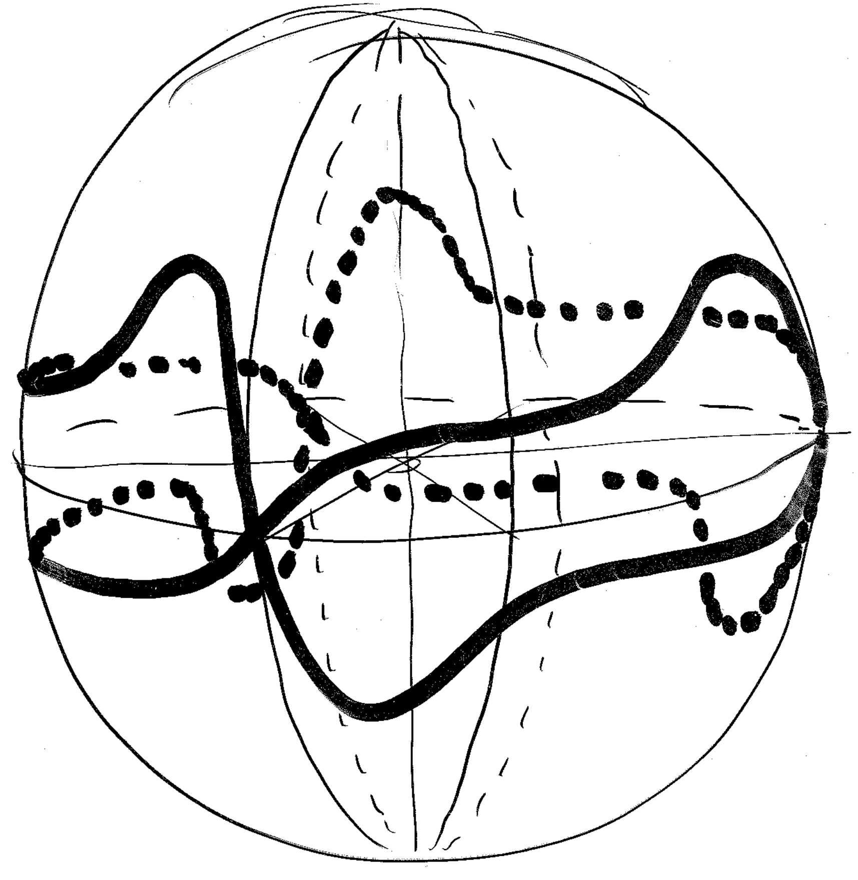


Transport // d'une connexion

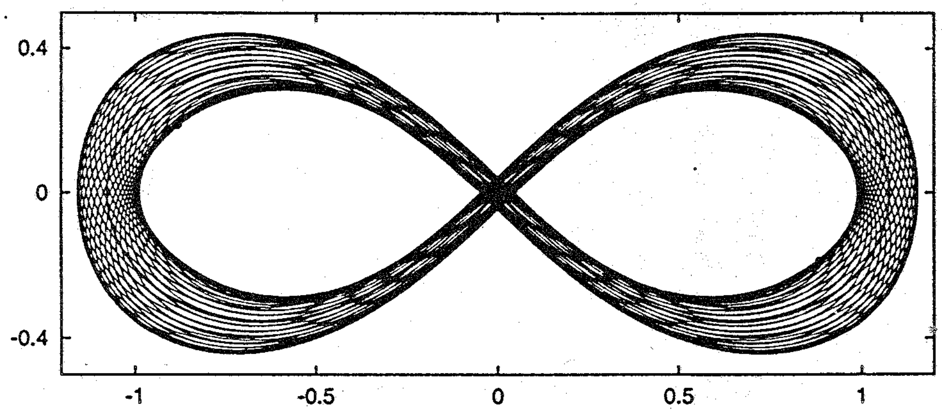
? Relaxation of the symmetry hypothesis

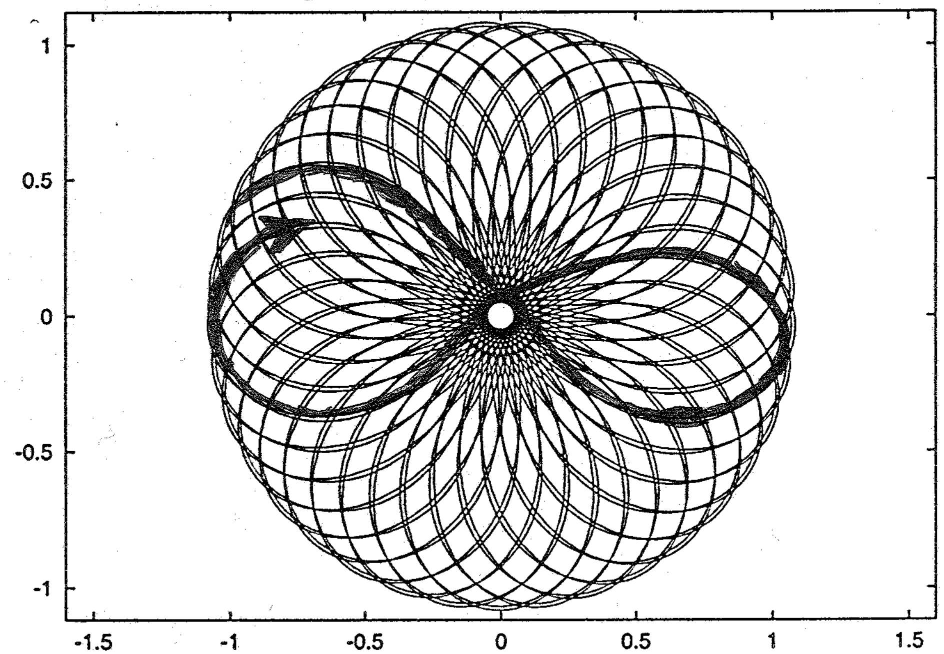
$\mathbb{Z}/6\mathbb{Z} = \{s\}$, $D_3 = \{s^2, \sigma\}$





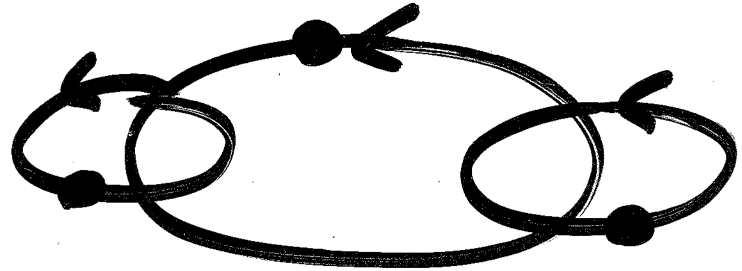
Satellites of the eight





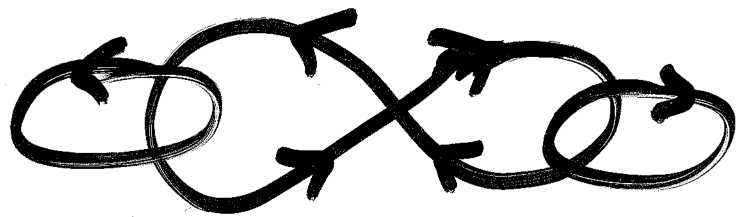
FIGURES BY C. SIMÓ

Open questions

? min $(1,0,1)$ =  ?

Broucke / Hénon

numerical (equal masses)

? min $(1,0,-1)$ = 

? $\exists$

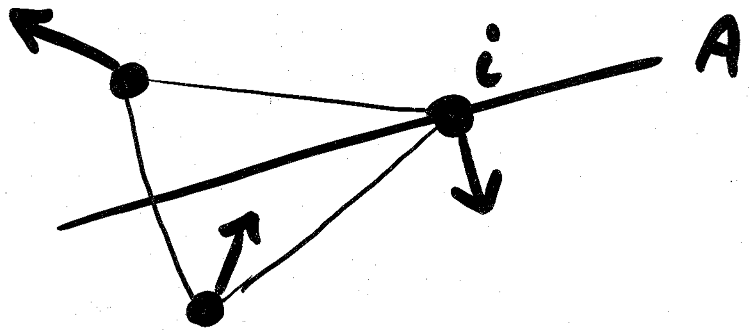
or collisions ?

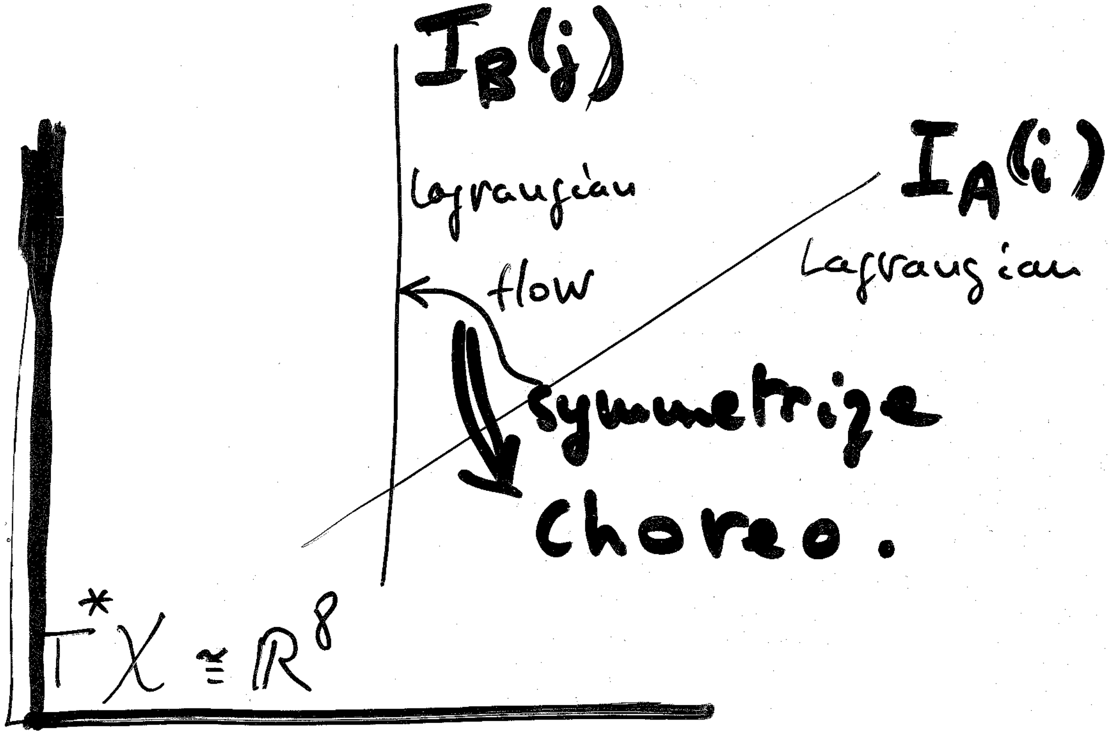
? min $(0,0,0)$ + $\mathbb{Z}/3$ sym = ∞ ?

More 3-body choreographies
(C. Simó 2001)

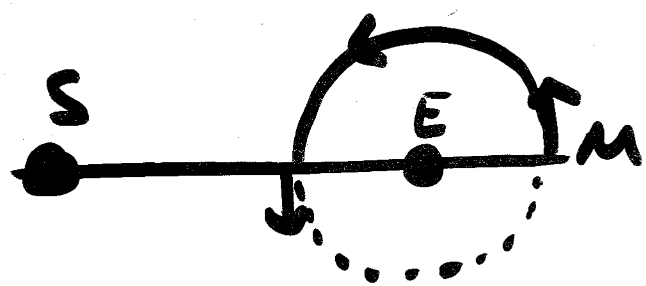
(NOT MIN.)

$I_A(i)$:





Compare Birkhoff:



Orbite 2   choreo-3.gnu

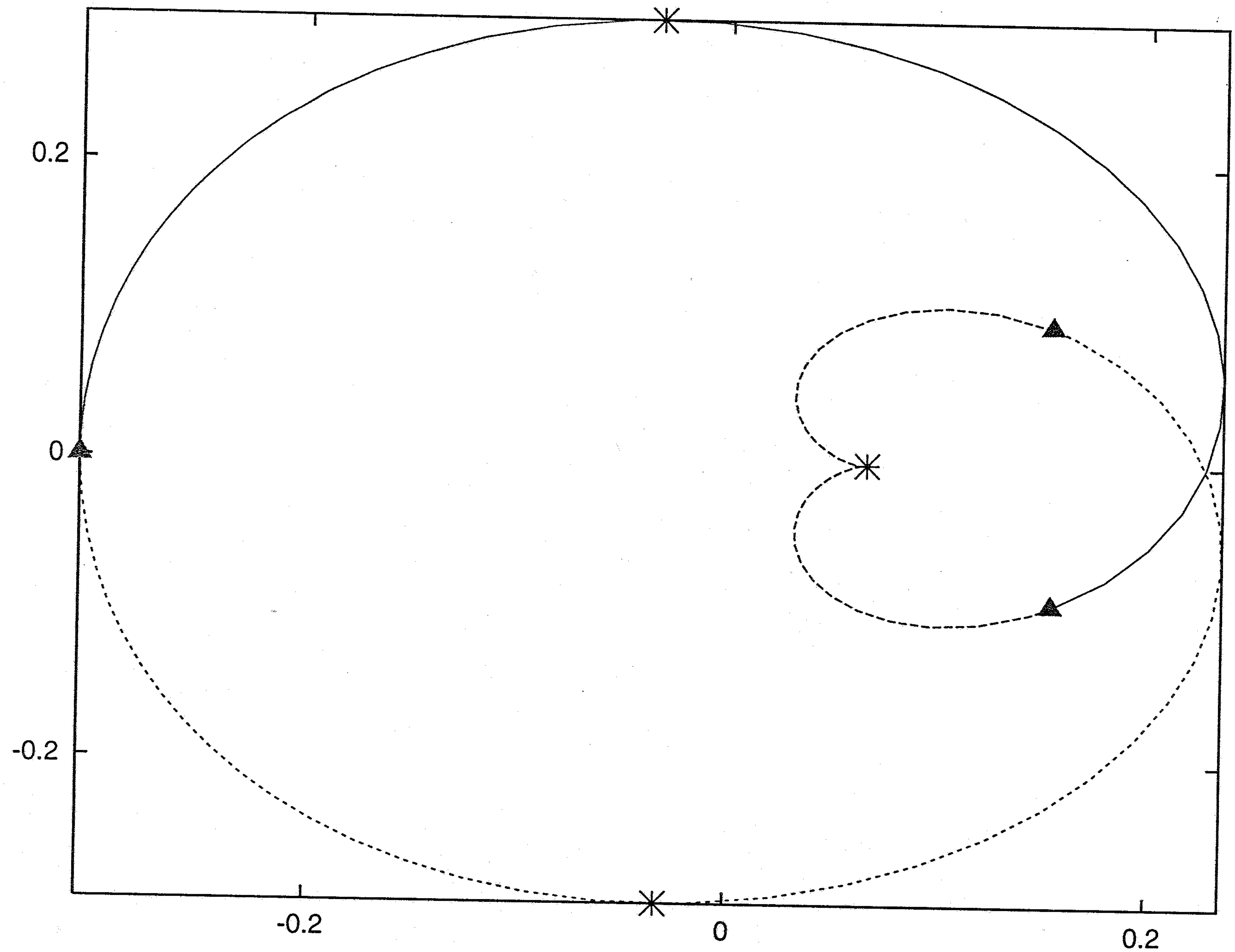
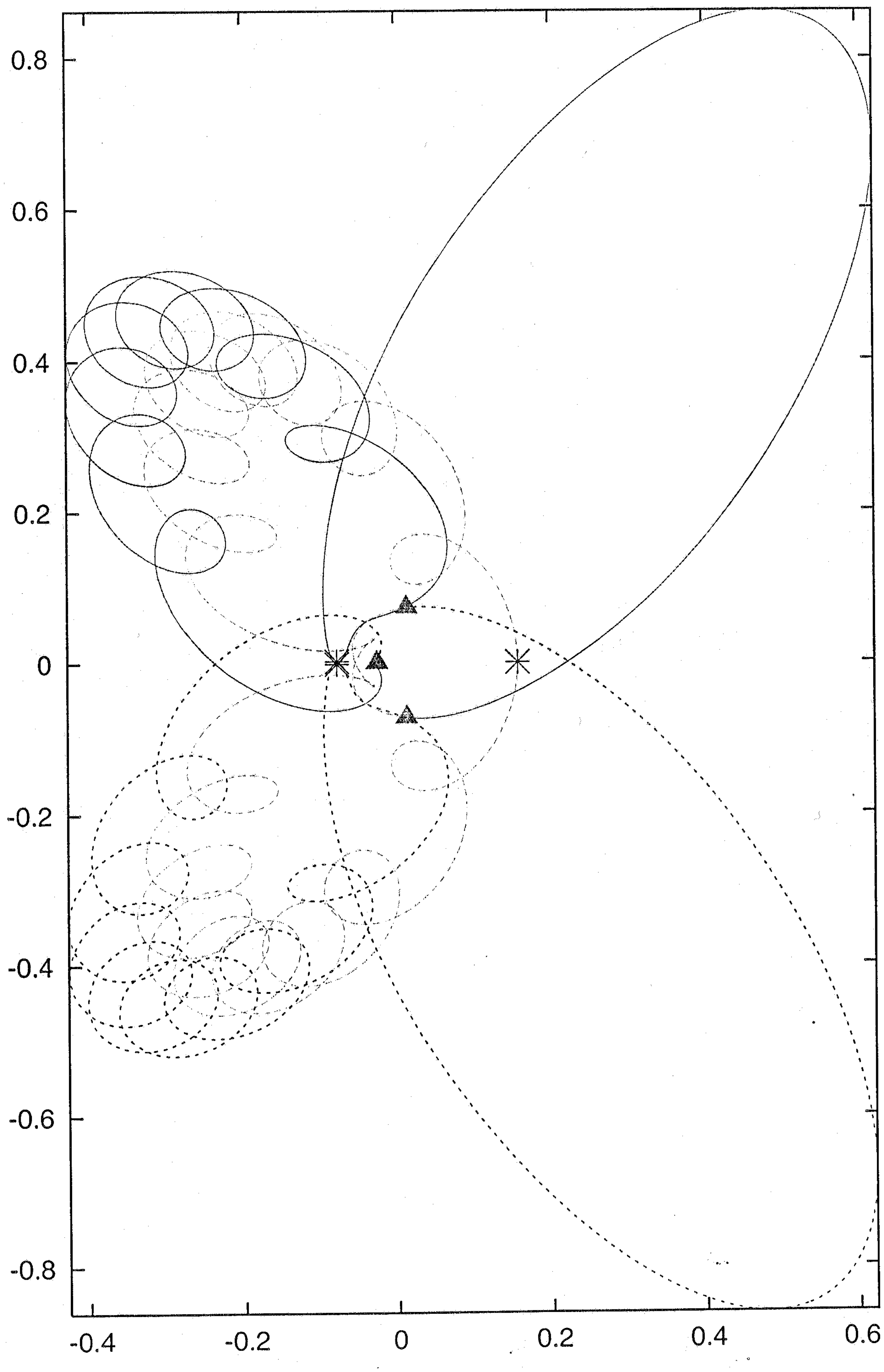
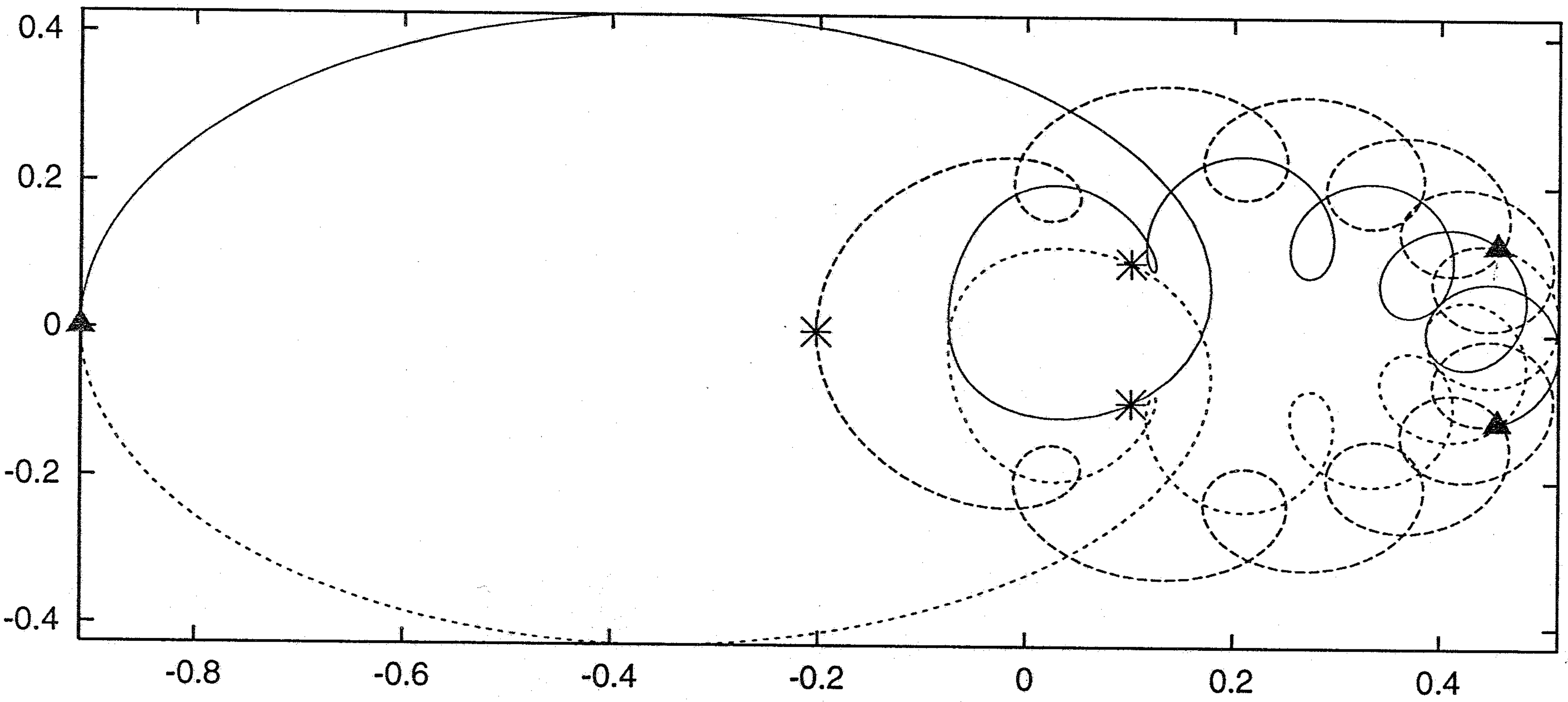
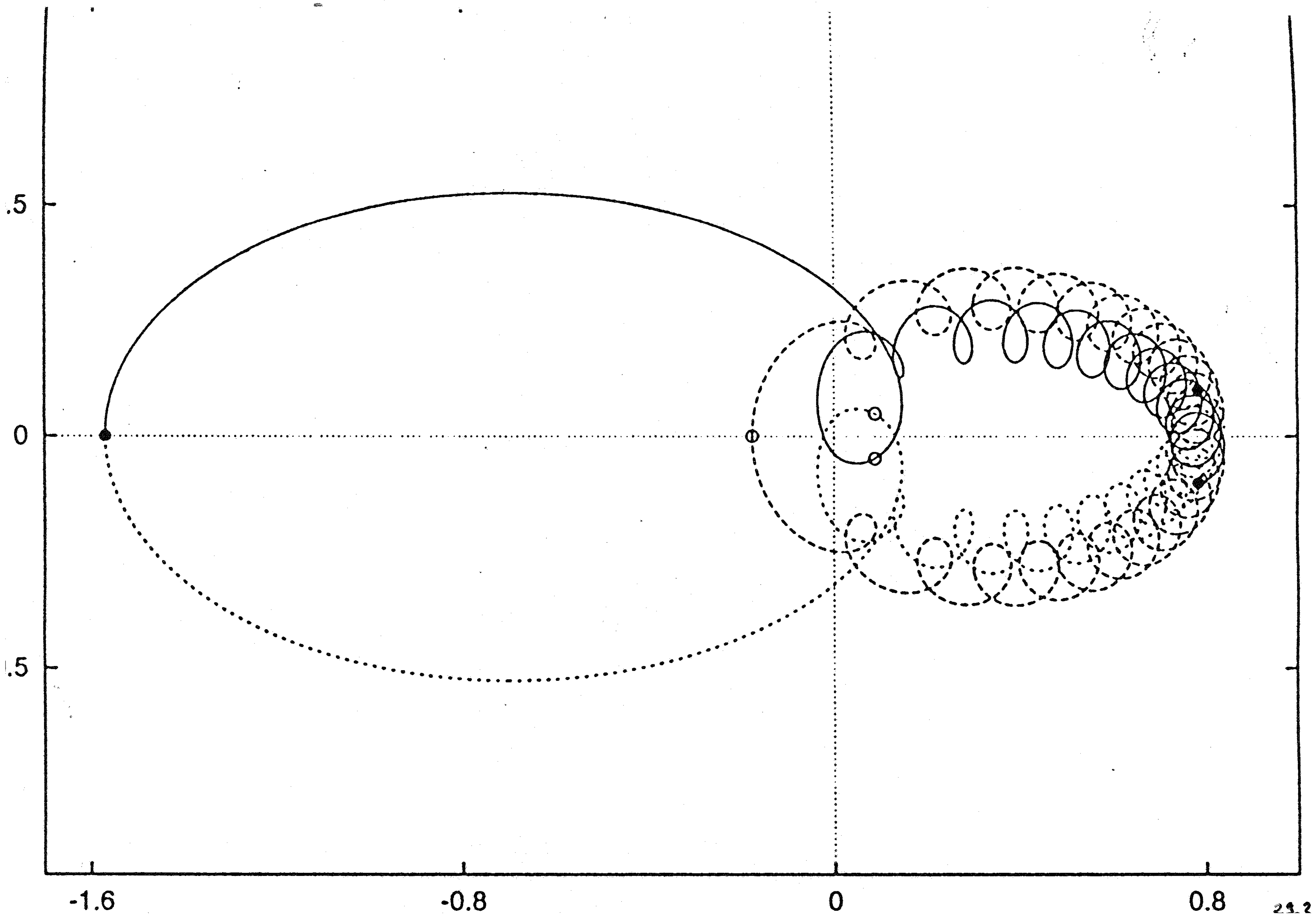


FIGURE BY C. SIMÓ


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.4
-0.2
0
0.2
0.4
0.6

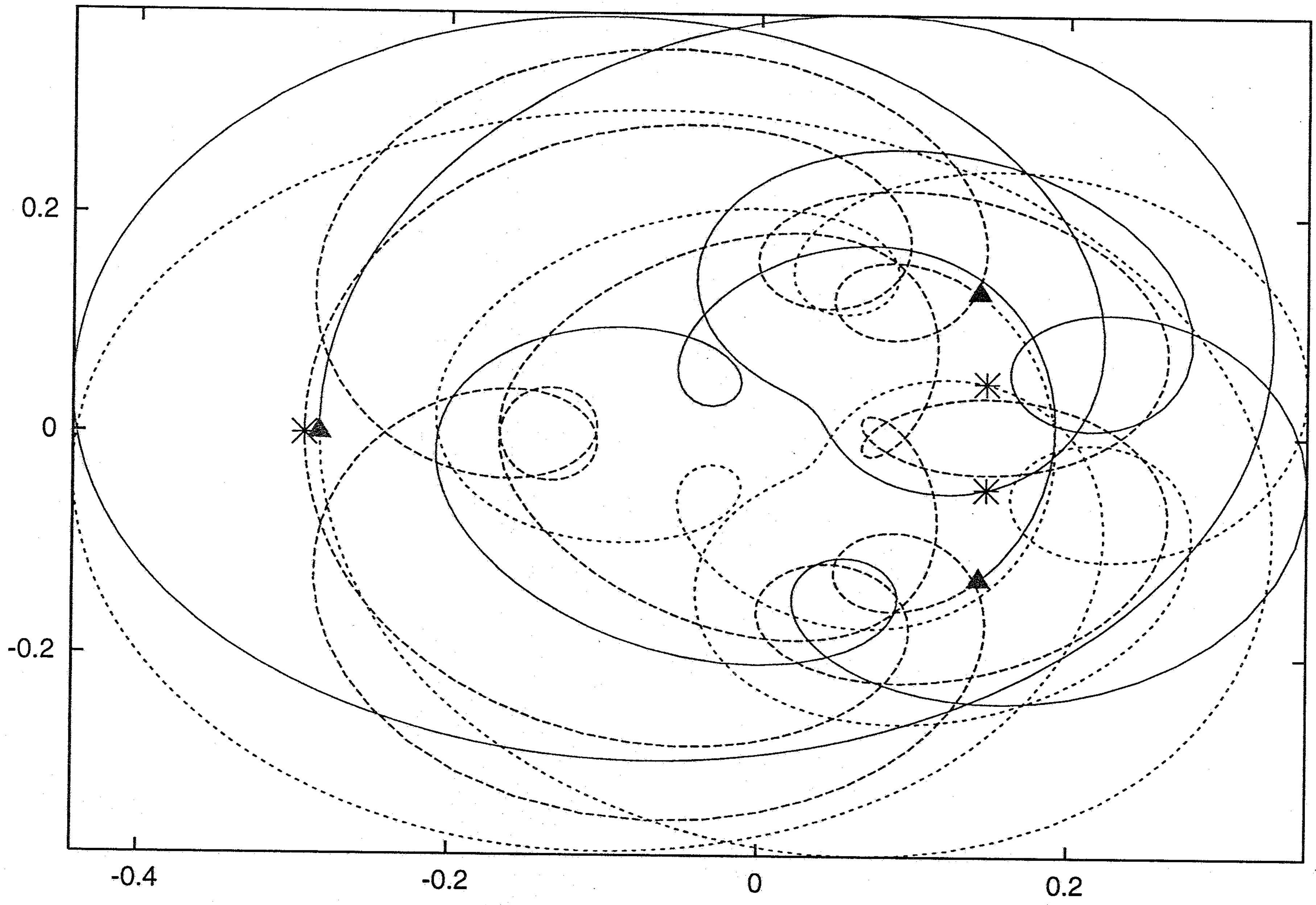
Orbite 50 choreo3.gnu





.5
0
.5
-1.6
-0.8
0
0.8

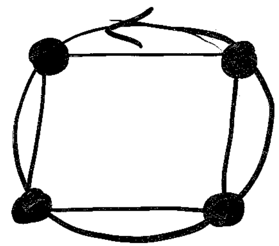
orbite 125 choreo3.gnu



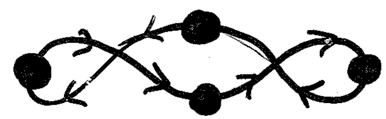
4 bodies in $\mathbb{R}^2$

EQUAL MASSES

Symmetry constraints

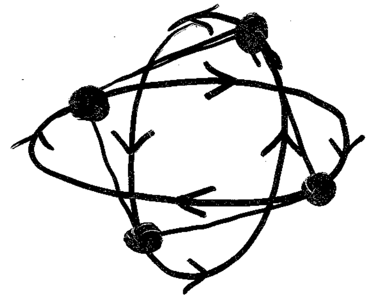
$\mathbb{Z}/2\mathbb{Z}$ $\quad x$ min of $S|_{\Lambda_a}$ $\Longleftrightarrow$  A.C. & N. Desolneux 1998

$D_4 \times \mathbb{Z}/2\mathbb{Z}$ J. Gerver 2000 (numerical)



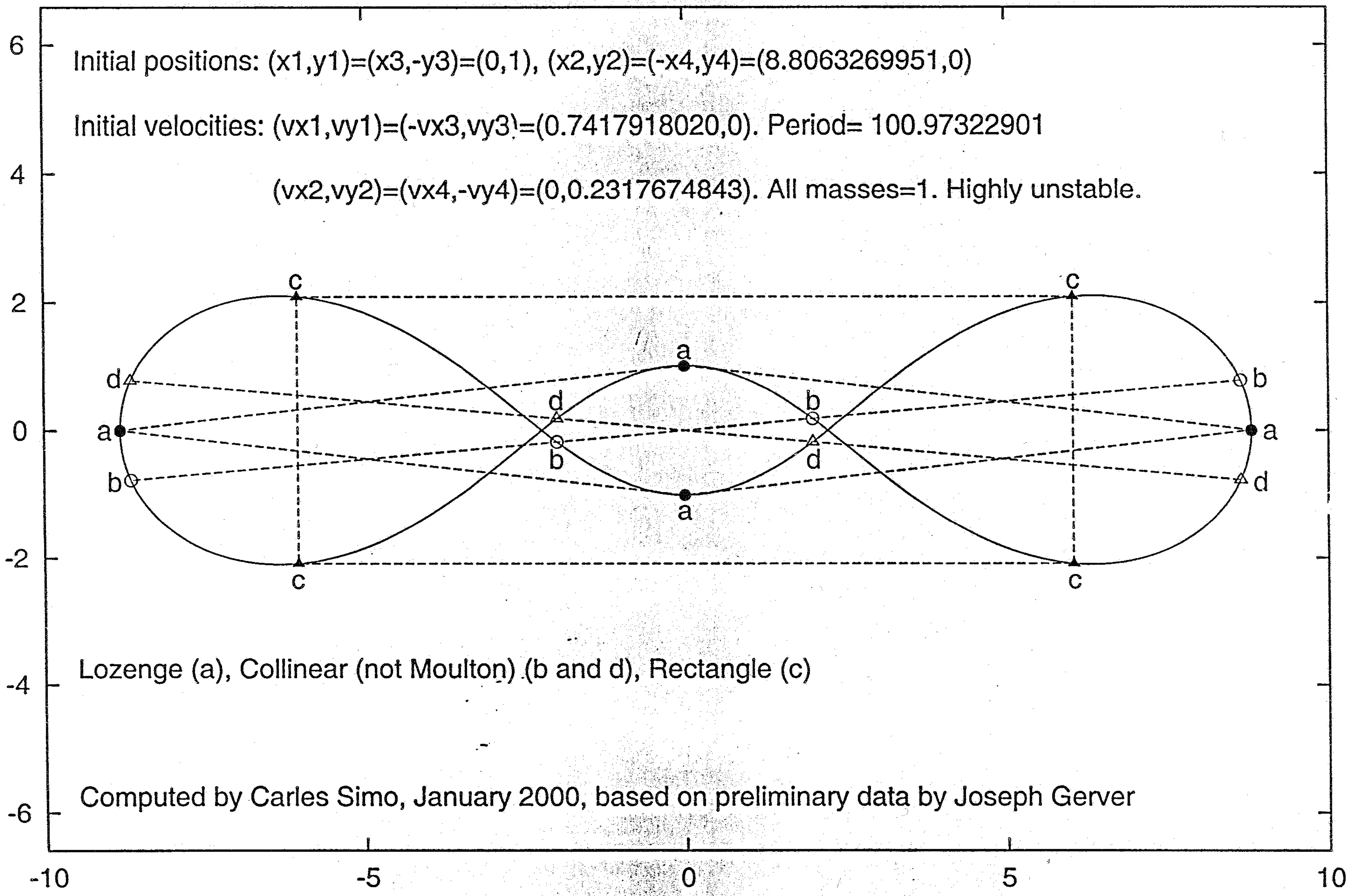
Relative min. ? Proof ?

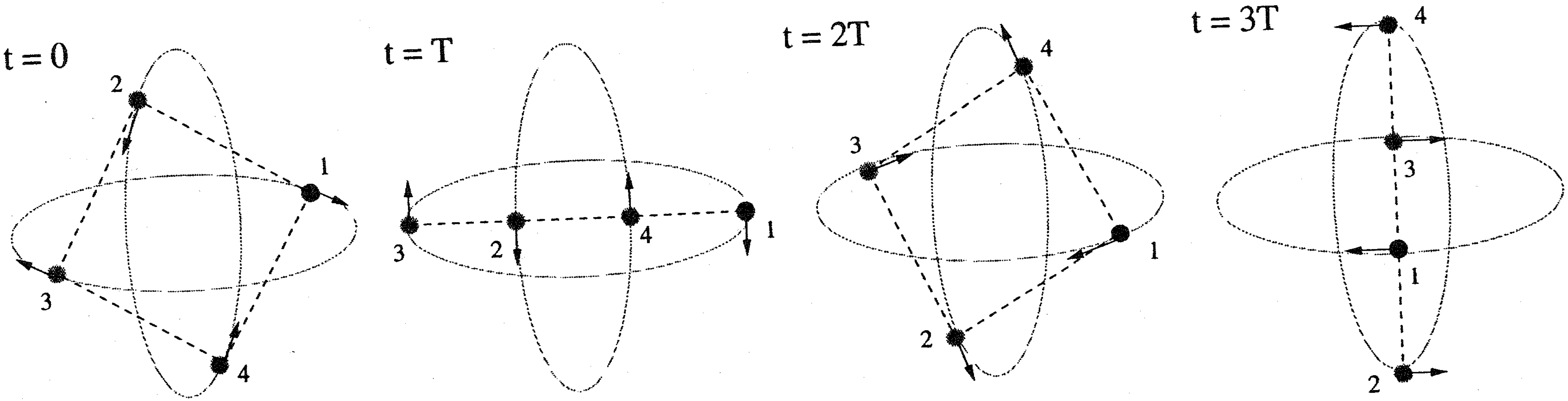
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ K.C. Chen 2000



$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Periodic orbit of the 4-body problem, equal masses travelling on the same path





t = 0
2
1
3
4
t = T
3
2
4
1
t = 2T
4
3
1
2
t = 3T
4
3
1
2

D12

fleur 4 corps.

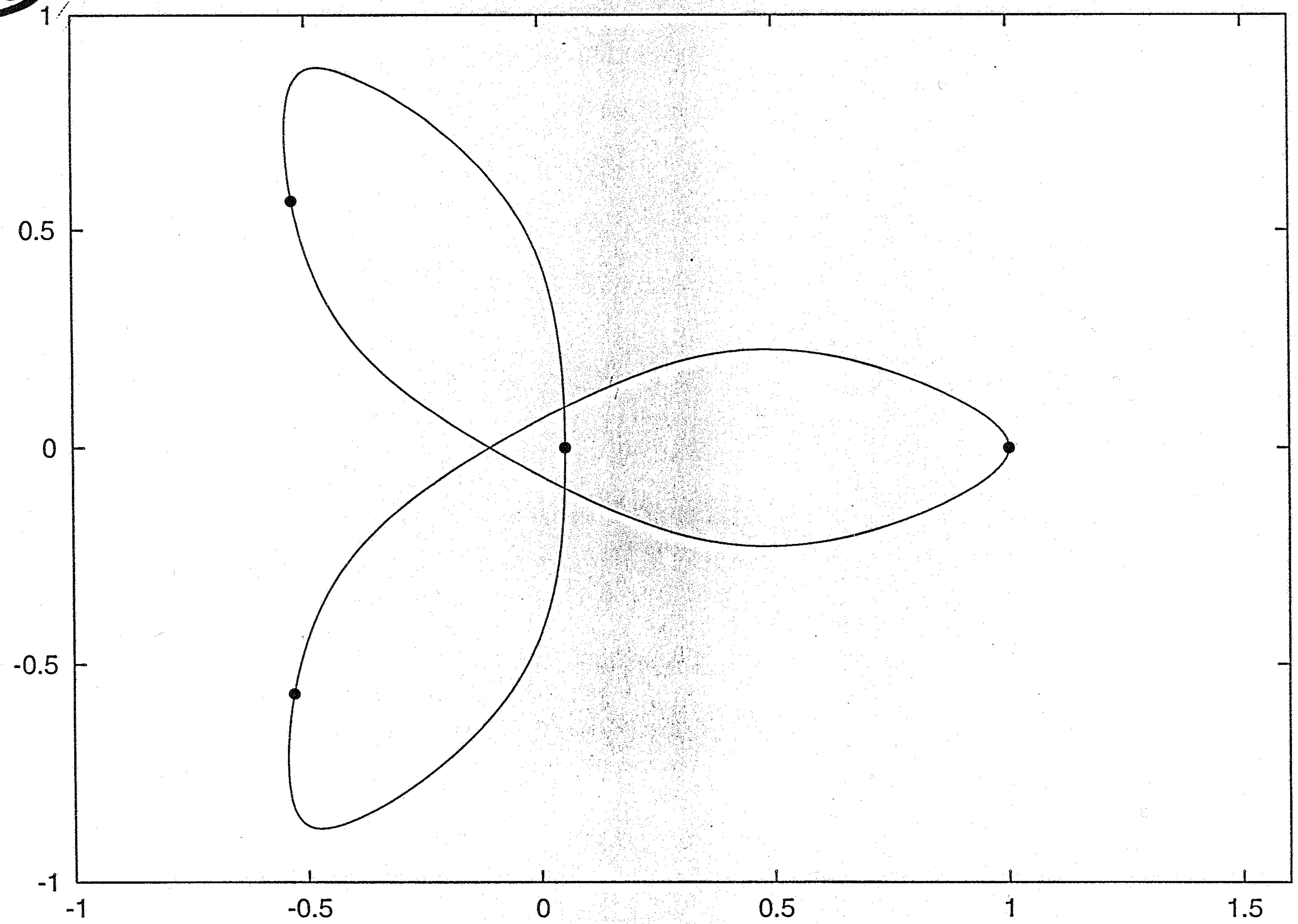
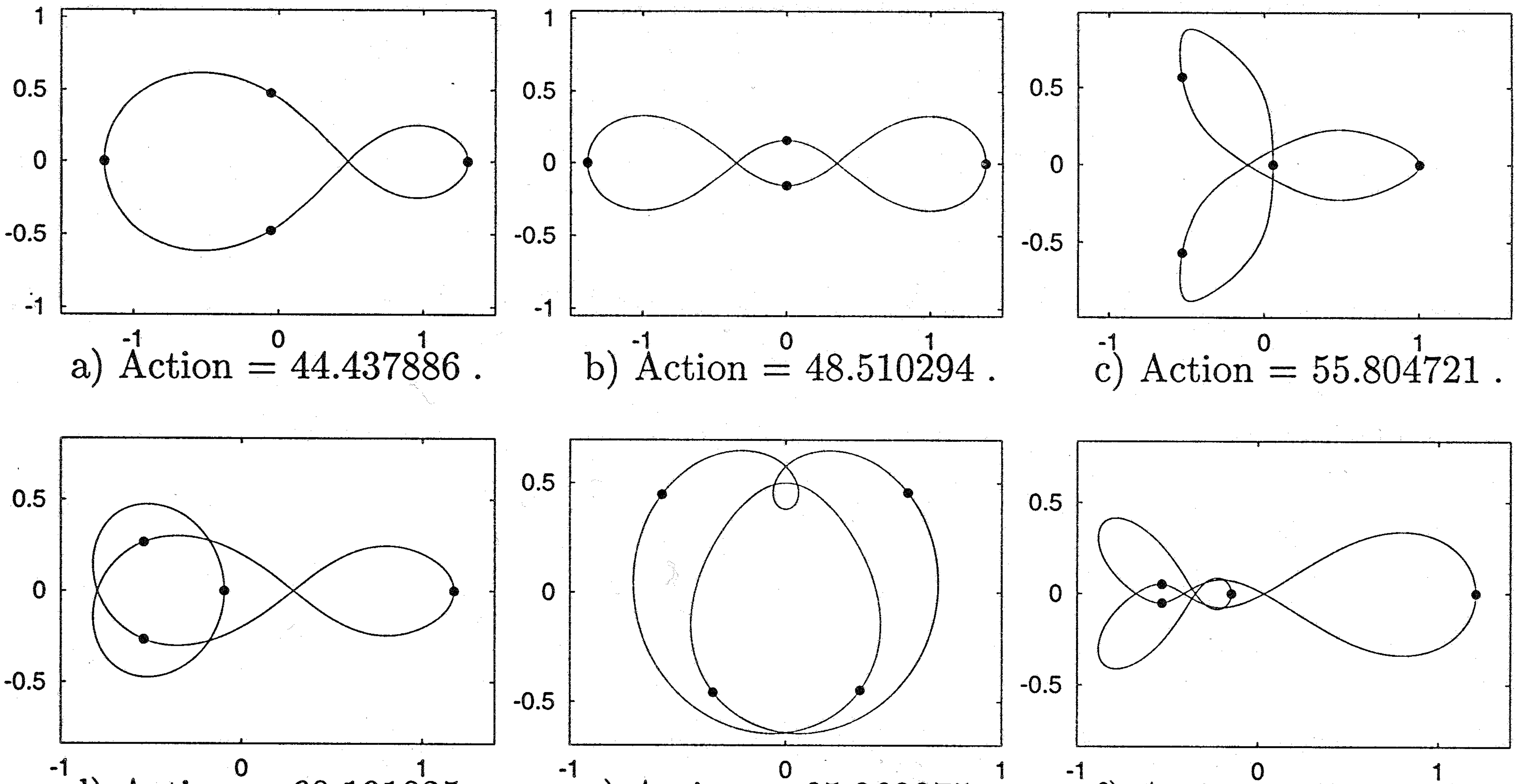


FIGURE BY C. SIMO



a) Action = 44.437886 .
b) Action = 48.510294 .
c) Action = 55.804721 .
d) Action = 60.191825 .
e) Action = 65.269875 .
f) Action = 67.186712 .

Figure 3: Simple choreographies for four bodies under the Newtonian potential.

4 bodies in $\mathbb{R}^3$

EQUAL MASSES

Symmetry constraints

$\mathbb{Z}/2\mathbb{Z}$ ~~Th~~. (A. Venturelli 2001, based on del Antonio)

‖ x min of $S|_{\Lambda_a}$ $\Longrightarrow$ no collision

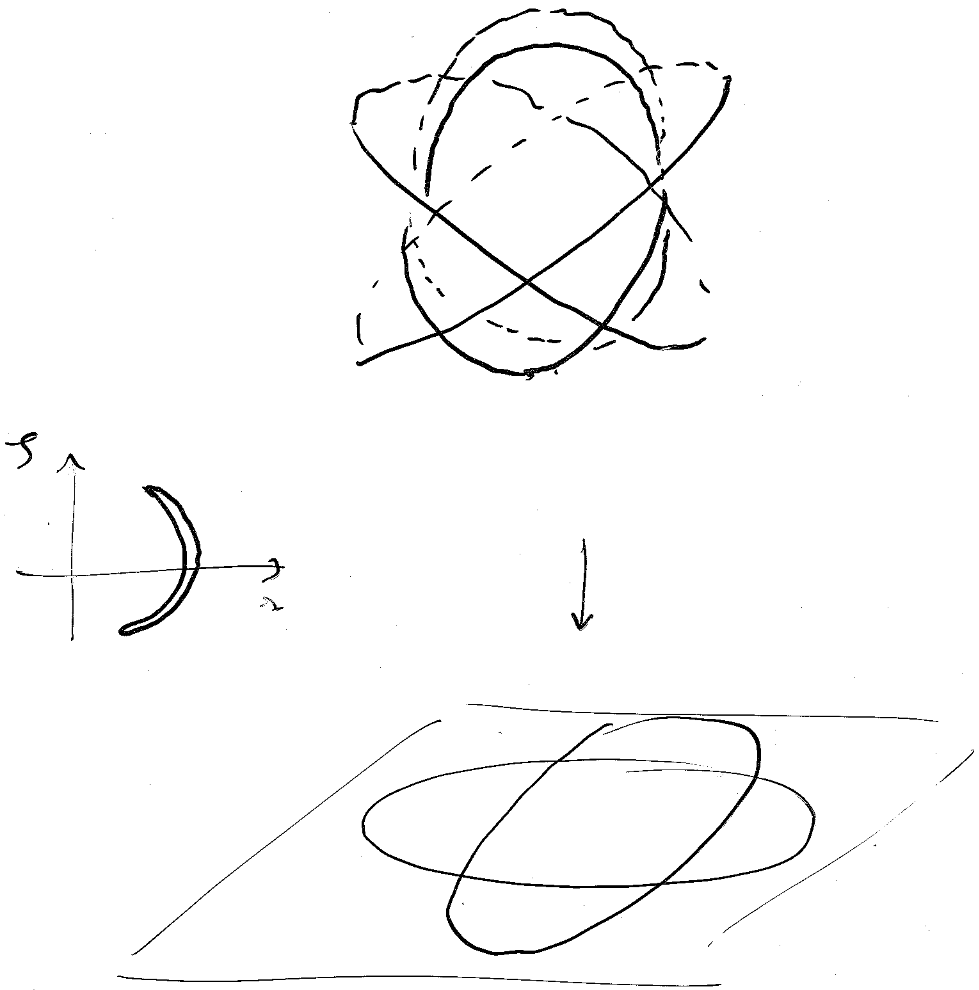
OK $\forall$ masses

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ action via orient. reversing isometry of $\mathbb{R}^3$:

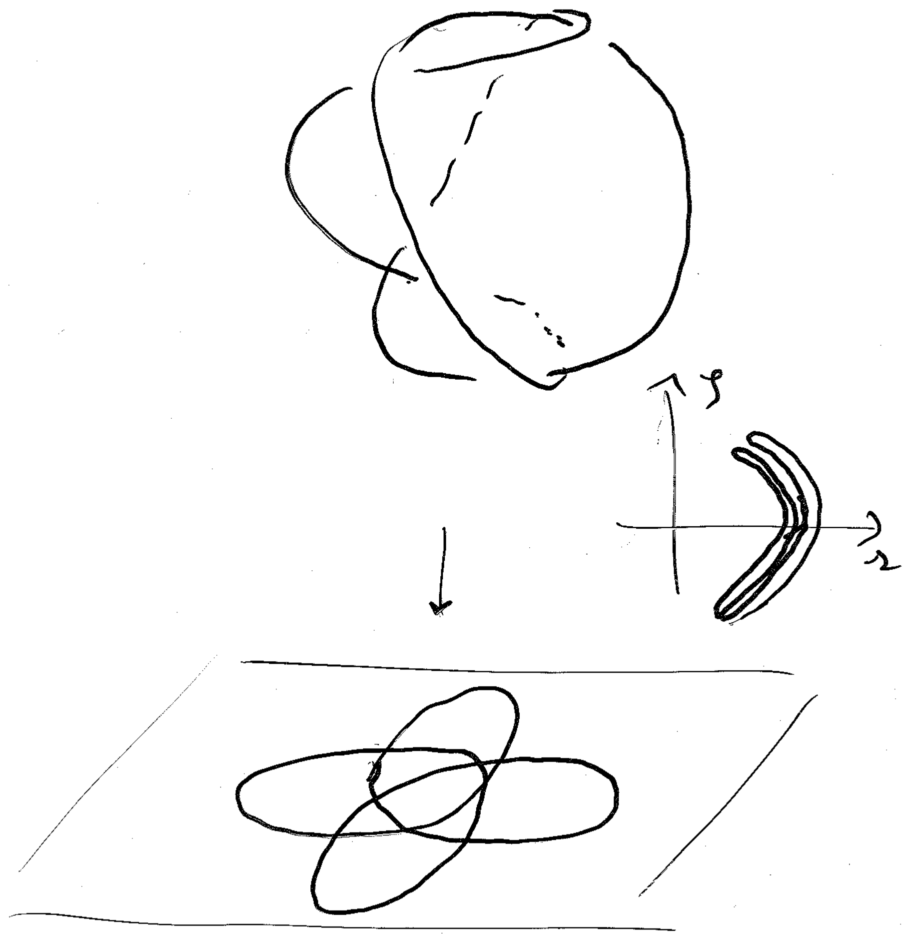
$(x\ y\ z) \longmapsto (-y\ \ x\ \ -z)$

Th (A.C. & A. Venturelli 2000):

x min of $S|_{\Lambda_{\mathbb{Z}_2 \times \mathbb{Z}_4}}$ $\Longleftrightarrow$ HIP-HOP



HIP HOP



Choreography
(G. Hoyuant)

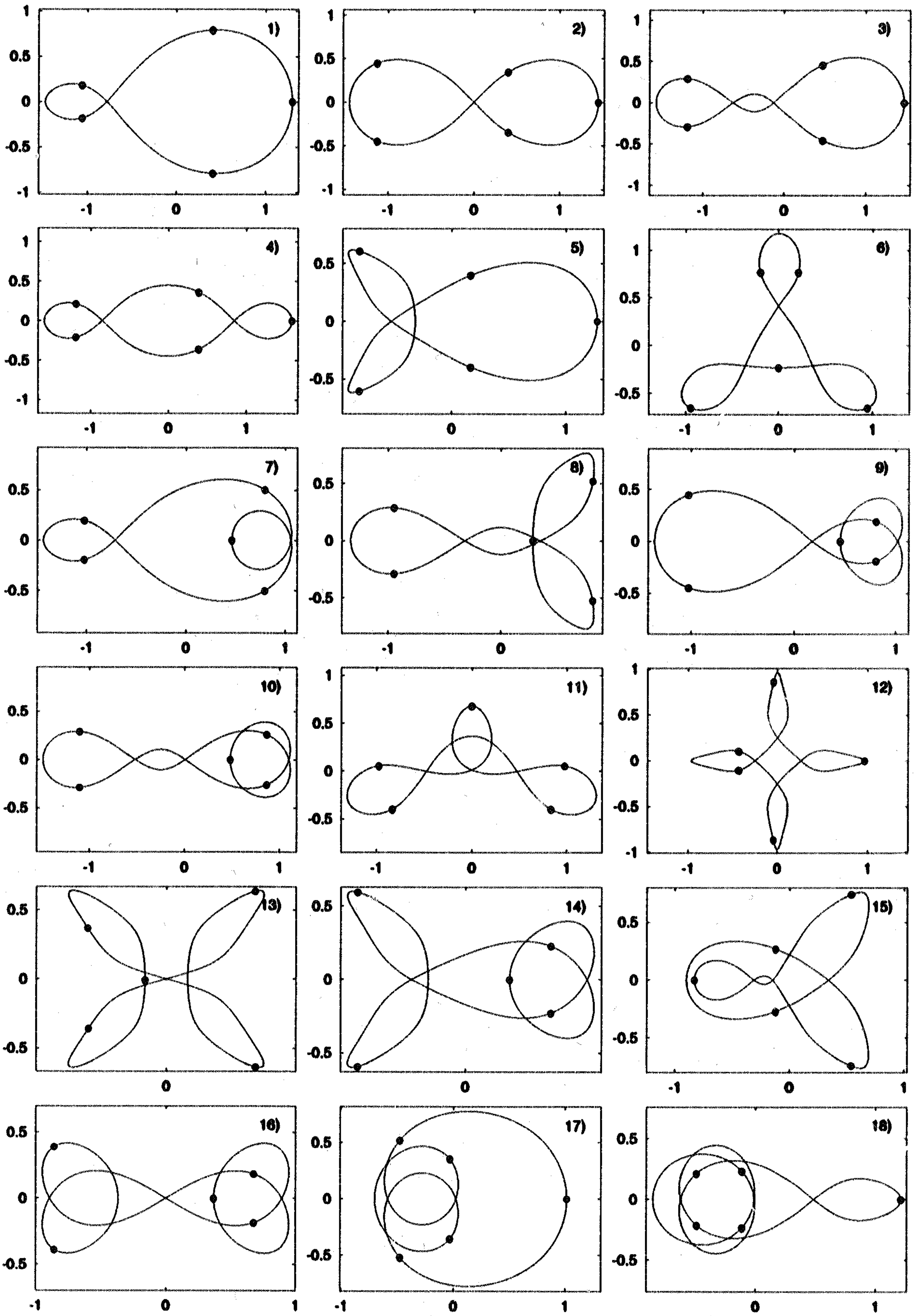


FIGURE 2. Choreographies found for 5 bodies. The dots denote initial conditions.

34

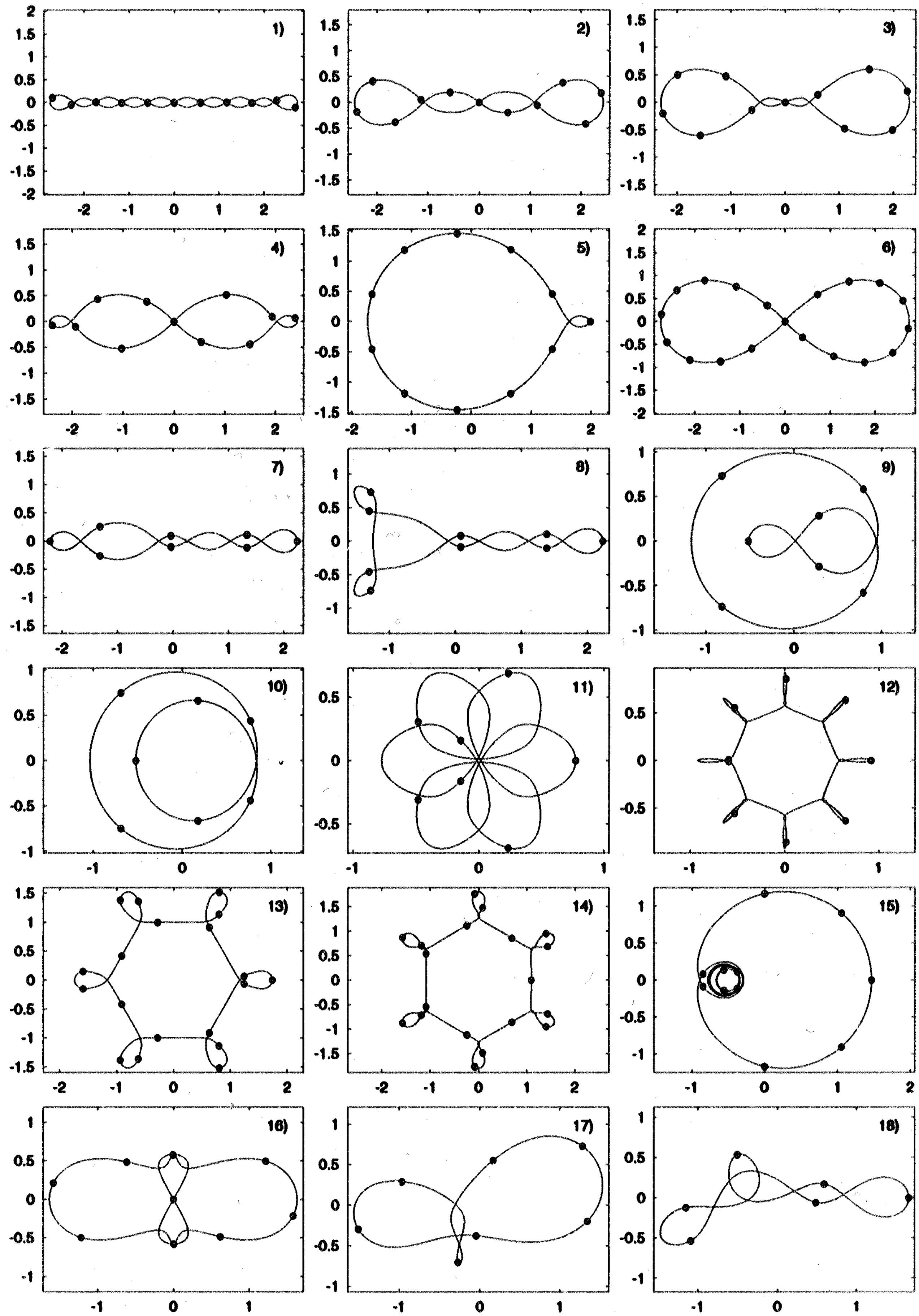


FIGURE 3. A sample of different choreographies.

~~Saut à~~ la corde
Métier à Tisser comme deux cordes s'interpénétrant

99 corps sur la chaîne $Z/2 * Z/2$ sym. Carlos 7-2.2000

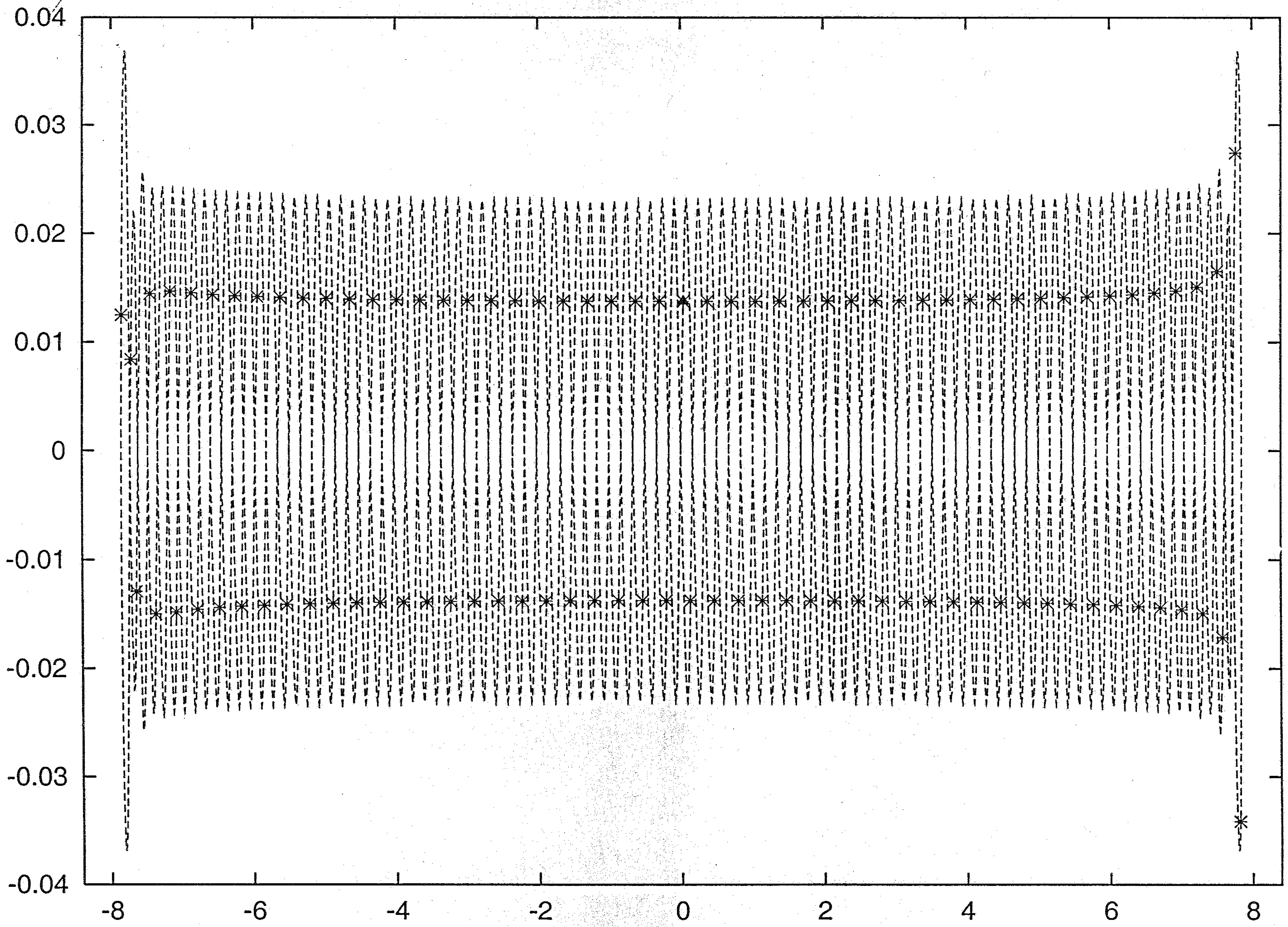


FIGURE BY C. SIMO

$N =$ Many bodies

Best candidate for global min: ∞ with odd # of bodies and D_{2N}-symmetry

...

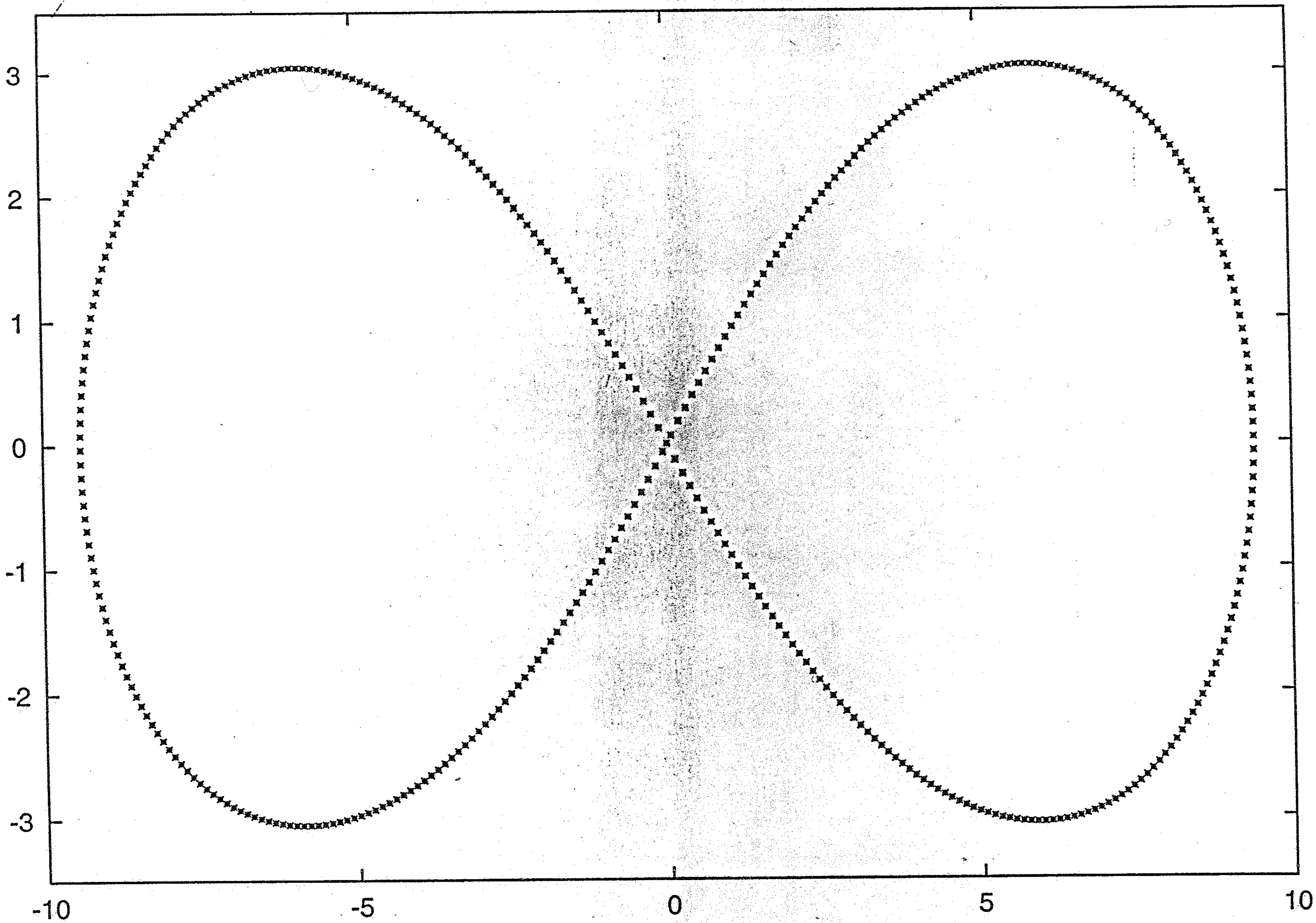
$$\alpha(s)(x_0, x_1, \dots, x_{N-1}) = \left(-\bar{x}_{\frac{N+1}{2}}, -\bar{x}_{\frac{N+3}{2}}, \dots, -\bar{x}_0, -\bar{x}_1, \dots, -\bar{x}_{\frac{N-1}{2}}\right)$$

$$\alpha(\sigma)(\quad\text{———}\quad) = (-x_0, -x_{N-1}, -x_{N-2}, \dots, -x_1)$$

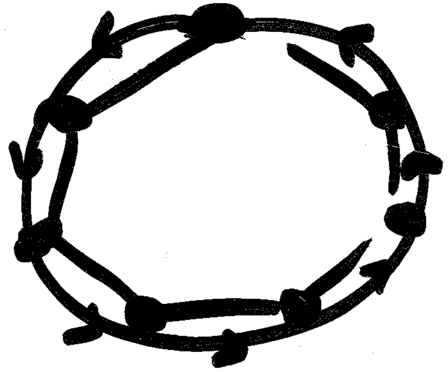
$$\beta(s)(t) = t + T/N$$

$$\beta(\sigma)(t) = -t$$

399 cdfs [illegible]



Open questions (2)

? min $\mathbb{Z}/n$-sym. =  ?

? lim. $n \to \infty$

? min $(0, \dots, 0)$ + $\mathbb{Z}/n$ symmetry

Q. Must the masses be equal for an n-body sol. of the form

$$\left(q(t),\, q\left(t+\frac{T}{n}\right),\, \dots\dots,\, q\left(t+\frac{(n-1)T}{n}\right)\right)$$

?

YES for $n \leq 5$

? for $n \geq 6$