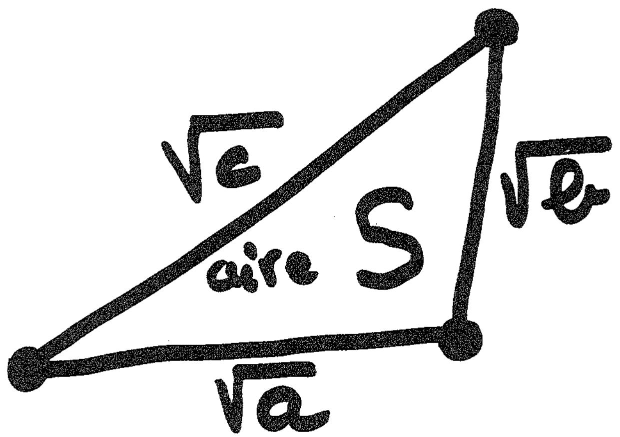
Pour Gilles

La forme de $\underline{n}$ corps

A. Chenciner, Paris

28.6.2001



"Metrica" (– 124)

Héron d'Alexandrie
(? Archimède ?)

$$16\,S^2 = \underbrace{(\sqrt{a}+\sqrt{b}+\sqrt{c})}_{2s}\underbrace{(-\sqrt{a}+\sqrt{b}+\sqrt{c})}_{2(s-\sqrt{a})}\underbrace{(\sqrt{a}-\sqrt{b}+\sqrt{c})}_{2(s-\sqrt{b})}\underbrace{(\sqrt{a}+\sqrt{b}-\sqrt{c})}_{2(s-\sqrt{c})}$$

$$= 2ab+2bc+2ca-a^2-b^2-c^2$$

$$= \det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c & b \\ 1 & c & 0 & a \\ 1 & b & a & 0 \end{pmatrix}$$

(Cayley-Menger det.)

4×4 ?

bizarre, bizarre...

Heath (Greek mathematics)

Book II gives the mensuration of certain solid figures, finding the solid content of a cone, a cylinder, a parallelepiped, a prism, a pyramid, a frustum of a pyramid and of a cone, a sphere and a segment thereof, a *spire* or *tore* (anchor-ring), the two special solids measured in Archimedes' *Method*, and the five regular solids.

Book III deals with the division of figures into parts having given ratios to one another, first plane figures, then solids, namely a pyramid, a cone and a frustum thereof, a sphere.

Among the cases of triangles in Book I the most interesting is that of the scalene triangle (acute angled or obtuse angled) in which the lengths of the three sides are given. This problem is solved in two ways.

(1) A perpendicular is drawn from a vertex (A) to the opposite side (BC), and the theorems of Eucl. II. 12 and 13 are used in order to find the lengths of the segments into which BC is divided by the perpendicular AD. The length of the perpendicular itself is then deduced, and the area ($=\frac{1}{2}AD.BC$) is thus found.

* (2) The second method is to use the formula which we write as

$$\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}},$$

where s is half the sum of the sides a, b, c; and Heron gives an admirable proof of the formula by pure geometry, as follows.

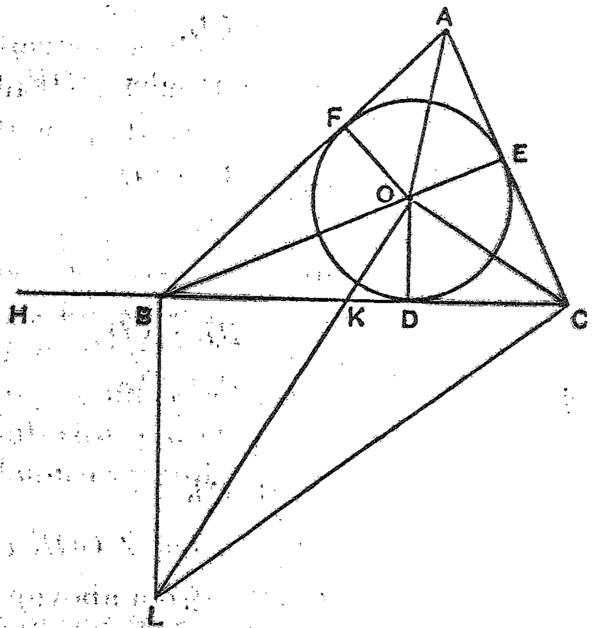
Let the sides of the triangle ABC be given in length. Inscribe the circle DEF, and let O be its centre.

Then
$$BC.OD = 2\Delta BOC,$$
$$CA.OE = 2\Delta COA,$$
$$AB.OF = 2\Delta AOB;$$

whence, by addition,

$$p.OD = 2\Delta ABC,$$

where p is the perimeter.



Produce CB to H so that $BH = AF$.

Then, since $AE = AF$, $BF = BD$, and $CE = CD$, we have $CH = BC + AF = \frac{1}{2}p = s$.

Therefore $\quad CH.OD = \Delta ABC.$

But $CH.OD$ is the 'side' of the product $CH^2.OD^2$, i.e. $\sqrt{(CH^2.OD^2)}$, so that

$$(\Delta ABC)^2 = CH^2.OD^2.$$

Draw OL at right angles to OC cutting BC in K, and BL at right angles to BC meeting OL in L. Join CL.

Then, since each of the angles COL, CBL is a right angle, $COBL$ is a quadrilateral in a circle.

Therefore $\quad \angle COB + \angle CLB = 2R.$

But $\angle COB + \angle AOF = 2R$, because AO, BO, CO bisect the angles round O, and the angles COB, AOF are together equal to the angles AOC, BOF, while the sum of all four angles is equal to $4R$.

Therefore $\angle AOF = \angle CLB$.

Accordingly, the right-angled triangles AOF, CLB are similar;

therefore
$$BC : BL = AF : FO$$
$$= BH : OD,$$

and, alternately,
$$CB : BH = BL : OD$$
$$= BK : KD;$$

whence, *componendo*, $CH : HB = BD : DK$.

It follows that

$$CH^2 : CH.HB = BD.DC : CD.DK$$
$$= BD.DC : OD^2, \text{ since } \angle COK \text{ is right.}$$

Therefore
$$(\Delta ABC)^2 = CH^2.OD^2 \text{ (from above)}$$
$$= CH.HB.BD.DC$$
$$= s(s-a)(s-b)(s-c).$$

The proposition itself, as we have seen (p. 340), is attributed to Archimedes.

The chapter (8) containing the above proof is otherwise of the highest interest because it explains a method of obtaining approximations to the value of the square roots of numbers which are not squares. This is the only classical method on record, and it enables us to understand how Archimedes may have arrived at the approximations to $\sqrt{3}$ which he merely states without any explanation (cf. pp. 309–10 above).

If A is a non-square number, and a^2 is the nearest square number to it, so that $A = a^2 \pm b$, Heron's rule amounts to saying that a first approximation (α_1) to $\sqrt{A}$ is

$$\alpha_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

He says further that a second approximation can be found by substituting for a in the above formula the first approximation α_1; thus

$$\alpha_2 = \tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Heron does not himself seem to make any direct use of the formula for a second approximation; but the method is general, and by continuing the process indefinitely we can find any number of successive approximations.

If we substitute in (1) the value $a^2 \pm b$ for A, we obtain the well-known formula

$$\alpha_1 = a \pm \frac{b}{2a}.$$

Examples in the *Metrica* are the following:

(1) $\sqrt{720}$. Since 729 ($= 27^2$) is the nearest square to 720, the first approximation to $\sqrt{720}$ is

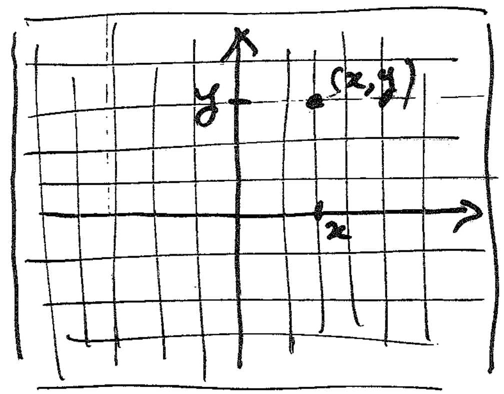
$$\alpha_1 = \tfrac{1}{2}(27 + \tfrac{720}{27}) = \tfrac{1}{2}(27 + 26\tfrac{2}{3}) = 26\tfrac{1}{2}\,\tfrac{1}{3}.$$

(2) $\sqrt{63}$, says Heron, is nearly $7\frac{1}{2}\,\frac{1}{4}\,\frac{1}{8}\,\frac{1}{16}$, which he would clearly obtain thus:

$$\alpha_1 = \tfrac{1}{2}(8 + \tfrac{63}{8}) = \tfrac{1}{2}(8 + 7\tfrac{7}{8}) = 7\tfrac{1}{2}\,\tfrac{1}{4}\,\tfrac{1}{8}\,\tfrac{1}{16}.$$

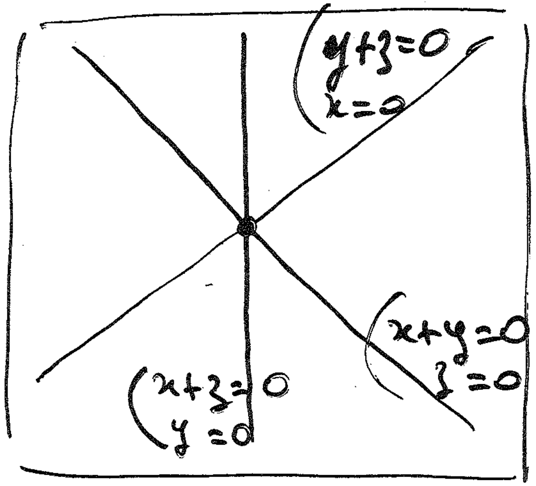
Heron has occasion to make many such approximations, especially in his mensuration of the regular polygons. In the case of the equilateral triangle he proves that the area Δ is given by $\Delta^2 = 3a^4/16$. In the particular case taken $a = 10$, so that $\Delta = \sqrt{1875} = 43\frac{1}{3}$, nearly. Sometimes a well-known approximation to $\sqrt{3}$ is used, as when, in

... en fait 2×2, <u>mais</u> pas de base privilégiée !

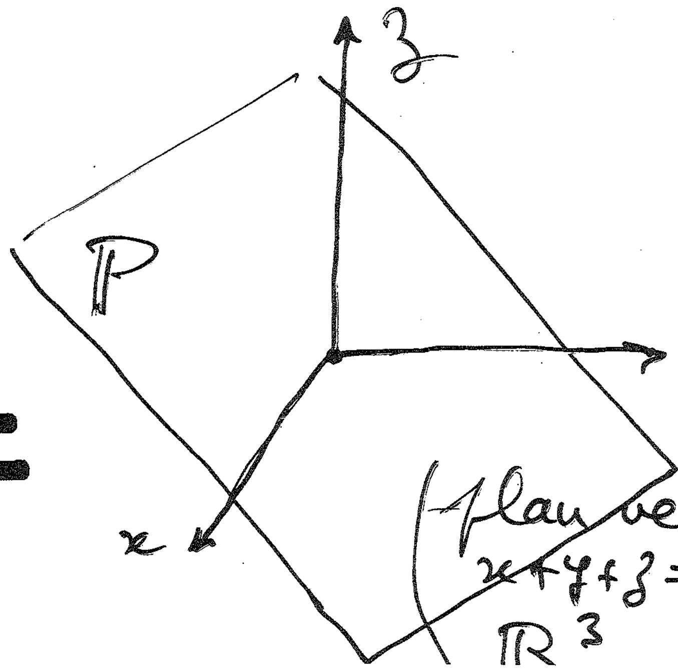


$\mathbb{R}^2$

$\neq$



$\mathbb{P}$ =



(plan vectoriel $x+y+z=0$ dans $\mathbb{R}^3$

Clé : réduction des symétries

(Réf : A. Albouy, A. Chenciner, le pb des n corps et les distances mutuelles) Inventiones Mathematicae 131, p. 151-184, 1998

S ne change pas par

$$\left.\begin{array}{l}\text{translation}\\ \text{rotation}\\ \text{symétrie}\end{array}\right\} = \begin{array}{l}\text{isométries}\\ \text{euclidiennes}\end{array}$$

donc

$$S : \left(\begin{array}{l}\text{"espace des triangles}\\ \text{à isométrie près}\end{array}\right) \longrightarrow \mathbb{R}_+$$

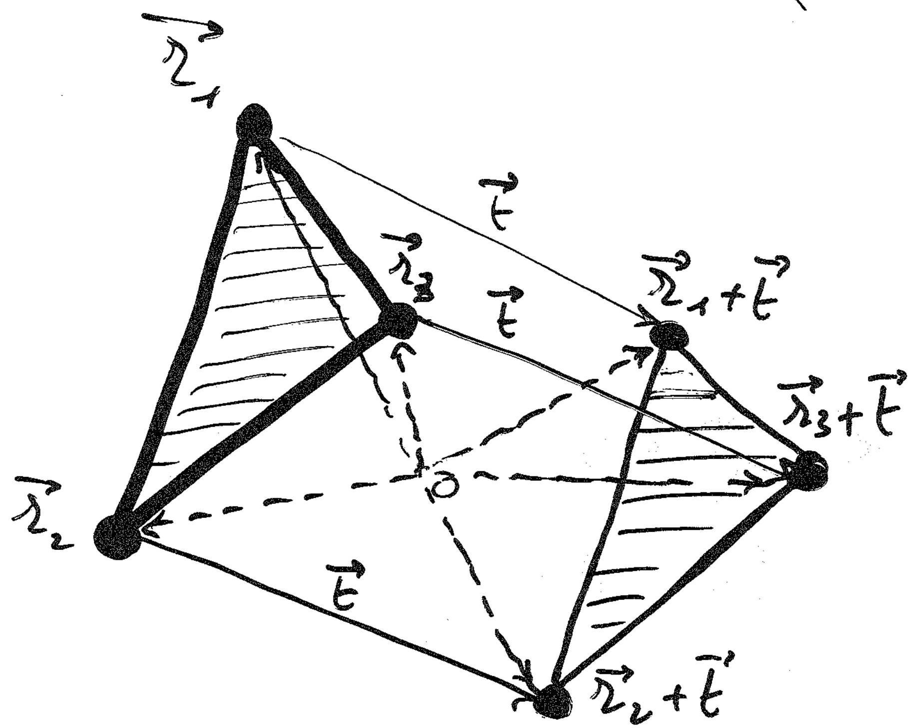
Problème : Comment calculer ?

$$\mathcal{D}^* \subset \mathbb{R}^n$$

$$\mathcal{D}^* = \left\{ (\xi_1, \dots, \xi_n),\ \sum_{i=1}^{n} \xi_i = 0 \right\}$$

$$x : \mathcal{D}^* \longrightarrow E \ \text{e.v.}$$

$$(\xi_1, \dots, \xi_n) \longmapsto \sum_{i=1}^{n} \xi_i \vec{r}_i = \sum_{i=1}^{n} \xi_i (\vec{r}_i + \vec{t})$$



[n corps dans E
à translation près

Si $E = \mathbb{R}^p$,

x est représenté par la matrice $p \times n$

$$\begin{array}{ccc} \vec{x}_1 & \ldots & \vec{x}_n \\ \end{array}$$
$$\begin{pmatrix} | & & | \\ | & \cdots & | \\ | & & | \end{pmatrix},$$

équivalente à

$$\begin{array}{ccc} \vec{x}_1 + \vec{t} & \ldots & \vec{x}_n + \vec{t} \\ \end{array}$$
$$\begin{pmatrix} | & & | \\ | & \cdots & | \\ | & & | \end{pmatrix}$$

ou par une matrice $p \times (n-1)$ si on a choisi une base de D^x

Action des rotations (ou symétries) :

$$R.(\vec{r}_1, \dots, \vec{r}_n) = (R.\vec{r}_1, \dots, R.\vec{r}_n)$$

Dans une base orthonormée de E :

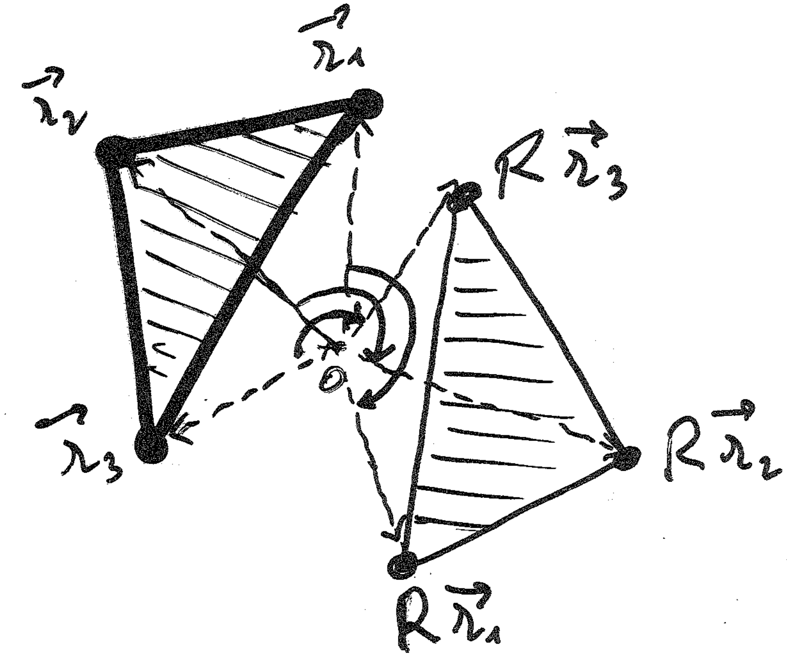
$${}^tR\,R = Id$$

$$\Downarrow$$

$$\begin{array}{cc} & \begin{array}{ccc} \vec{r}_1 & \cdots & \vec{r}_n \end{array} \\ \begin{array}{c} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{array} & \begin{pmatrix} \text{———} \\ \vdots \\ \text{———} \end{pmatrix} \begin{pmatrix} | & \cdots & | \end{pmatrix} \end{array}$$

$$\parallel$$

$$\Big(\langle \vec{r}_i, \vec{r}_j \rangle_E \Big)$$ invariant (Construction de Gram)



$$\beta(\xi,\eta) = \sum_{i,j} \langle \vec{r}_i , \vec{r}_j \rangle_E \, \xi_i \eta_j$$

$$= \sum_{i,j} \left(-\frac{1}{2} r_{ij}^2\right) \xi_i \eta_j$$

||| sur D^*, pas sur $\mathbb{R}^n$!

(car $r_{ij}^2 \overset{\text{déf.}}{=} |\vec{r}_i - \vec{r}_j|^2 = |\vec{r}_i|^2 + |\vec{r}_j|^2 - 2\langle \vec{r}_i , \vec{r}_j \rangle$).

forme quadratique sur D^*

La "forme" β contient toute l'information sur la "forme" des n corps $\vec{r}_1, \ldots, \vec{r}_n$

<u>Théorème</u> (Schoenberg 1938, Menger)

> Les r_{ij} sont les distances mutuelles de n corps plongés dans E euclidien
>
> $$\Updownarrow$$
>
> La forme β est ≥ 0

(i.e. $\forall \xi, \eta \in \mathcal{D}^*$, $\sum_{i,j} \left(-\frac{1}{2} r_{ij}^2\right) \xi_i \eta_j \geq 0$).

Complément : on peut choisir $\dim E = p$

$$\Updownarrow$$

rang $\beta = p$.

TRANSACTIONS vol 44, 1938 -

METRIC SPACES AND POSITIVE DEFINITE FUNCTIONS*

BY

I. J. SCHOENBERG

3. Conditions for isometric imbedding in Hilbert space in terms of positive definite functions. It was pointed out by K. Menger and by the author (for references see [10]) that a necessary and sufficient condition that a separable space $\mathfrak{S}$ be imbeddable in $\mathfrak{H}$ is that for any $n+1$ points of $\mathfrak{S}$, ($n \geqq 2$), we have

$$\sum_{i,k=1}^{n} (\overline{P_0P_i}^2 + \overline{P_0P_k}^2 - \overline{P_iP_k}^2)\rho_i\rho_k \geqq 0,$$

for arbitrary real ρ_i.* Let us now put this condition in a slightly more symmetrical form. By summing over the three terms separately, we may write this as

$$2\sum_1^n \rho_k \cdot \sum_1^n \overline{P_0P_k}^2\rho_k - \sum_1^n \overline{P_iP_k}^2\rho_i\rho_k \geqq 0,$$

* This was proved for the case when $\mathfrak{S}$ is a separable *semi-metric* space; that is, when the metric $\overline{PP'}$ satisfies the additional condition (3) $\overline{PP'}>0$ if $P \neq P'$, whereas we postulated only that (1) $\overline{PP'}=\overline{P'P} \geqq 0$, (2) $\overline{PP}=0$. However, our quadratic inequality, for $n=2$, insures the triangle inequality $\overline{PQ}+\overline{QR} \geqq \overline{PR}$ for any three points of $\mathfrak{S}$. If we now *identify* with P all points Q such that $\overline{PQ}=0$ (which is now allowed, since $\overline{PQ}=0$ implies $\overline{RP}=\overline{RQ}$ for any R, on account of the triangle inequality) and do this for all points of $\mathfrak{S}$, we get a new space which is not only semi-metric but even metric.

and if we set $\rho_0 = -\sum_1^n \rho_k$, this last inequality is equivalent to

$$-\rho_0^2\overline{P_0P_0}^2 - 2\sum_0^n \overline{P_0P_k}^2\rho_0\rho_k - \sum_1^n \overline{P_iP_k}^2\rho_i\rho_k \geqq 0,$$

or, finally,

$$\sum_{i,k=0}^{n} \overline{P_iP_k}^2\rho_i\rho_k \leqq 0. \tag{5}$$

Hence the inequality (5), as a consequence of the relation

$$\sum_{i=0}^{n} \rho_i = 0, \tag{6}$$

is equivalent to the above stated condition of imbeddability.

D'où vient $\mathcal{D}^*$?

dual des "dispositions" $\mathcal{D} = \mathbb{R}^n / (1,\dots,1)\mathbb{R}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\{$n-uples sur $\mathbb{R}$ à translation près$\}$

$\forall \alpha \in \mathbb{R}, \ (x_1, \dots, x_n) \sim (x_1+\alpha, \dots, x_n+\alpha)$

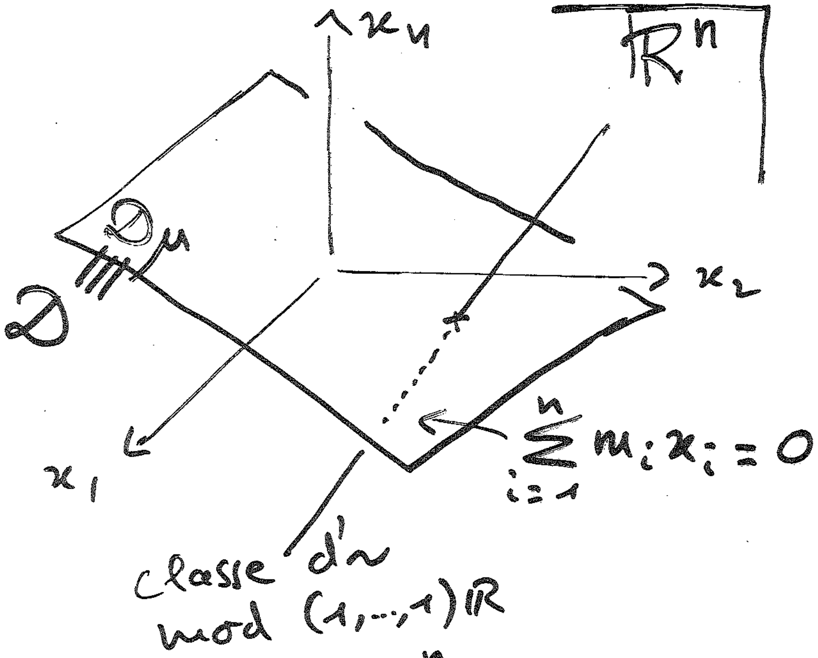
Dualité $\quad \mathcal{D}^* \times \mathcal{D} \longrightarrow \mathbb{R}$

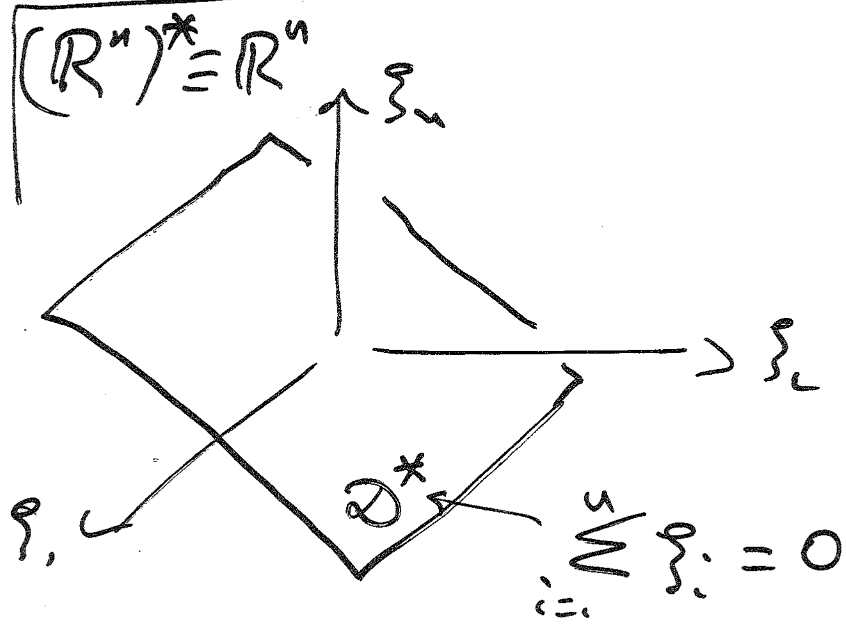
$$\Big(\underbrace{(\xi_1, \dots, \xi_n)}_{\sum \xi_i = 0}, (x_1, \dots, x_n)\Big) \longmapsto \sum_{i=1}^n \xi_i x_i = \sum \xi_i (x_i + \alpha)$$

$$\mathrm{Hom}(\mathcal{D}^*, E) = \mathcal{D} \otimes E$$

$$\beta : \mathcal{D}^* \xrightarrow{x} E \xrightarrow{\varepsilon} E^* \longrightarrow \mathcal{D}$$

Masses, volumes, inerties

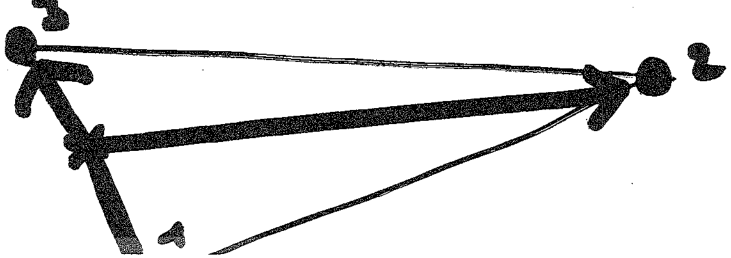




$$\mu(x,y) = \sum_{i=1}^n m_i x_i y_i \qquad \mu^{-1}(\xi,\eta) = \sum_{i=1}^n \frac{1}{m_i} \xi_i \eta_i$$

Masses $\Leftrightarrow$ Produits scalaires sur $\mathcal{D}$ et $\mathcal{D}^*$

Bases orthogonales $\longleftrightarrow$ coordonnées de Jacobi



β forme quadratique sur $\mathcal{D}^*$ $\xrightarrow{\mu}$ $B = \mu \circ \beta$ endomorphisme de $\mathcal{D}^*$

$$\begin{tikzcd} \mathcal{D}^* \arrow[r, "\beta"] \arrow[rr, bend right, "B"'] & \mathcal{D} \arrow[r, "\mu"] & \mathcal{D}^* \end{tikzcd}$$

Proposition : $\det(\mathrm{Id}_{\mathcal{D}^*} - \lambda B) =$

$$1 - \sigma_1 \lambda + \dots + (-1)^{n-1} \sigma_{n-1} \lambda^{n-1}$$

$$\sigma_{k-1} = \frac{1}{\sum m_i} \sum_{i_1 < \dots < i_k} m_{i_1} \dots m_{i_k} \, \mathrm{vol}^2_{i_1 \dots i_k}$$

∀ parallélépipède

= $(k-1)!$ vol. simplex

en particulier,

$$\eta_1 = \text{trace}\, B = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j r_{ij}^2 \underset{\substack{\uparrow \\ \text{Leibnitz}}}{=} \sum m_i \left| \vec{r}_i - \vec{r}_G \right|^2$$

(= moment d'inertie I par rapport au c.d.g.)

$$\eta_{n-1} = \det B = \frac{m_1 \cdots m_n}{\sum m_i} \left((n-1)!\ \text{vol. simplexe} \right)^2$$

$\Longrightarrow$ Héron pour $n=3$ (dim $D^* = 2$)

Pb : pas de ~~base~~ privilégiée

$\Longrightarrow$ pour calculer, det 4×4 !

L'endomorphisme $\mathcal{B}$ comme "inertie intrinsèque"

$$\mathcal{D} \underset{\equiv}{\xrightarrow{\mu}} \mathcal{D}^* \qquad E \underset{\equiv}{\xrightarrow{\varepsilon}} E^*$$

du côté des corps | du côté de l'espace

$\mathcal{B} = {}^tXX$ | $\mathcal{J} = X\,{}^tX$

ellipsoïde d'inertie

même information spectrale

INVARIANT $\quad$ si $X \mapsto X' = RX$ $\quad$ COVARIANT

$\mathcal{B}' = {}^tX\,{}^tRRX = \mathcal{B}$ | $\mathcal{J}' = R\mathcal{J}\,{}^tR$

Et si les corps deviennent célestes ...

$$U = \sum_{i<j} \frac{m_i m_j}{(\lambda_{ij}^2)^{1/2}} = U(B)$$

Théorème : Les pts critiques de

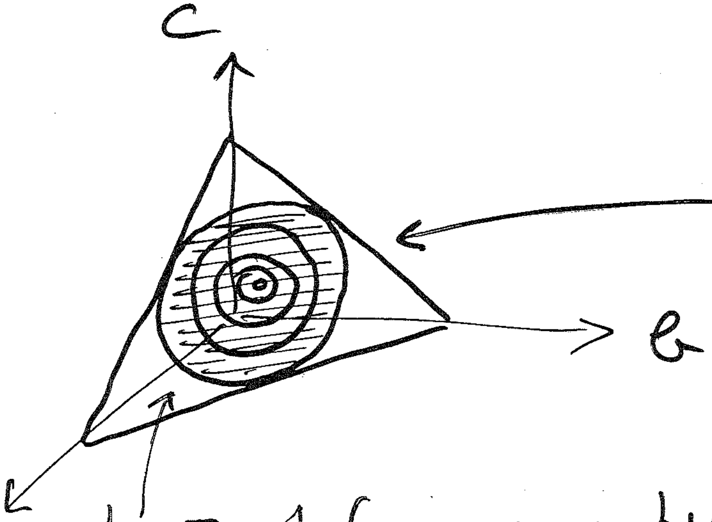
$$U|_{\text{Isospectrale}(B)}$$

sont les configurations admettant un mouvement d'équilibre relatif dans un E euclidien (Config. équilibrées)

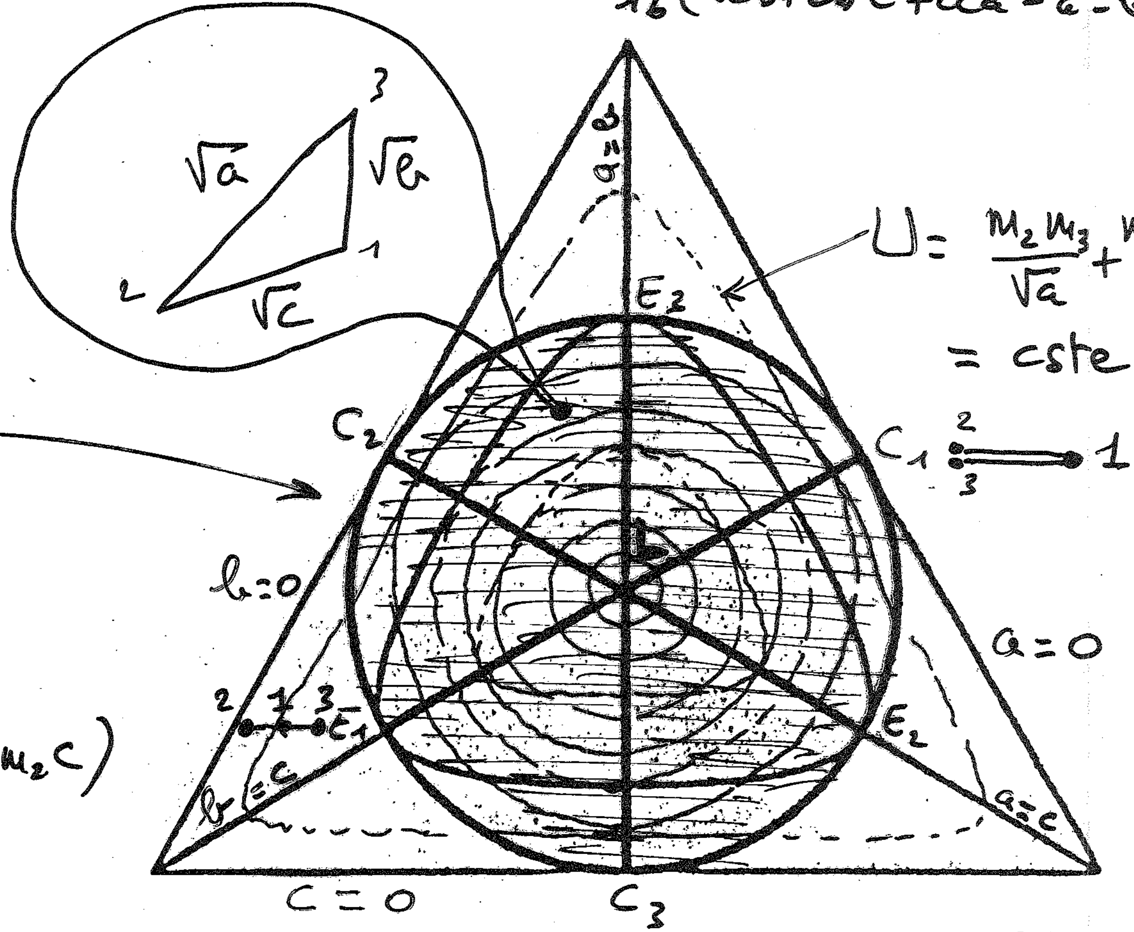
$n=3$ L'espace des triangles

Vrais triangles $\underset{\text{Schoenberg}}{\Longleftrightarrow}$ $\eta_2 = $ cste $\underset{\text{Héron}}{S^2} \geq 0$.

$\| \frac{1}{16}(2ab+2bc+2ca-a^2-b^2-c^2)$



$1 = \eta_1 = I = \frac{1}{\sum m_i}(m_2 m_3 a + m_3 m_1 b + m_1 m_2 c)$



$U = \frac{m_2 m_3}{\sqrt{a}} + \frac{m_3 m_1}{\sqrt{b}} + \dots$

$=$ cste

$m_1 = m_2 = m_3$, $I = 1$

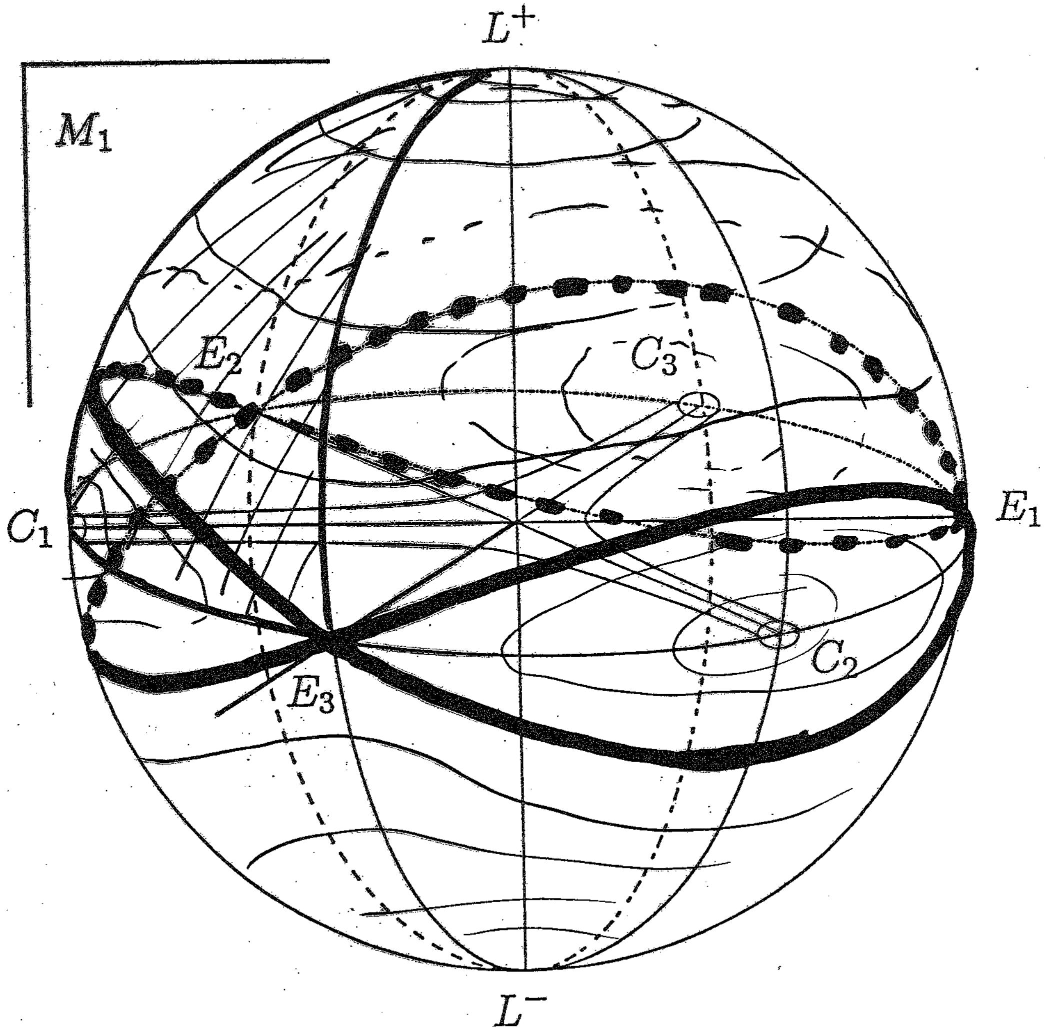
Et si on oriente les triangles du plan ?

2 copies de  =  $\pm$

$$\begin{tikzcd}
(\mathbb{R}^2)^3 \arrow[r] & \mathbb{R}^4 \overset{\text{Jacobi}}{=} \mathbb{C}^2 \arrow[d, "/SO(2)", "\text{HOPF}"'] & \underset{I=1}{\supset} & S^3 \arrow[d, "\text{Hopf}"] \\
 & \mathbb{R}^3 = \mathbb{R}\times\mathbb{C} & \underset{I=1}{\supset} & S^2 =
\end{tikzcd}$$

$(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ $\mathbb{R}^4 = D \otimes \mathbb{R}^2$ (z_1, z_2)

$(|z_1|^2 - |z_2|^2,\ 2\bar{z}_1 z_2)$

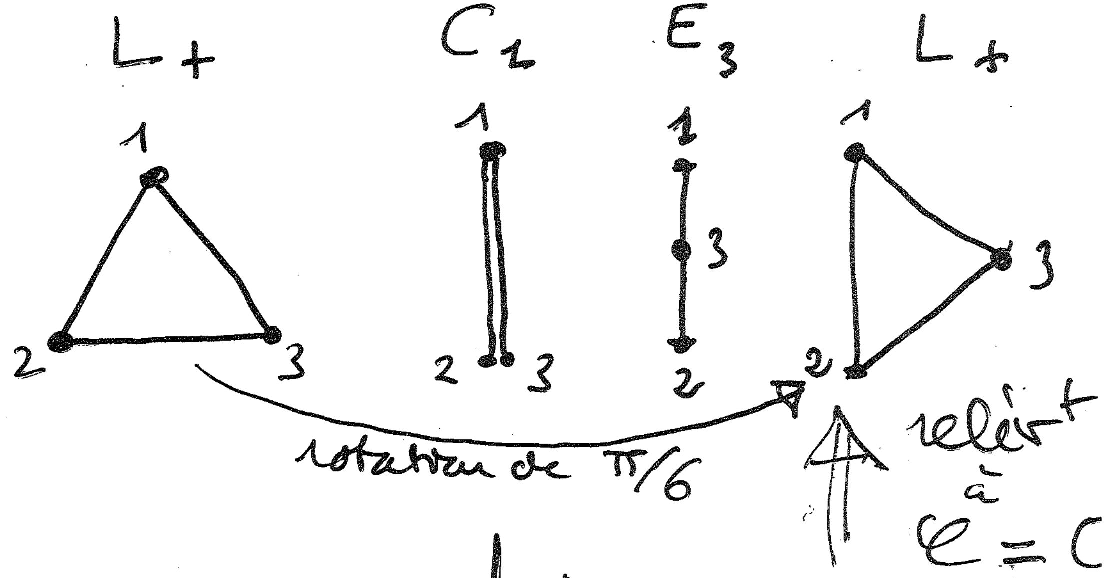


Il subsiste une symétrie D_6

(~~[illegible]~~ (par isométries si les masses sont égales)

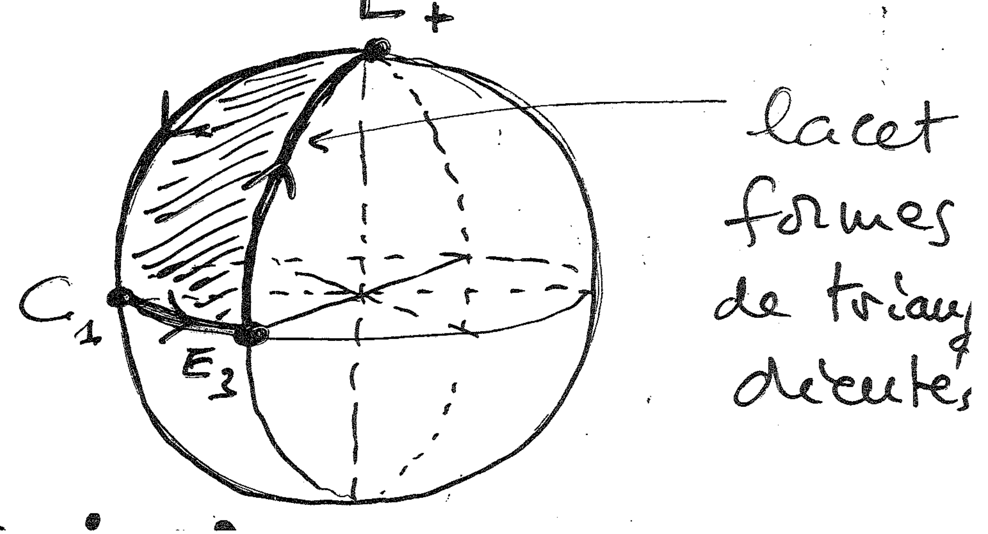
Tourner sans rotation, ou commen[t] un chat retombe sur ses pattes :

$\left(\begin{array}{l}\text{Triangles } {}_2\overset{1}{\Delta}{}_3 \\ \text{dans } \mathbb{R}^2\end{array}\right) = \mathbb{R}^4$



$\Big\downarrow$ quotient par les rotations

$\left(\begin{array}{l}\text{Formes de} \\ \text{triangles orientés}\end{array}\right) = \mathbb{R}^3 \supset$



∃ Une solution du pb des 3 corps (masses égales) qui possède l'entière symétrie D_6 de l'espace des triangles :