ICMP 03 (LISBOA)

SYMMETRIES AND "SIMPLE" SOLUTIONS IN THE CLASSICAL N-BODY PROBLEM

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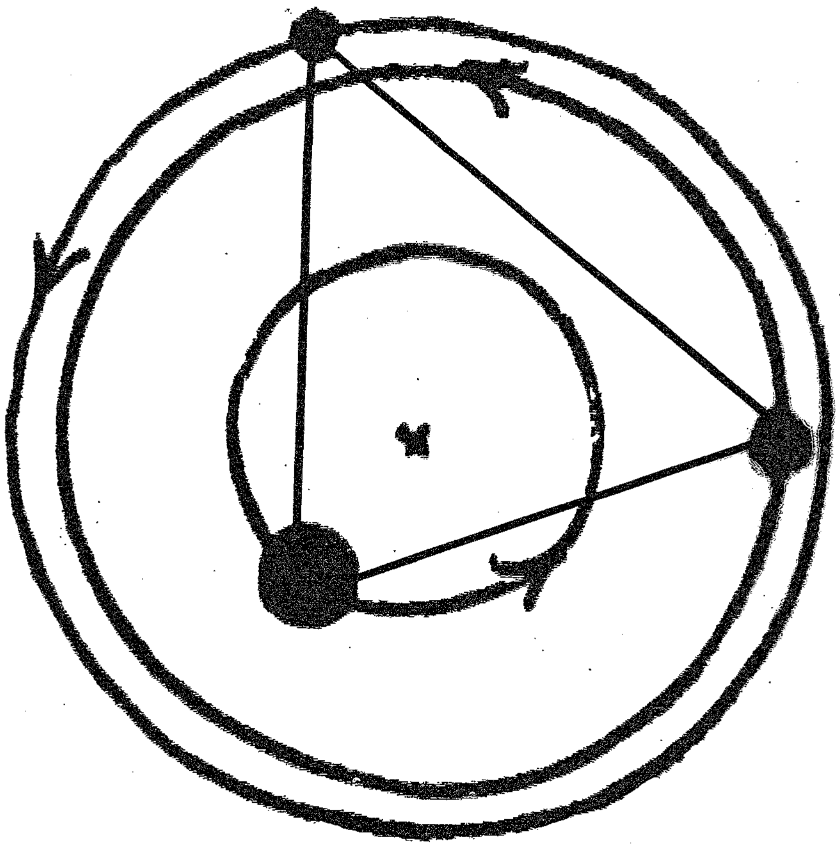
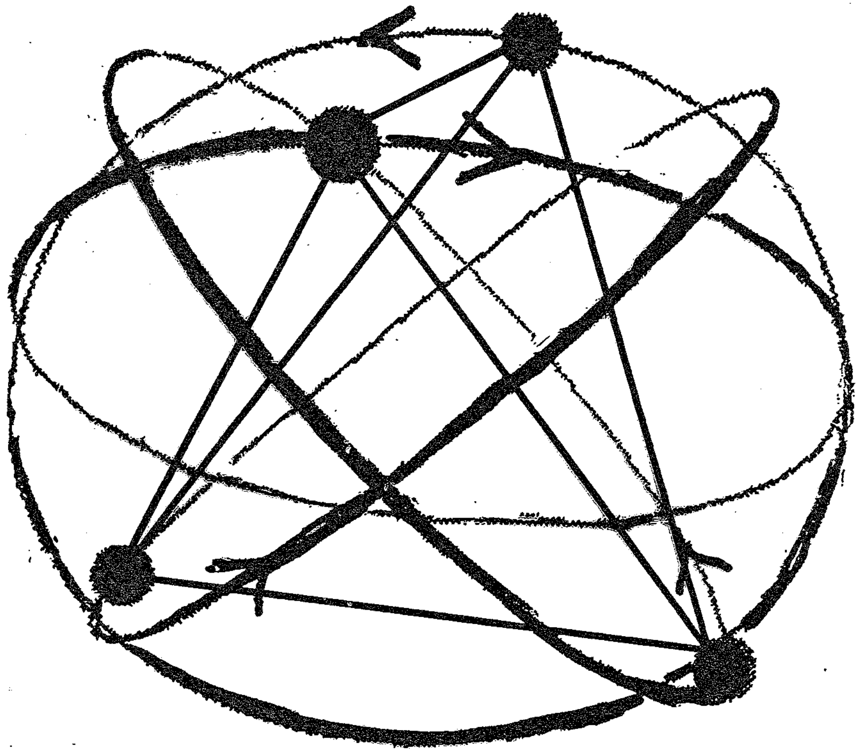
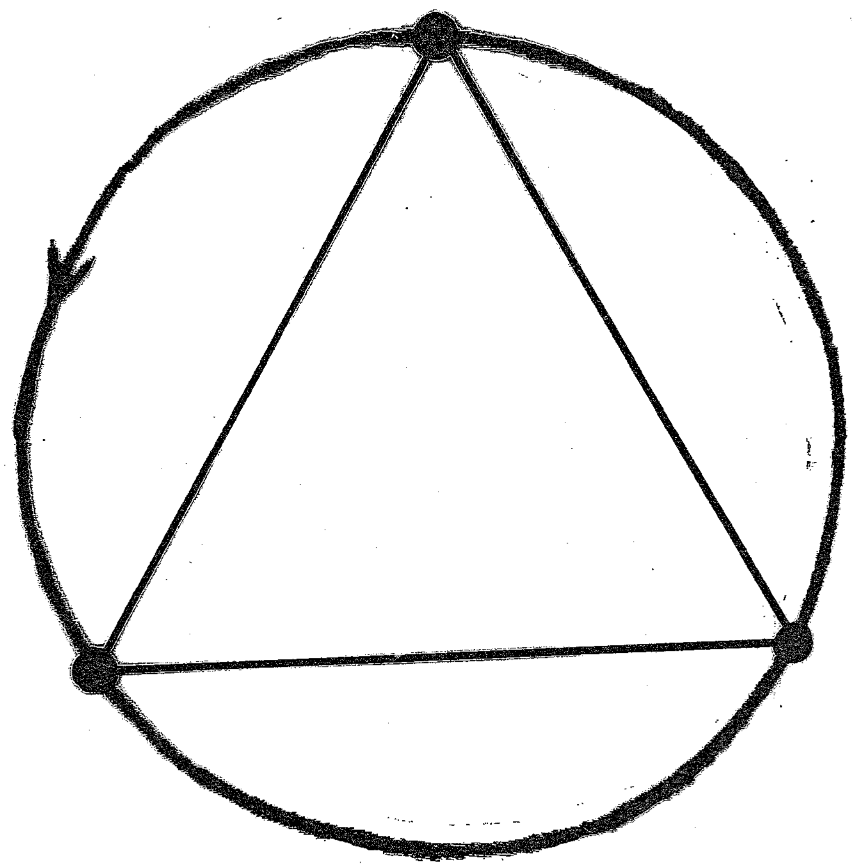
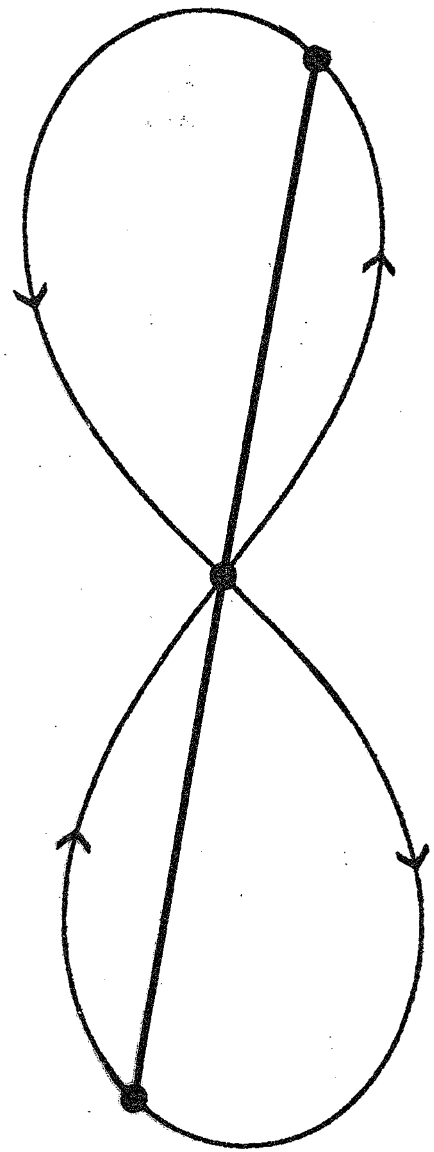
$\mathbb{Z}/2\mathbb{Z}$





$\mathbb{Z}/3\mathbb{Z}$

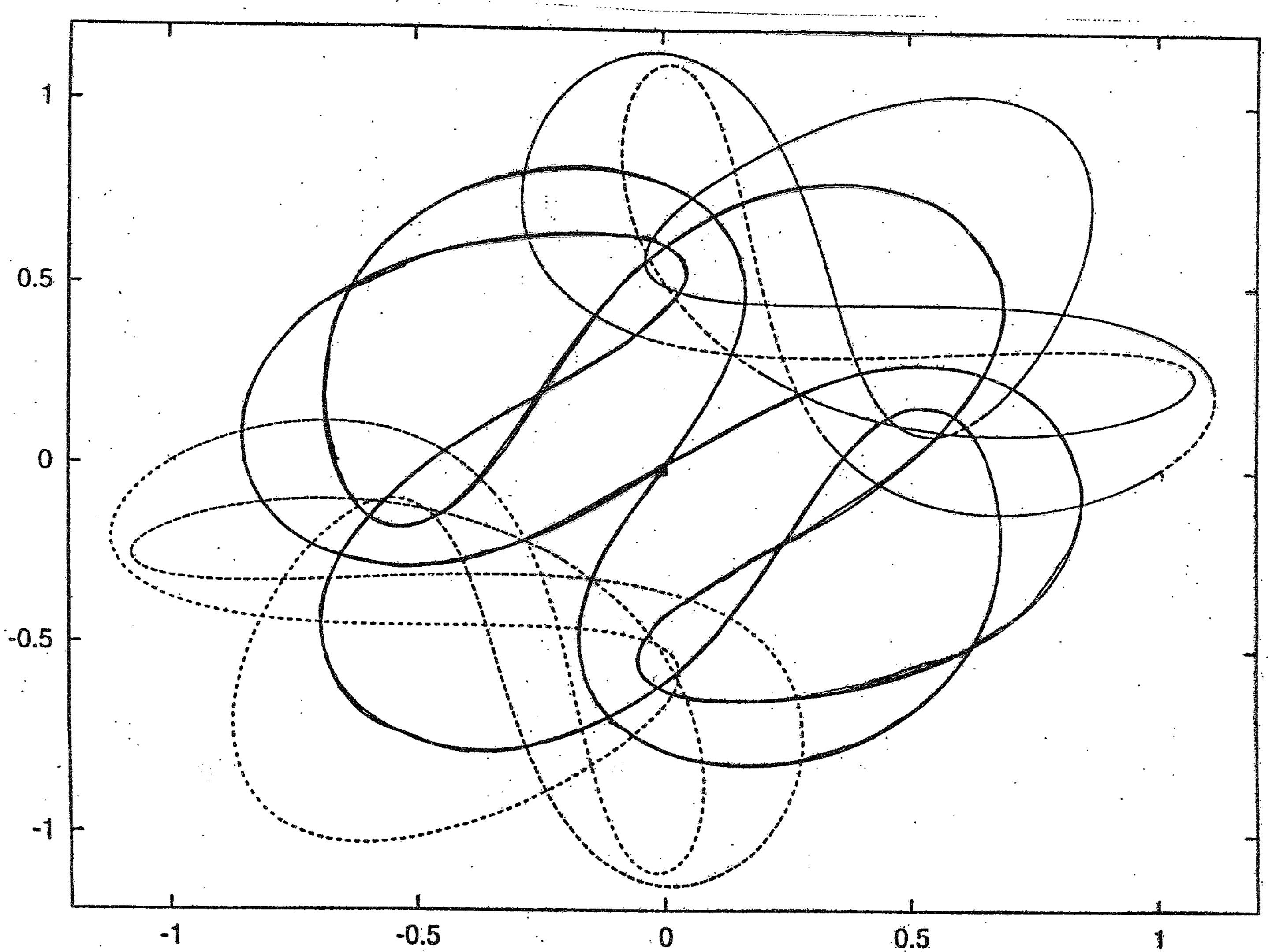




Common features in these pairs of periodic solutions?

- Symmetric
- Simple

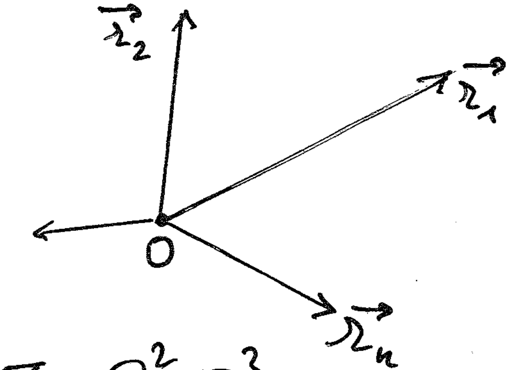
What means "simple" ?



Carlès Simó
déc. 1999

The classical n-body problem

$$\mathcal{X} = \left\{ x = (\vec{r}_1, \dots, \vec{r}_n),\ \sum_{i=1}^{n} m_i \vec{r}_i = \vec{0} \right\}$$



$E = \mathbb{R}^2, \mathbb{R}^3, \dots$ euclidean

$$\ddot{x} = \nabla U(x)$$

gradient for the kinetic energy metric:

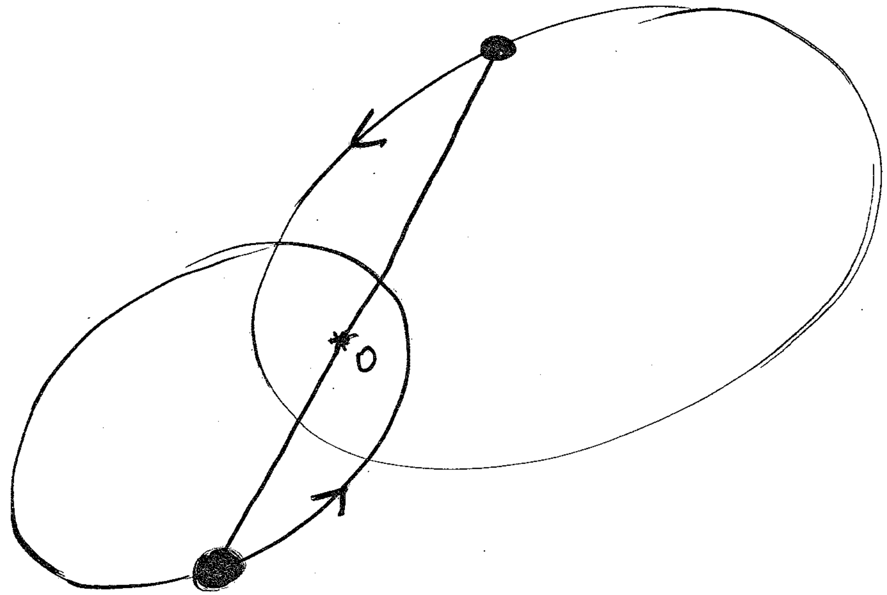
$$|x|^2 = I(x) = \sum_i m_i |\vec{r}_i|_E^2$$

$$= \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\vec{r}_i - \vec{r}_j|_E$$

$$U(x) = \sum_{i<j} g \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

First criterion of simplicity :

"Distance" to 2-body solutions



Measure it by <u>decomposition</u> of <u>velocity</u>

$$K = |\dot{x}|^2 = \underset{\substack{\| \\ J^2/I}}{K_{scale}} + \underset{\substack{\geq \\ c^2/I}}{K_{rot.}} + \underset{\substack{\geq \\ 0}}{K_{def}}$$

$= \text{ if } E = \mathbb{R}^2$

$I = |x|^2$, $J = x \cdot \dot{x} = \frac{1}{2}\dot{I}$

$c = |\mathcal{C}|$, $\mathcal{C} = \sum m_i \vec{r}_i \wedge \dot{\vec{r}}_i$ angular momentum

$\Rightarrow$ Sundman's $\leq$: $IK - J^2 \geq c^2$

$\Updownarrow$

One forgets the variations of the <u>shape</u>

= for "Keplerian motions"

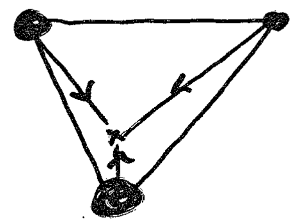
Keplerian motions = homographic motions

$\Updownarrow$ constant shape up to similarity

Only possible for

Central configurations

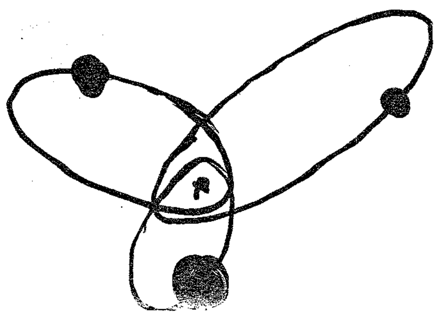
$$\underbrace{\nabla U(x)}_{\text{forces}} = \lambda \underbrace{x}_{\text{configuration}} = \frac{\lambda}{2} \nabla I(x)$$

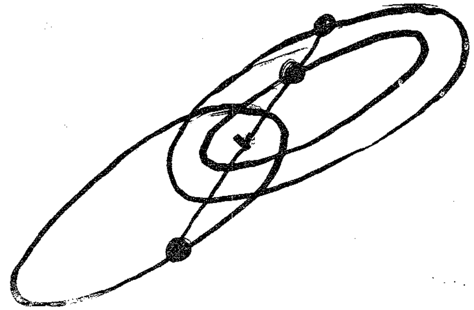


$\Leftrightarrow$ x critical pt of $U|_{I=cste}$

$\Leftrightarrow$ x critical pt of $\tilde{U} = \sqrt{I}\,U$

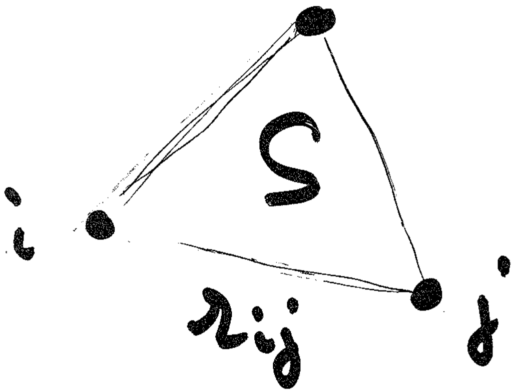
3 bodies: Euler Lagrange





$0 < e < 1$

$\forall m_i$, $\exists\, 1!$ Central conf. of 3 bdies which is non collinear



r_{ij}^2 = indep. variables
(open cond. $S^2 = P(r_{ij}^2) \geq 0$)

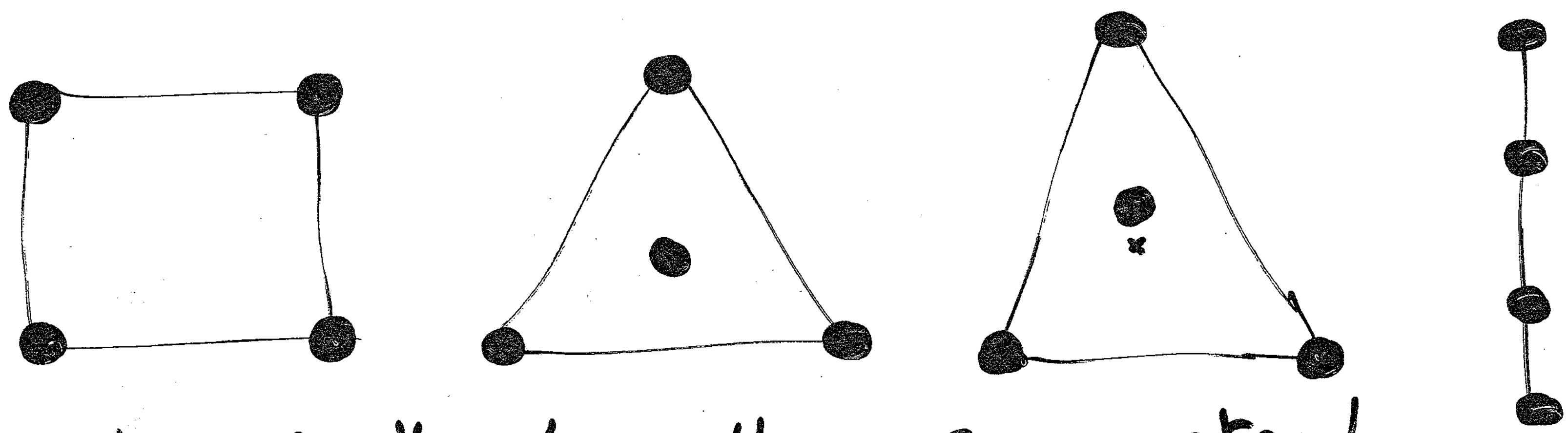
$$I = \sum_i m_i |\vec{r}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j r_{ij}^2$$

$$U = g \sum_{i<j} m_i m_j \left(r_{ij}^2\right)^{-1/2}$$

$\nabla U \mathbin{//} \nabla I \implies$ all $r_{ij}^2 =$

EQUILATERAL

To determine central configurations for $n \geq 4$ bodies is a very hard problem: even finiteness of the # of solutions is not known (except for 4 equal masses [Albouy]).



<u>Key</u> of Albouy's result: Sy<u>mmetry</u>!

Homographic motions are closely related to symmetries:

$e=0$ <u>Rel. equilibrium</u> = singularities of equations reduced by <u>isometries</u>

$e=1$ <u>homothetic motions</u>

$\longrightarrow$ "reduction by homothety"

$$\left(x(t) \text{ sol.} \Longrightarrow \lambda^{-2/3} x(\lambda t) \text{ sol.} \right)$$

$$\left(\ddot{x} = \nabla U(x)\right) = \text{Euler-Lagrange}$$

equations of the <u>action</u>

$$\mathcal{A}(x(t)) = \int_{t_0}^{t_1} \left[\frac{1}{2}|\dot{x}(t)|^2 + U(x(t))\right] dt$$

$$\mathcal{A} : \underbrace{H^1([t_0,t_1], \mathcal{X})}_{\Lambda} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$$

$\Lambda = \{$paths in configuration space
or loops if $[t_0,t_1]$ replaced by $S^1_T = \mathbb{R}/T\mathbb{Z}\}$

Next criterion of simplicity:

$x(t)$ action minimizer

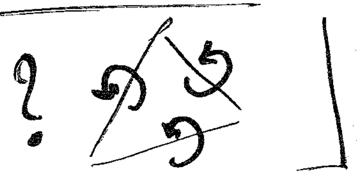
Z $\mathcal{A}:\Lambda\to\mathbb{R}_+\cup\{+\infty\}$

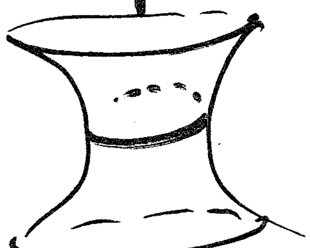
has min. = 0 at ∞ty (non coercive)

$\Rightarrow$ need constraints

homological, homotopical

Poincaré 1896





|SYMMETRY|

much nicer!

SUR LES SOLUTIONS PÉRIODIQUES
ET LE PRINCIPE DE MOINDRE ACTION

Comptes rendus de l'Académie des Sciences, t. 123, p. 915-918 (30 novembre 1896).

La théorie des solutions périodiques peut, dans certains cas, se rattacher au principe de moindre action.

Supposons trois corps se mouvant dans un plan et s'attirant en raison inverse du cube des distances ou d'une puissance plus élevée de ces distances; j'appelle a, b, c ces trois corps.

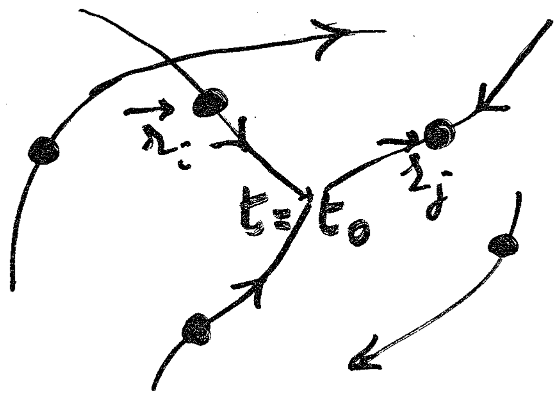
L'énergie cinétique T est essentiellement positive et il en est de même de la fonction des forces U, qui est égale à une somme de termes de la forme $\frac{kmm'}{r^n}$, où k est une constante positive, m et m' les masses de deux des trois corps, r leur distance et n un exposant au moins égal à 2.

L'action hamiltonienne

$$J = \int_{t_0}^{t_1} (T + U)\, dt$$

sera donc essentiellement positive.

The problem of collisions



Sundman 1913

$$|\vec{r}_i - \vec{r}_j| = O\left(|t - t_0|^{2/3}\right)$$

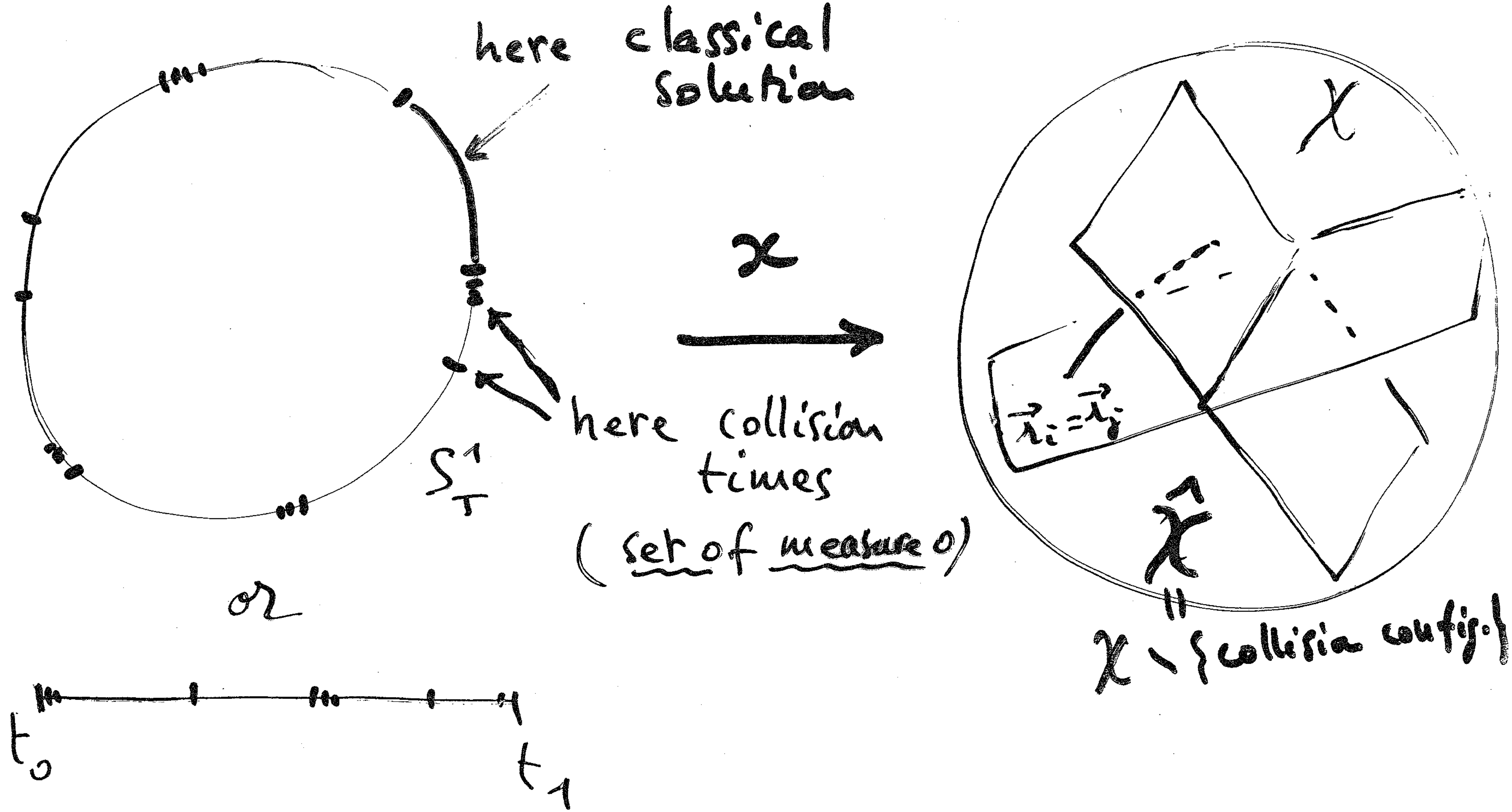
$$|\dot{\vec{r}}_i - \dot{\vec{r}}_j| = O\left(|t - t_0|^{-1/3}\right)$$

$$\Rightarrow \mathcal{A} = \int_t^{t_0} O\left(|t - t_0|^{-2/3}\right) < +\infty \quad !!!$$

$\Downarrow$

not a priori excluded in minimizers

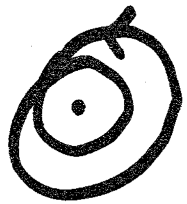
A priori structure of minimizers



This is not an academic problem:

ex: Kepler problem (= 1 fixed center) in the plane

$$\mathcal{A} = \int_0^T \left(\frac{|\dot{x}|^2}{2} + \frac{1}{|x|} \right) dt$$

min $\mathcal{A}\Big|$  realized only by 

coll. ej.

$\wedge$

$\mathcal{A}\Big($  $\times 2\Big)$

Discrete symmetries of the action

$$G \times H^1(S^1_T, \mathcal{X}) \xrightarrow[(\tau,\sigma,\rho)]{} H^1(S^1_T, \mathcal{X})$$

$\forall g \in G,$

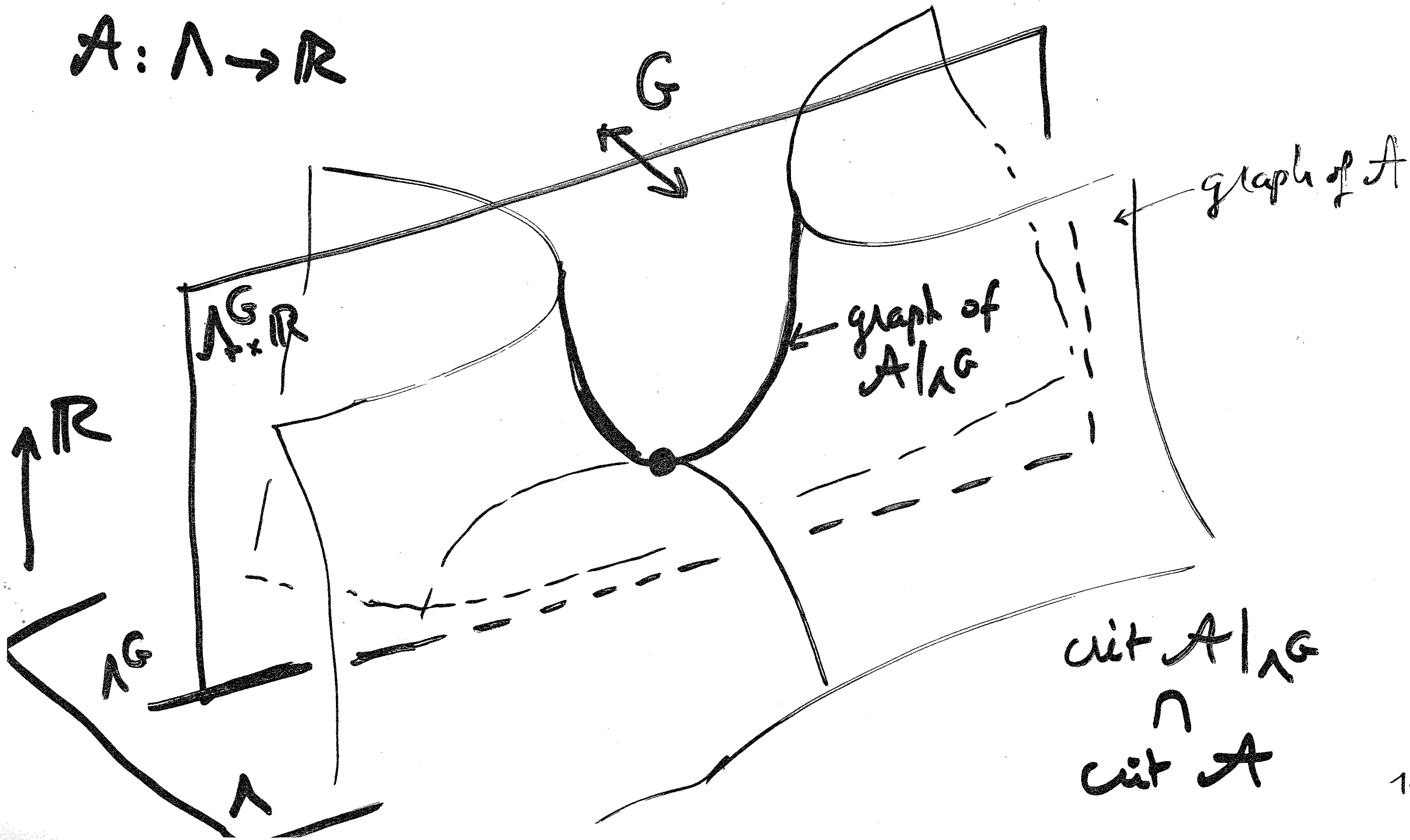
$$\begin{tikzcd}
S^1_T \arrow[r, "x"] \arrow[d, "\tau(g)"'] & \mathcal{X} \arrow[r, hook] \arrow[d] & (\{1,2,\dots,n\} \arrow[r] \arrow[d, "\sigma(g)"] & E) \arrow[d, "\rho(g)"] \\
S^1_T \arrow[r, "g.x"] & \mathcal{X} \arrow[r, hook] & (\{1,2,\dots,n\} \arrow[r] & E)
\end{tikzcd}$$

i.e. $g.(\vec{x}_1(t),\dots,\vec{x}_n(t)) = \left(\rho(g)\vec{x}_{\sigma(g^{-1})1}(\tau(g^{-1})t),\dots,\rho(g)\vec{x}_{\sigma(g^{-1})n}(\tau(g^{-1})t)\right)$

$\mathcal{A}$ invariant provided $\left\{\begin{array}{l} \tau : G \to O(2) \\ \sigma : G \to \mathfrak{S}(n) \leftarrow \text{if only equal masses are permuted} \\ \rho : G \to O(E) \end{array}\right.$

Palais' symmetric criticality principle

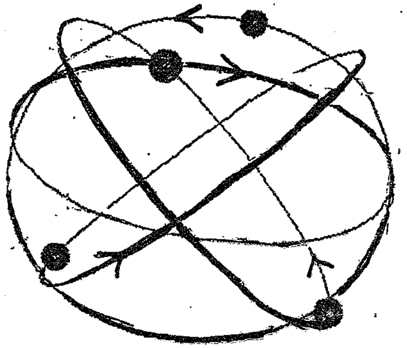
$\mathcal{A}: \Lambda \to \mathbb{R}$



$$\operatorname{crit} \mathcal{A}|_{\Lambda^G} \subset \operatorname{crit} \mathcal{A}$$

BASIC EXAMPLES

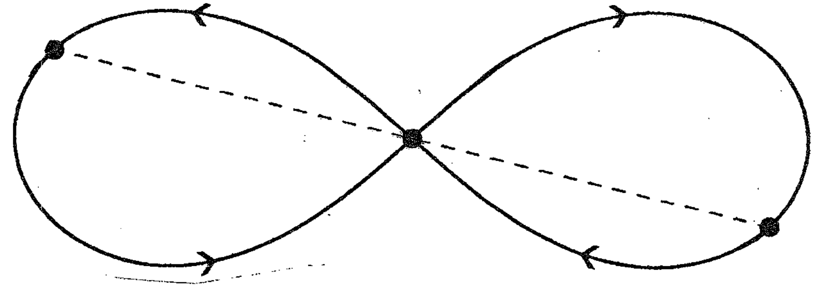
HIP-HOP (4 equal masses in $\mathbb{R}^3$)



$$G = \{ g_1, g_2 \mid g_1^2 = g_2^4 = 1,\ g_1 g_2 = g_2 g_1 \} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

	g_1	g_2
τ	$t \xmapsto{R_\pi} t + T/2$	$t \xmapsto{Id} t$
σ	$\begin{matrix}1&2&3&4\\ &\downarrow Id& & \\1&2&3&4\end{matrix}$	$\begin{matrix}1&2&3&4\\ &\downarrow& & \\2&3&4&1\end{matrix}$
ρ	$(x,y,z) \xmapsto{-Id} (-x,-y,-z)$	$(x,y,z) \mapsto (-y,x,-z)$

EIGHT (3 equal masses in $\mathbb{R}^2$)



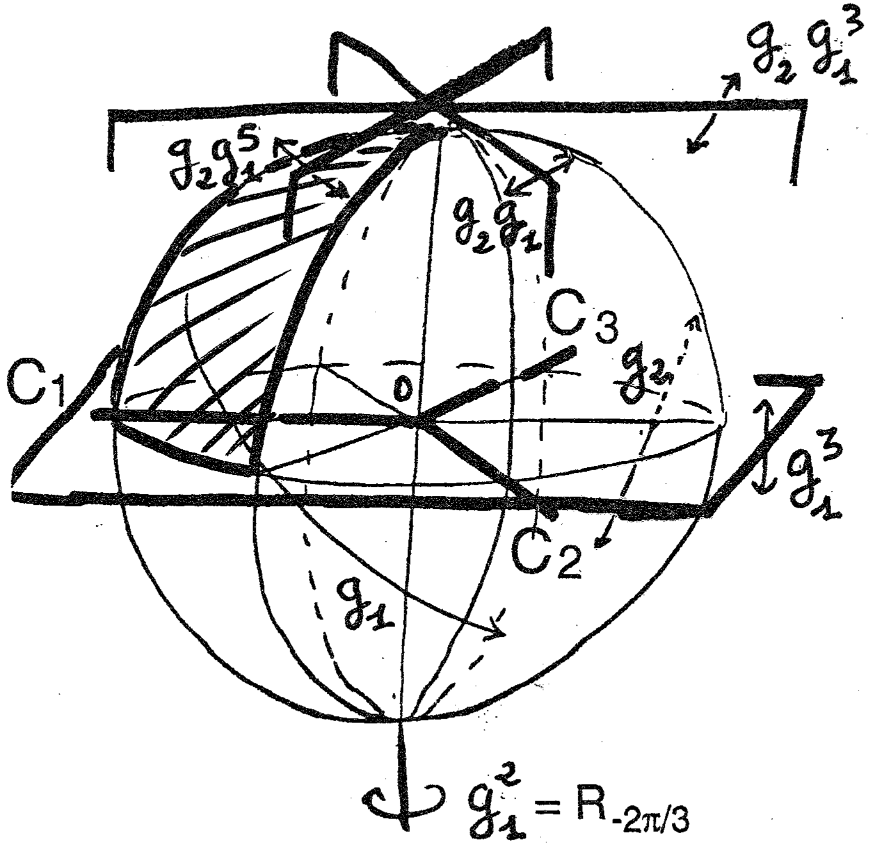
$$G = \{ g_1, g_2 \mid g_1^6 = g_2^2 = 1,\ g_1 g_2 = g_2 g_1^{-1} \} \cong D_6$$

	g_1	g_2
τ	$t \xmapsto{R_{\pi/3}} t + T/6$	$t \longmapsto -t$
σ	$\begin{matrix}1&2&3\\ &\downarrow& \\2&3&1\end{matrix}$	$\begin{matrix}1&2&3\\ &\downarrow& \\1&3&2\end{matrix}$
ρ	$(x,y) \mapsto (-x,y)$	$(x,y) \mapsto (-x,-y)$

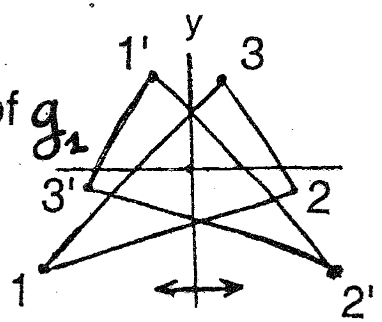
Origin of the D_6-action:

the symmetries of the SHAPE SPHERE

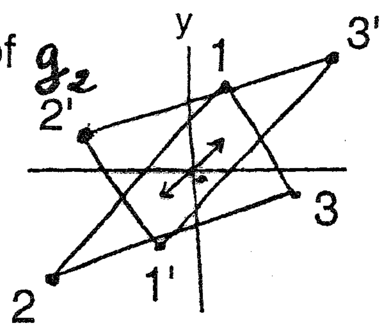
$$= (X \setminus \{0\}) / \text{oriented similarities}$$



action of g_1



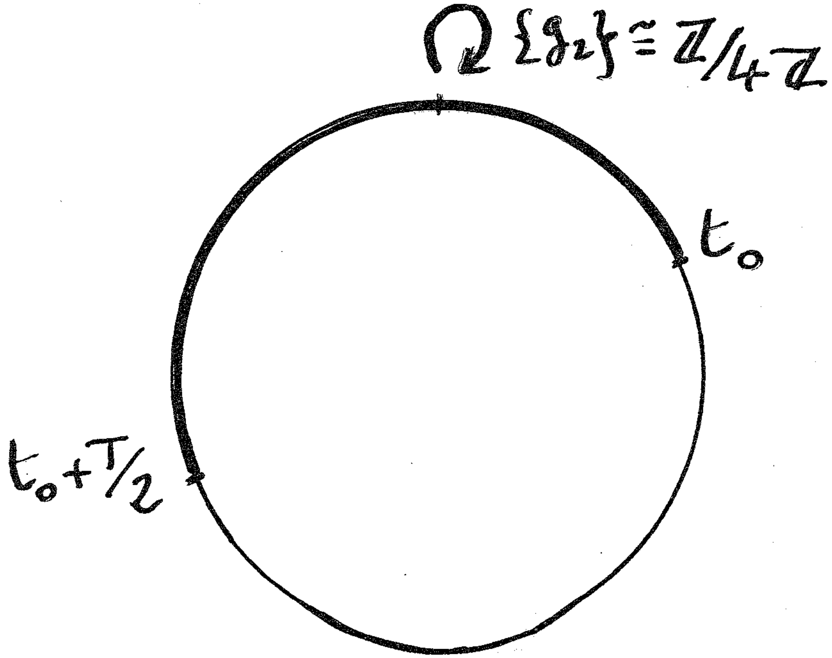
action of g_2



Symmetry constraints behave better because they are related to the FIXED ENDS PROBLEM

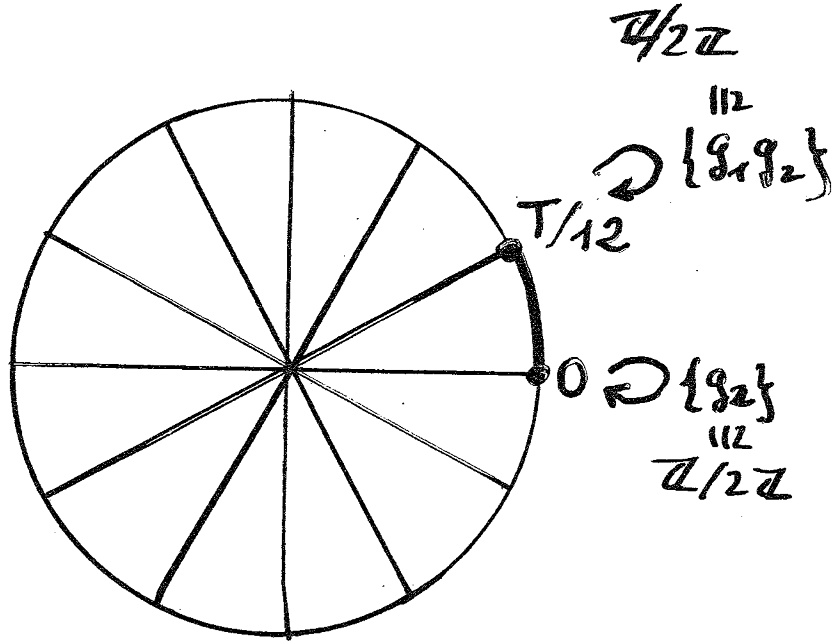
$\mathbb{I} \subset S^1_T$: fundamental domain of τ.action

HIP-HOP



$\mathbb{I} \longrightarrow \chi^{\ker \tau} = \chi^{\mathbb{Z}/4\mathbb{Z}}$

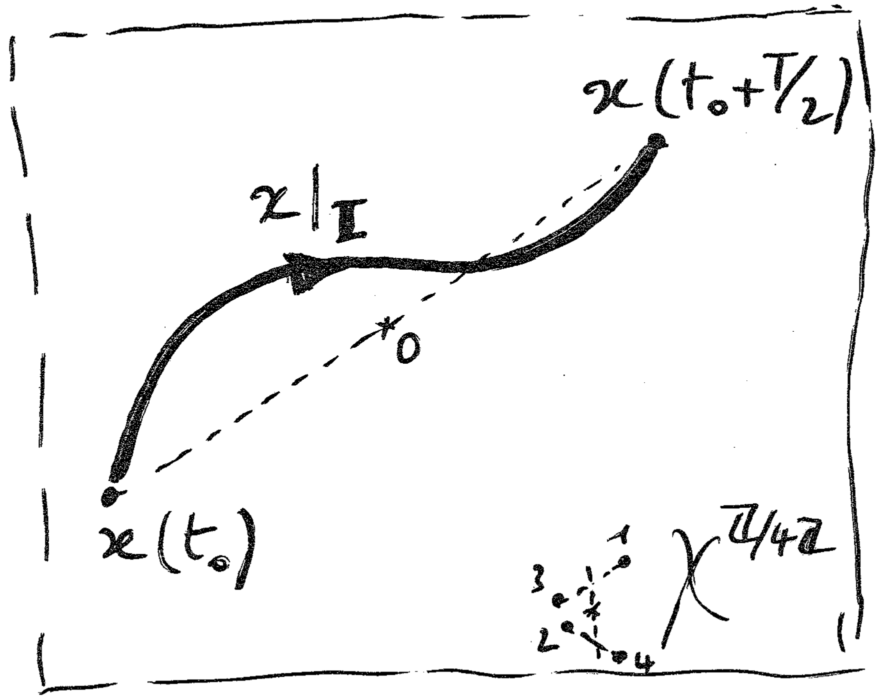
EIGHT

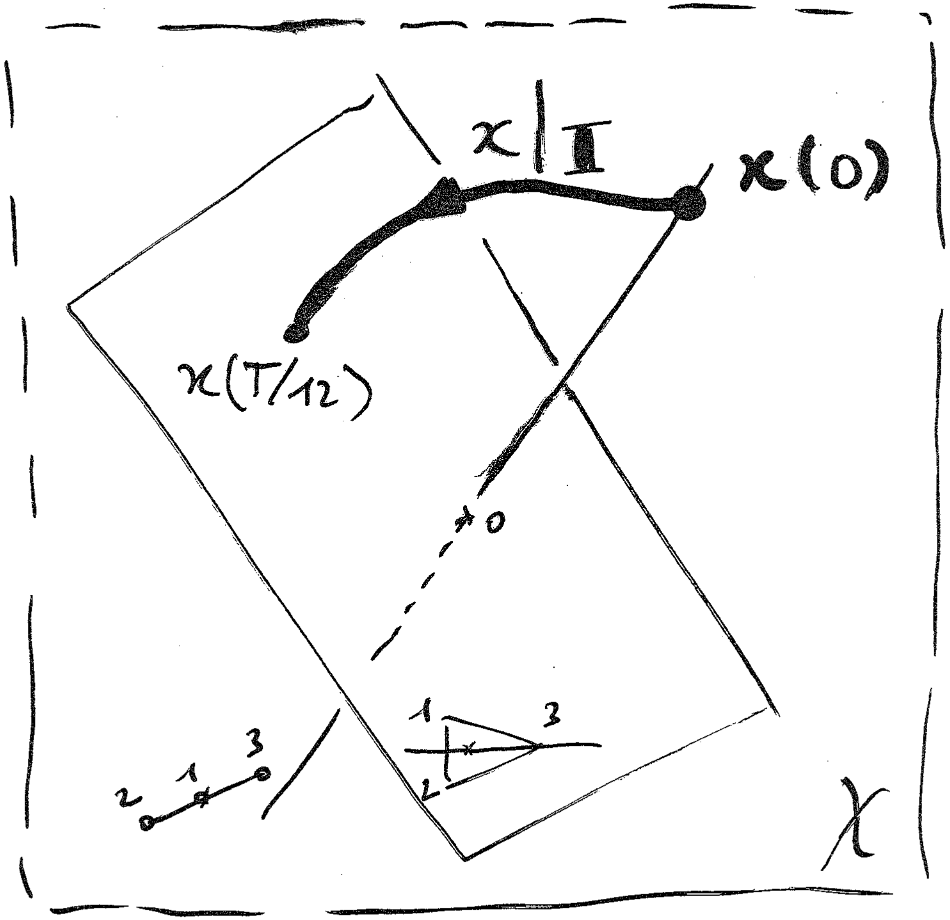


$(\mathbb{I}, 0, T/_{12}) \longrightarrow (\chi, \chi^{\{g_2\}}, \chi^{\{g_1 g_2\}})$

x minimizes in $\Lambda^G \Rightarrow x|_{I}$ minimizes with fixed ends

HIP-HOP	EIGHT

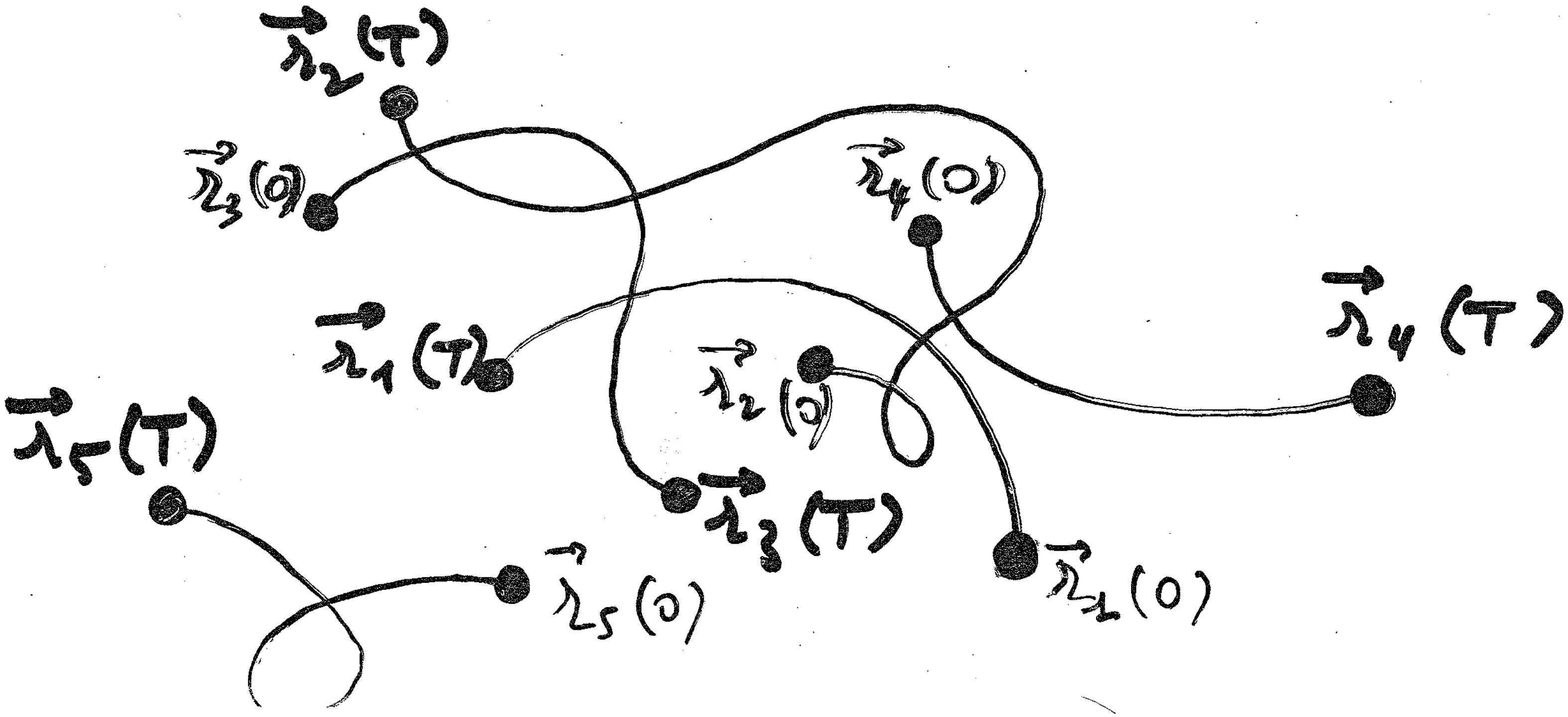




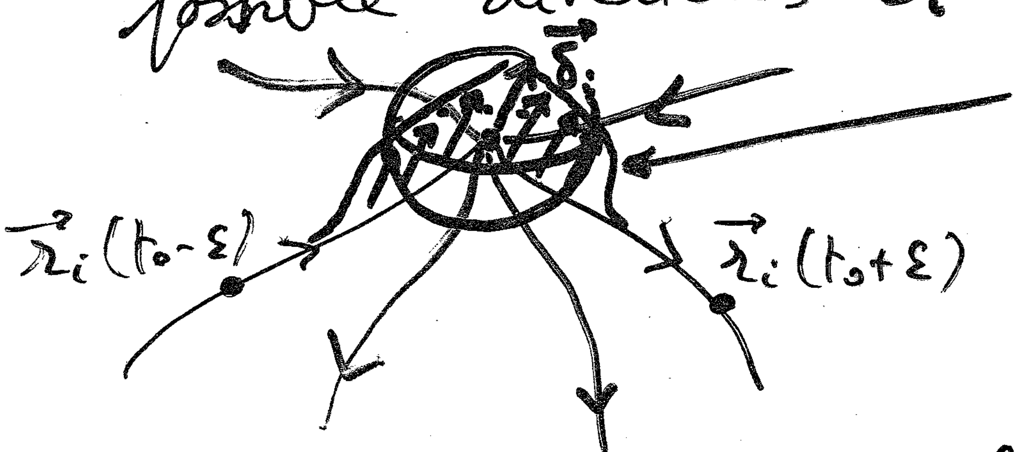
MARCHAL'S THEOREM $\quad E = \mathbb{R}^2, \mathbb{R}^3, \ldots$

$\forall T > 0$, $\forall$ configurations $x', x'' \in \mathcal{X}$
(possibly with collisions!)

any such x
minimizing $\mathcal{A}$ with fixed endpoints and time
is <u>COLLISION FREE</u> in $]0,T[$



Key idea: estimate average of action after moving one colliding body i in all possible directions $\vec{\delta}_i$



Works if { isolated collision at t_0 with limit configuration

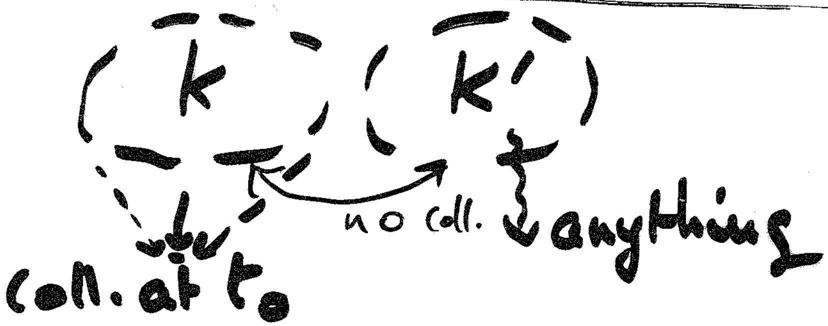
Fact:
average < initial action
easy in $\mathbb{R}^3$:
[amounts to truncating potential of body i

Complete proof uses

- Montgomery & Venturelli: to reduce to isolated collisions
- Terracini & Venturelli to reduce to homothetic motions (blow up)

) cf. A.C. ICM03 & Venturelli's thesis

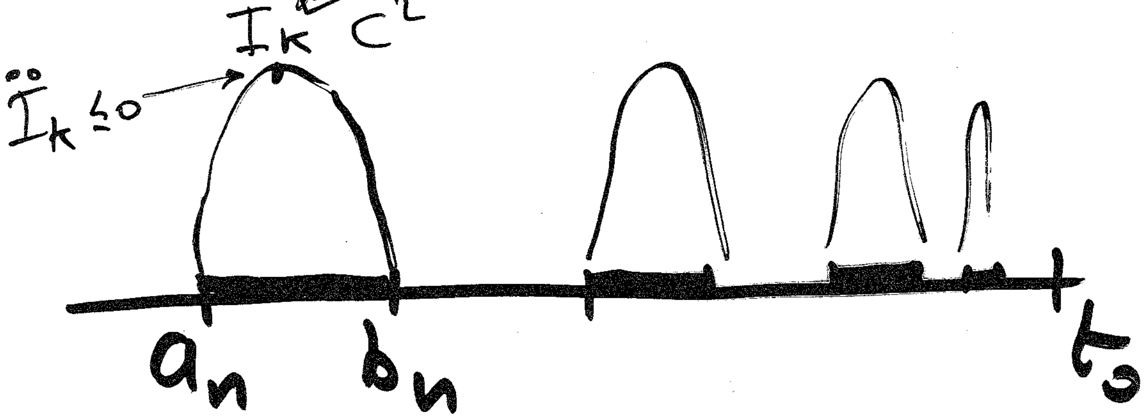
I. Collision with (loc.) min. # of bodies in a minimizer $\Longrightarrow$ ISOLATED

① 

$\Longrightarrow$ H_k is a.c.

Idea: internal variations

② If no subcluster collision $\Longrightarrow$ NO ACCUMULATION



Lagrange – Jacobi

$$\ddot{I}_k = \underbrace{4H_k}_{\text{bded}} + 2\underset{\downarrow\,\infty}{U_k} + \text{bded}$$

Now, if <u>isolated</u> collision at t_0,

$$\frac{x(t)}{|x(t)|} \longrightarrow \{\text{Set of C.C.}\}$$

???

continua ?

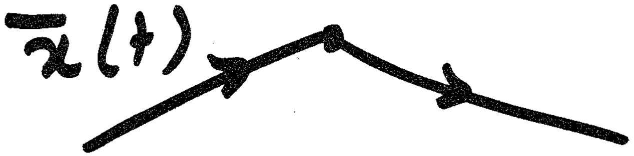
infinite spin ?

II. Blow up $\Longrightarrow$ reduces isolated collision to parabolic homothetic collision-ejection

$\downarrow$

$$x_\lambda(t) = \lambda^{-2/3}\, x(\lambda t)$$

$\downarrow \ \lambda_n \to 0$



C.C.

$$\bar{x}(t) = \begin{cases} (t_0 - t)^{2/3}\, \bar{x}_0 & t < t_0 \\ (t - t_0)^{2/3}\, \bar{x}_1 & t > t_0 \end{cases}$$

$\Longrightarrow$ Marchal's <u>argument</u> applies!

Ferrario - Terracini's <u>invariant</u> version of Marchal's theorem :

<u>Main remark</u>: enough to take average on <u>circles</u>

<u>Problem</u> : need to move several bodies in order to preserve invariance under group action.

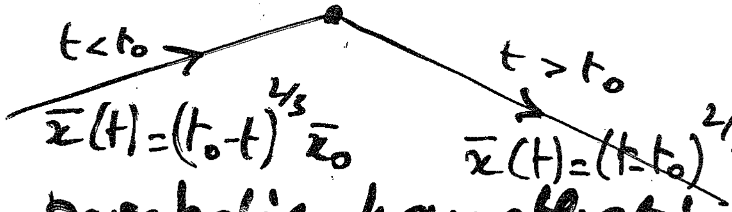
<u>Result</u> : Marchal's thm works in $\mathcal{X}^K$ provided K. action has the <u>ROTATING CIRCLE PROPERTY</u>

FORMULAS (collision at $t = t_0$)

BLOW UP:

$$\boxed{x_\lambda(t) = \lambda^{-2/3}\, x\big(t_0 + \lambda(t - t_0)\big)}$$

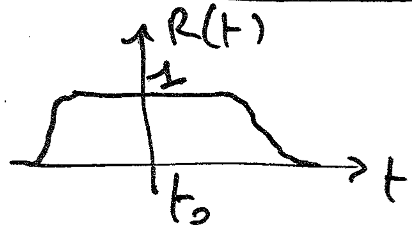
$\exists\, \lambda_n \to 0$, $\quad x_\lambda \xrightarrow[\text{uniformly}]{} \bar{x}$



$\bar{x}(t) = (t_0 - t)^{2/3}\, \bar{x}_0 \qquad \bar{x}(t) = (t - t_0)^{2/3}\, \bar{x}_1$

parabolic homothetic

PERTURBATION:



$$\boxed{x_{new}(t) = \bar{x}(t) + R(t)\underbrace{(\vec{\delta}_1, \dots, \vec{\delta}_n)}_{\delta}}$$

x min fixed en

$\Downarrow$

$\bar{x}$ min fixed en

$$\mathcal{A}\big(x_{new}(t)\big) - \mathcal{A}\big(x(t)\big) = |\delta|^{1/2} \sum_{i<j} S\Big(\vec{r}_i - \vec{r}_j, \frac{1}{|\delta|}\underline{(\vec{\delta}_i - \vec{\delta}_j)}\Big) + O(|\delta|),$$

where $S(\vec{r}, \vec{s}) = \displaystyle\int_0^\infty \left[\frac{1}{|t^{2/3}\vec{r} + \vec{s}|} - \frac{1}{|t^{2/3}\vec{r}|}\right] dt$

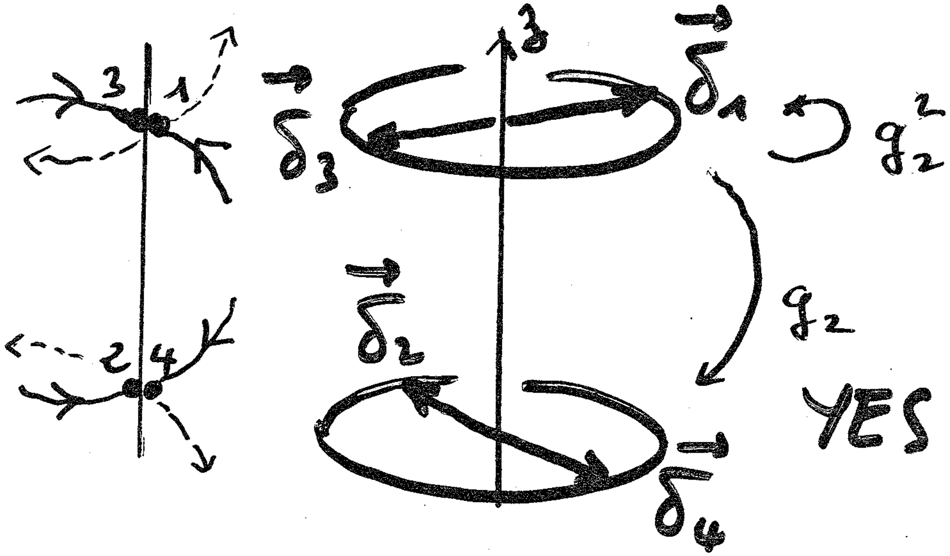
KEY FACT:

$$\int_{\vec{r} \in \text{circle}} S(\vec{r}, \underline{\vec{s}})\, d\vec{r} < 0$$

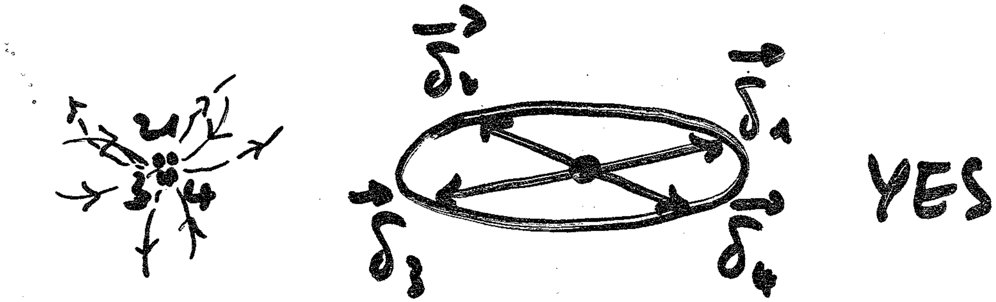
ROTATING CIRCLE PROPERTY :

HIP-HOP

Simultaneous double collision at t

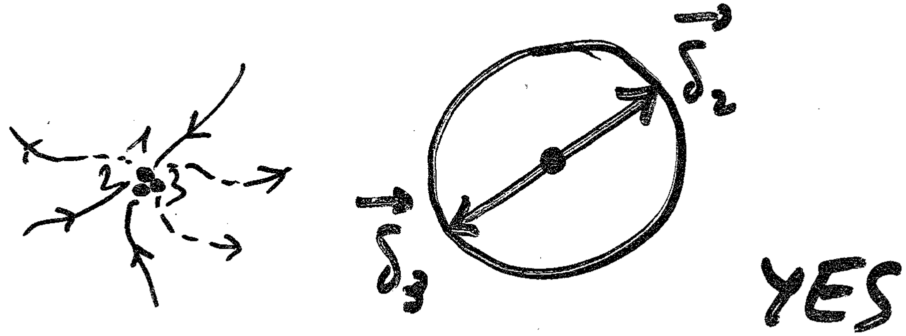


Total collision at t

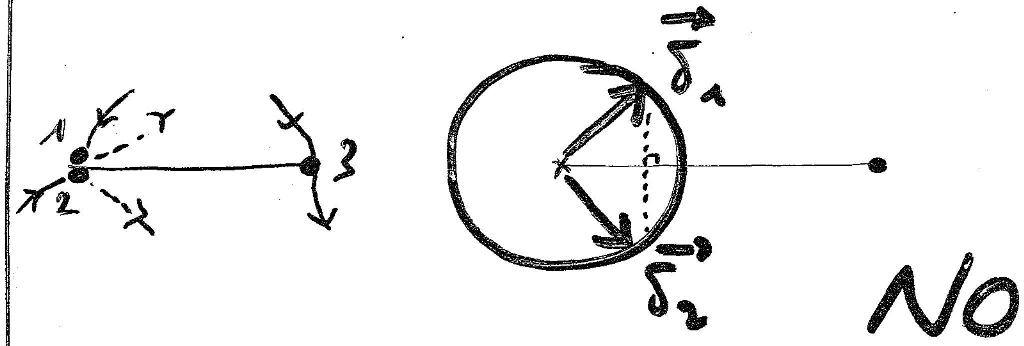


EIGHT

Triple collision at $t=0$



Double collision at $t=T/12$



What can be done when the group action does not possess the Rotating Circle Property ???

1) Find explicit action decreasing deformations of colliding paths

Needs understanding of C.C. $\Rightarrow$ only for 3 (4=) bodies.

Italian school : Bessi & Coti Zelati, Serra & Terracini, del'Antonio, Sbano ...

A.C. & A. Venturelli for the Hip-Hop

2) Estimate action of paths with collision in Λ^G and compare to model with low action $\longleftarrow$ TOOL : compare to 2-body actions

Examples : Eight (A.C. & R. Montgomery)

Hill retrograde (K.C. Chen)

Estimation of the action of collisions in Λ^{D_6}

1. Leibniz formula $\displaystyle\sum_{i=1}^{n} m_i |\dot{\vec{r}}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\dot{\vec{r}}_i - \dot{\vec{r}}_j|^2$

provided $\sum_{i=1}^{n} m_i \dot{\vec{r}}_i = 0$

$\Downarrow$

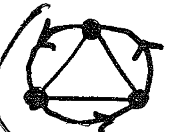
3-body action ($\forall i$, $m_i = 1$)

$$\mathcal{A}(x(t)) = \frac{1}{3} \sum_{i<j} \underbrace{\int_0^T \left[\frac{|\dot{\vec{r}}_i - \dot{\vec{r}}_j|^2(t)}{2} + \frac{3}{|\vec{r}_i - \vec{r}_j|(t)} \right] dt}_{\text{Kepler action}}$$

2. Invariance under g_1^3 : $x(t)$ and $x(t+T/2)$ symmetric$/Oy$

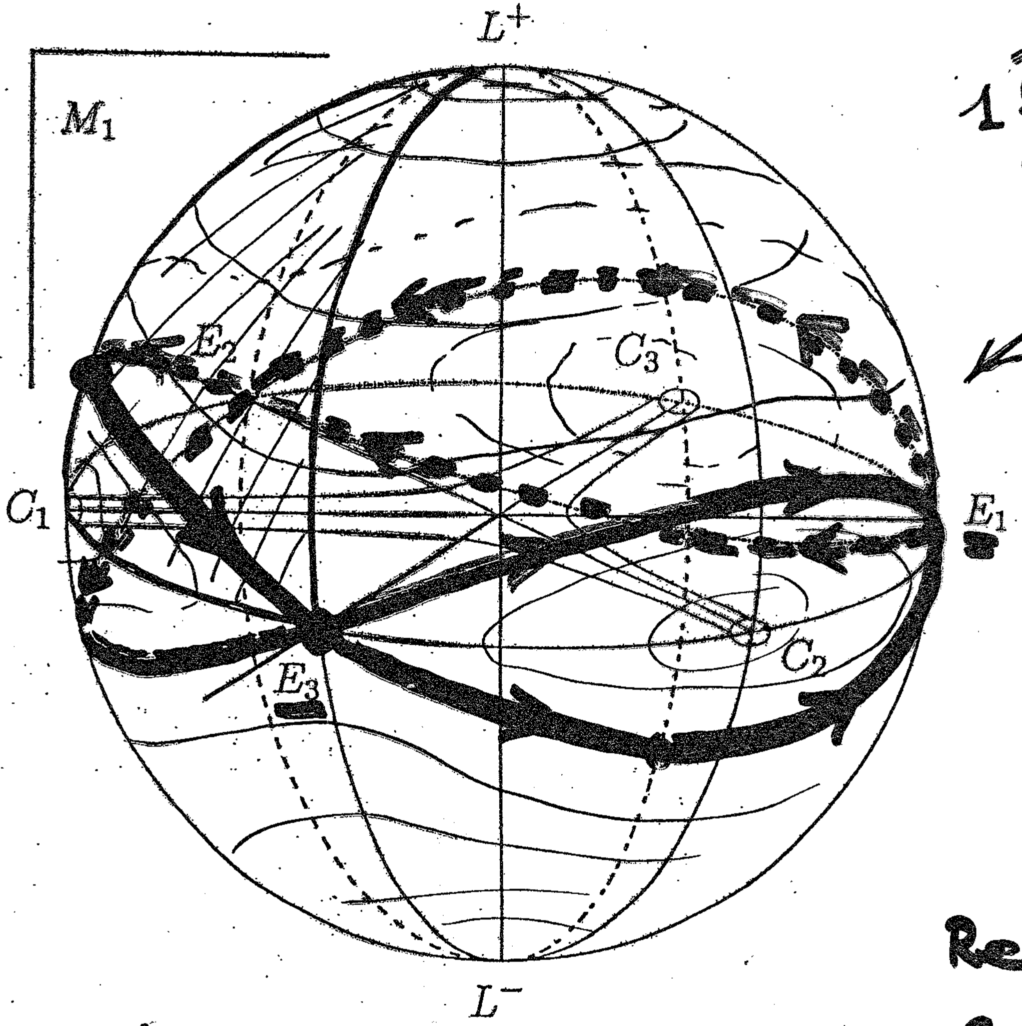
$\Downarrow$

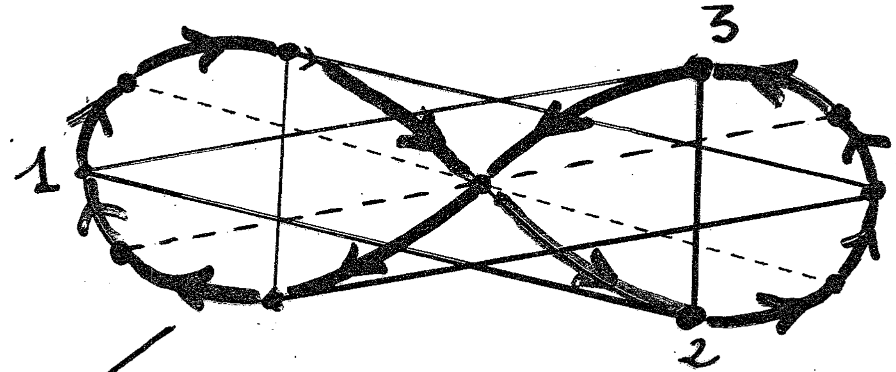
if collision at t_0, also collision at $t_0 + T/2$ (same bodies)

$1+2 \Rightarrow \mathcal{A}\left(x(t) \text{ in } \Lambda^{D_6} \text{ with collision} \right) \geq 2 \times 3\, \mathcal{A}\Big($  ej. coll per. $T/2$, Kepler ($g=3$) $\Big) = 2\mathcal{A}\Big($  per. $T/2$ $\Big)$

Equipotential loop as model in Λ^{D_6}

(purely geometrical)





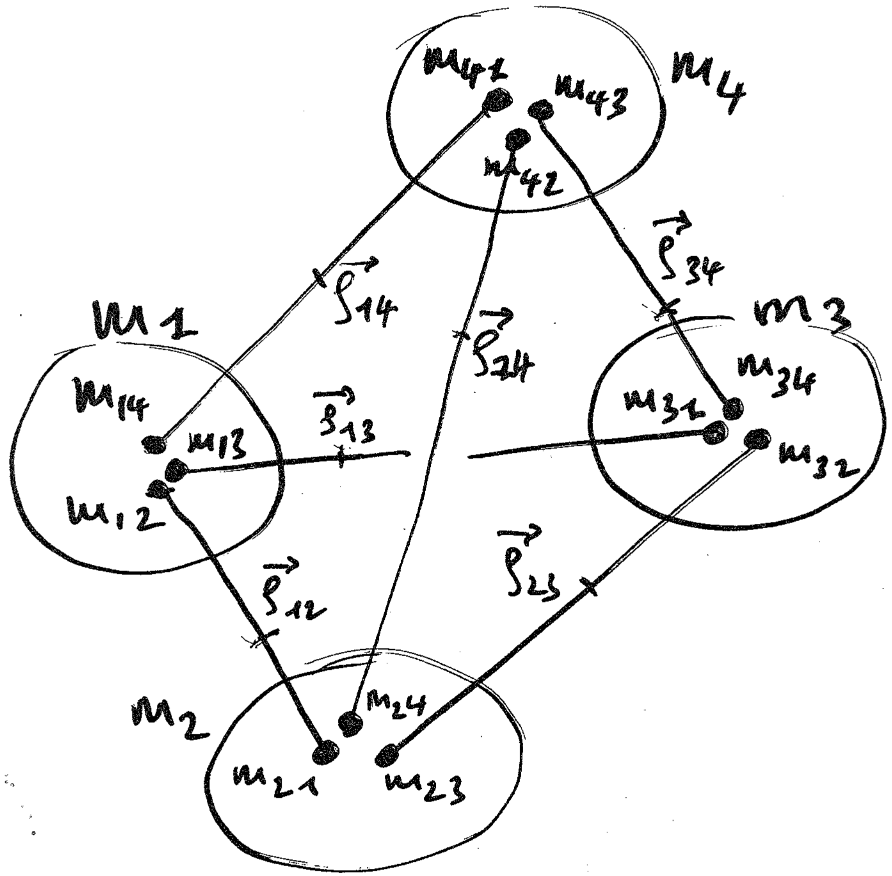
$I = cst$, $U = cst$, $K = cst$

Close to a counterexample to Saari's conjecture but NOT a solution!

Recently proved by R. Moeckel for 3 bodies in $\mathbb{R}^2$.

FACT: $\mathcal{A}_{model} < \mathcal{A}_{coll}$.

More generally, decompose n-body action into sum of $\underline{n(n-1)}$ 2-body actions (Chen)



$$\vec{\rho}_{ij} = \frac{m_{ij}\vec{r}_i + m_{ji}\vec{r}_j}{m_{ij}+m_{ji}}$$

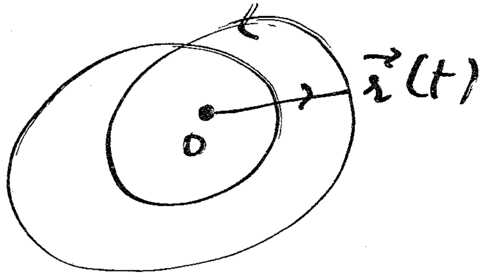
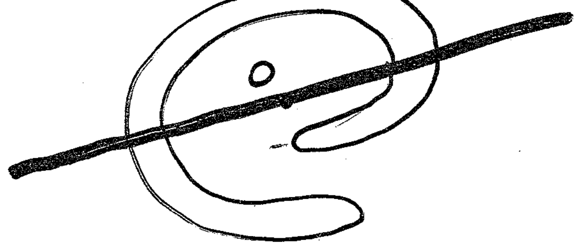
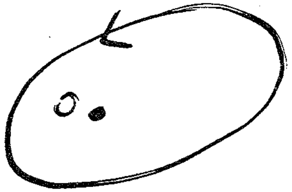
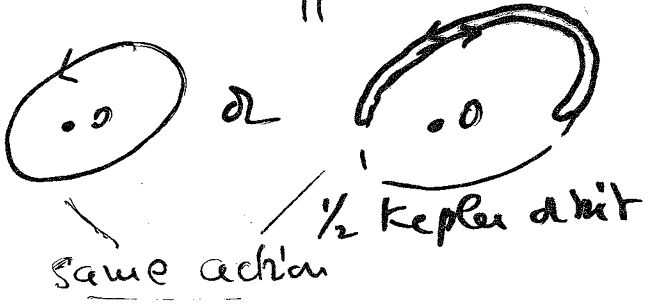
$$\sum_{j\neq i} m_{ij} = m_i$$

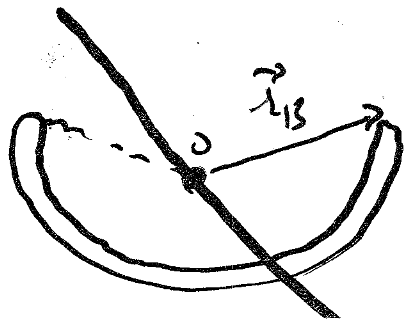
$$\mathcal{A} = \sum_{i<j}\int_0^T \left(\frac{m_{ij}m_{ji}}{2(m_{ij}+m_{ji})}\left|\dot{\vec{r}}_i - \dot{\vec{r}}_j\right|^2 + \frac{G\lambda_{ij}\, m_i m_j}{|\vec{r}_i - \vec{r}_j|}\right) dt$$

$$+ \sum_{i<j}\int_0^T \left(\frac{m_{ij}+m_{ji}}{2}\left|\dot{\vec{\rho}}_{ij}\right|^2 + \frac{G(1-\lambda_{ij})\, m_i m_j}{|\vec{r}_i - \vec{r}_j|}\right) dt$$

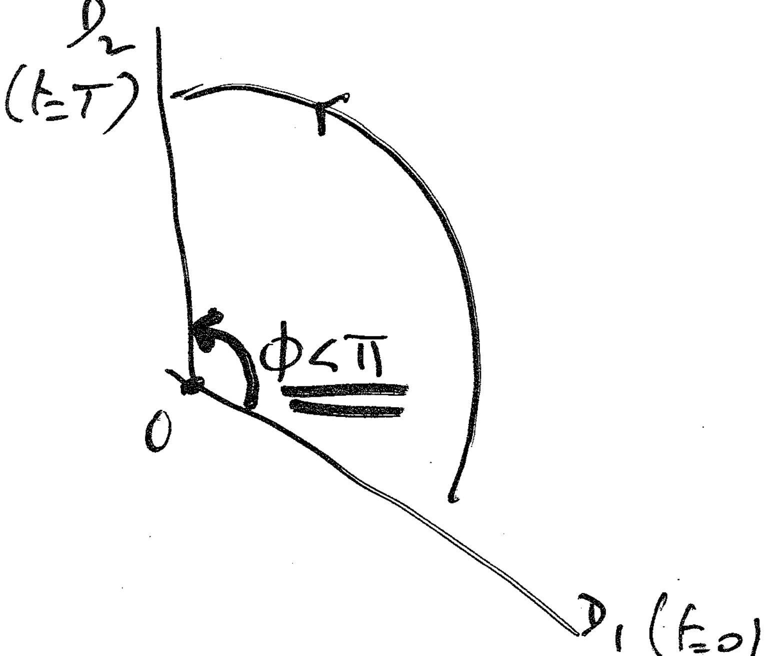
$0 \leq \lambda_{ij} = \lambda_{ji} \leq 1$ parameters

Then use estimates of $\mathcal{A} = \int_0^T \left(\frac{1}{2}|\dot{\vec{r}}|^2 + \frac{g}{|\vec{r}|}\right) dt$

GORDON	K.C. CHEN
	
min. among loops of <u>non zero index</u>	min. among loops <u>cutting</u> <u>every</u> <u>line</u> through attract. center
$\parallel$	$\parallel$
 Keplerian orbit	

<u>Ex</u> : Eight  

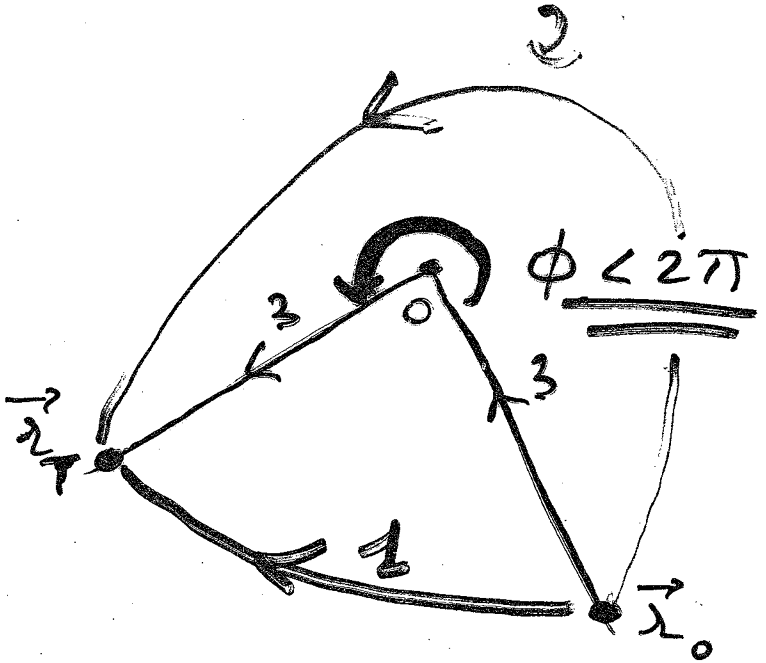
σ2



min. action of paths betw. D_1, D_2 = circle of per. $\frac{2\pi T}{\phi}$

(K.C. CHEN)

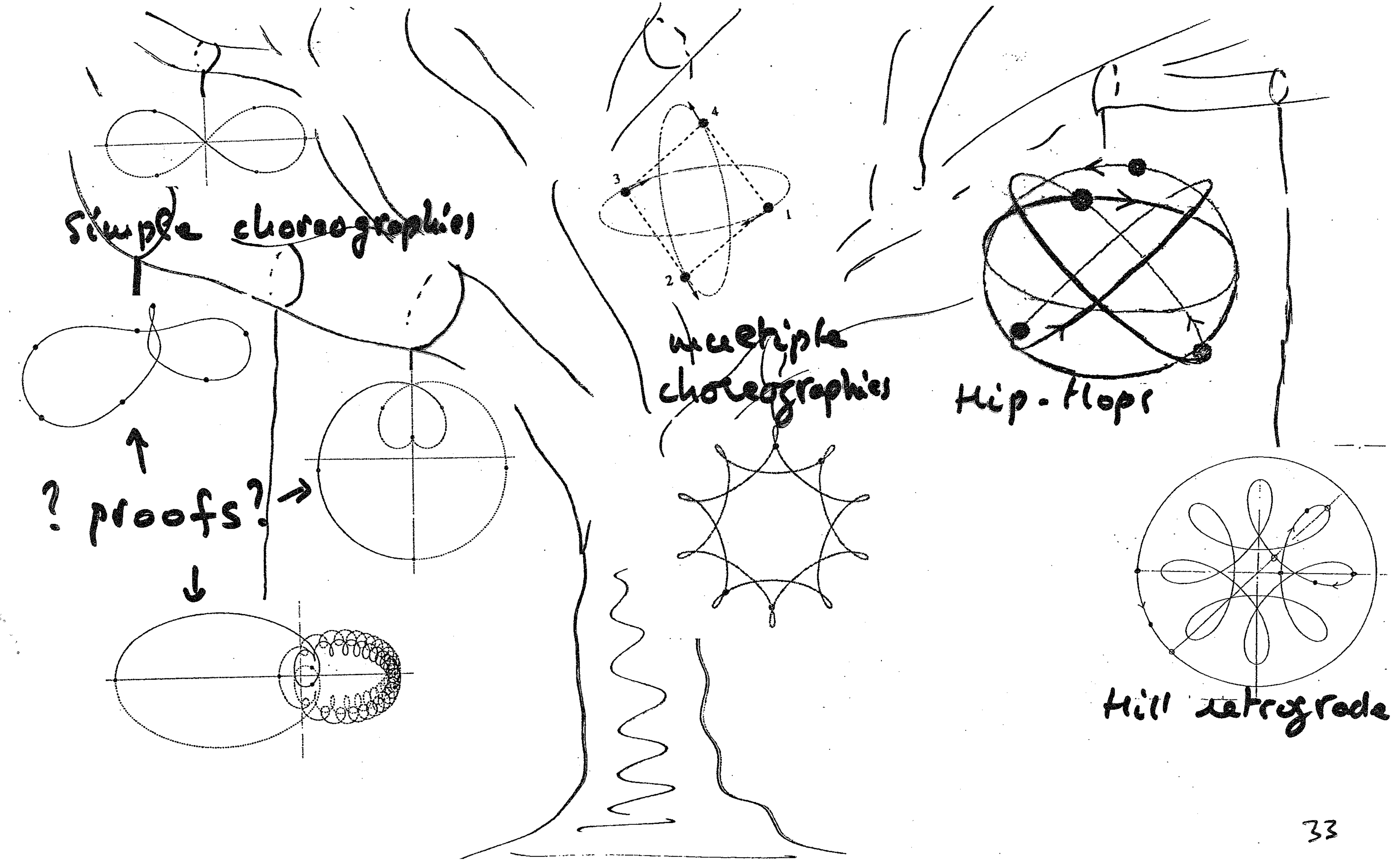
σ2



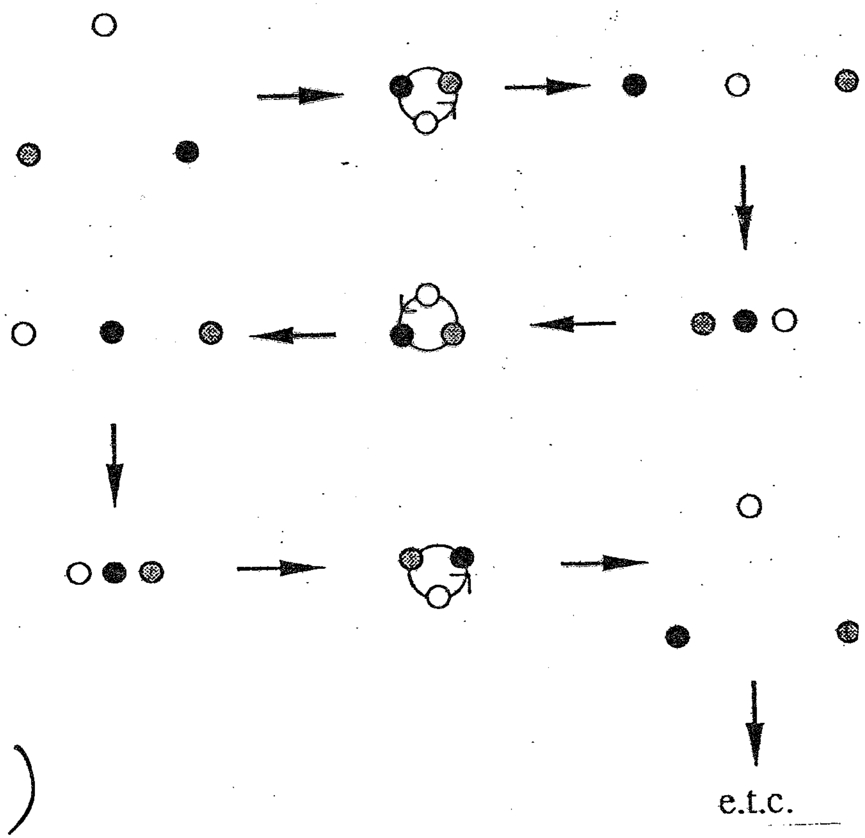
$\mathcal{A}(1) < \mathcal{A}(2) < \mathcal{A}(3)$

(C. MARCHAL)

Finally, ...


simple choreographies
? proofs?
4
3
1
2
multiple choreographies
flip-flops
Hill retrograde

Question: Most of these new "simple" solutions are unstable. Can one use them as a skeleton to build a symbolic dynamics?



(Figure from R. Moeckel
Same qualitative features of the 3-body problem)