ICM 2002

Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry

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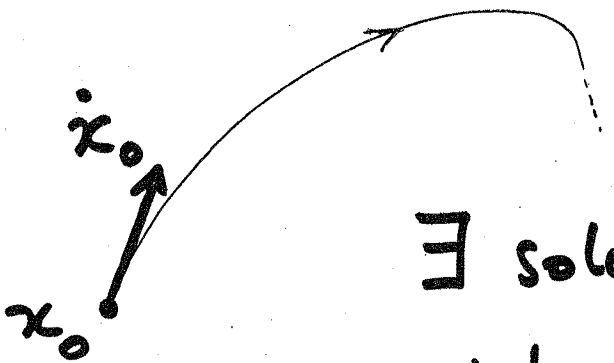
dedicated to the memory of Nicole Desolneux

$$\ddot{x} = \nabla f(x) \quad , \quad x \in \mathbb{R}^d$$

f "regular"

Cauchy problem



$\exists$ solution $x(t)$, $x(0) = x_0$

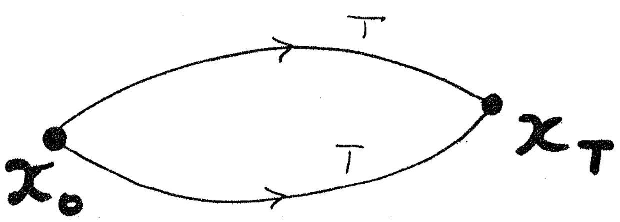
$1\,!$

$$\ddot{x} = \nabla f(x) \quad , \quad x \in \mathbb{R}^a$$

f "regular"

Dirichlet problem

<u>bounded below</u>



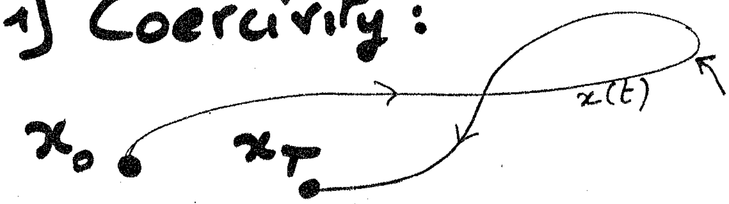
$\exists$ solution $x(t)$, $x(0) = x_0$, $x(T) = x_T$

in general not $1\,!$

Proof: minimize action $\mathcal{A} = \int_0^T \left(\frac{|\dot{x}|^2}{2} + f(x) \right) dt$

among paths $x \in \underline{H^1}\big(([0,T],0,T),(\mathbb{R}^d, x_0, x_T)\big)$

1) Coercivity:



if goes far, $\int_0^T \frac{|\dot{x}|^2}{2} dt$ big

2) (Tonelli 1920): $\mathcal{A}$ is **l.s.c.**

in the weak topology

Newton equations

$$\cancel{m_i}\, \ddot{\vec{r}}_i = - \sum_{j \neq i} \cancel{m_i}\, m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

$$\Updownarrow$$

$$\ddot{x} = \nabla U(x),$$

where ∇ = grad. for the "mass metric"

$$\|x\|^2 = I$$

N bodies in $E \cong \mathbb{R}^k$

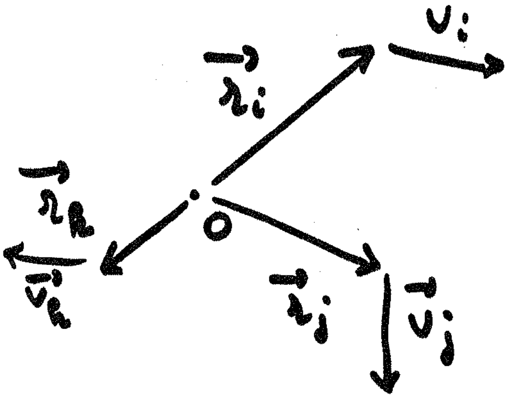
Configuration space $\hat{X} = X \setminus \text{coll.}$

$$X = \left\{ x = (\vec{r}_1, \dots, \vec{r}_n) \in E^N,\ \sum m_i \vec{r}_i = \vec{0} \right\}$$

$$\text{coll.} = \left\{ x \in X,\ \exists\, i \neq j,\ \vec{r}_i = \vec{r}_j \right\}$$

Phase space $T\hat{X} \equiv \hat{X} \times X$

$$(x,\ y)$$



$$x = (\vec{r}_1, \dots, \vec{r}_n)$$

$$y = (\vec{v}_1, \dots, \vec{v}_n)$$

Moment of inertia /0

$$I(x,y) = \sum_i m_i |\vec{r}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\vec{r}_i - \vec{r}_j|^2$$

2 × Kinetic energy

$$K(x,y) = \sum_i m_i |\vec{v}_i|^2 = \frac{1}{\sum m_i} \sum_{i<j} m_i m_j |\vec{v}_i - \vec{v}_j|^2$$

$$I = \|x\|^2, \quad K = \|y\|^2$$

Newtonian potential

$$U(x,y) = \sum_{i<j} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

Lagrangian

$$L = \frac{K}{2} + U > 0$$

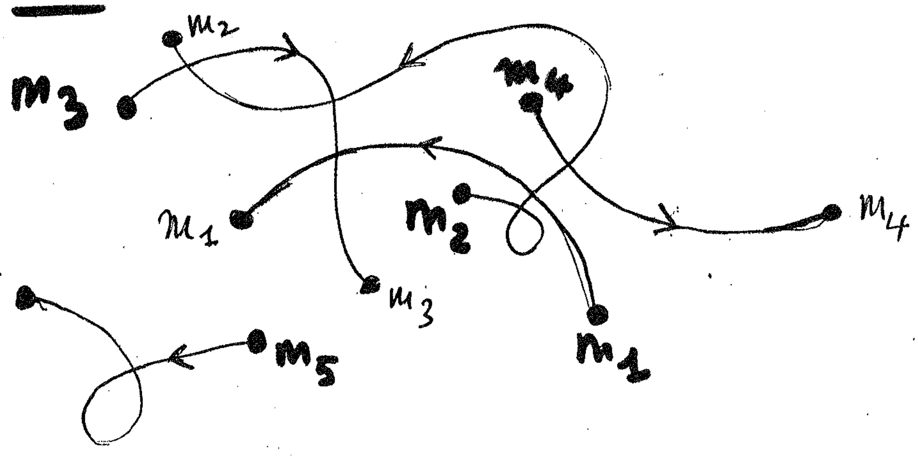
$$\text{Action } \mathcal{A} : H^1([0,T],\mathcal{X}) \longrightarrow \mathbb{R}\cup\{+\infty\}$$
$$x \longmapsto \int_0^T L(x,\dot{x})\,dt$$

$E = \mathbb{R}^2$ or $\mathbb{R}^3$, $m_1, \dots, m_n > 0$ fixed.

Theorem (C. Marchal + R. Montgomery, S. Terracini, A. Venturelli, A.C.)

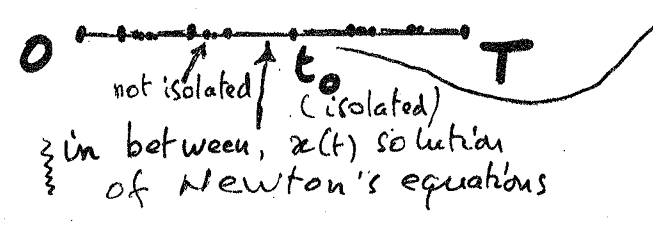
$\forall T > 0$, $\forall x_0, x_T \in \mathcal{X}$ (eventually with collisions), a minimizer of $\mathcal{A}$ in $H^1(([0,T],0,T),(\mathcal{X},x_0,x_T))$ exists and is collision-free in $]0,T[$

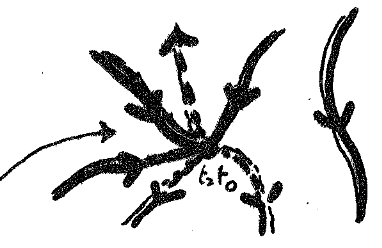
Corollary (Dirichlet problem): $\forall x_0, x_T$ configurations of n given masses in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\forall T > 0$, $\exists$ at least one solution (collision-free) of Newton's equations s.t.
$x(0) = \underline{\underline{x_0}}$, $x(T) = \underline{\underline{x_T}}$.



Proof. $\exists$ of a minimizer $x(t)$ follows, as in the regular case, from l.s.c. of $\mathcal{A}$ in weak topology.

A PRIORI $\exists$ closed set of measure 0 of collision times





Why are collisions a priori possible in a minimizer?

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② If a solution of Newton's equations is such that $\vec{x}_i(t_0) = \vec{x}_j(t_0)$,

$$\begin{cases} |\vec{x}_i(t) - \vec{x}_j(t)| \sim \text{cste}\, |t-t_0|^{2/3} \\ |\dot{\vec{x}}_i(t) - \dot{\vec{x}}_j(t)| \sim \text{cste}\, |t-t_0|^{-1/3} \end{cases}$$

(explicit for 2 bodies, Sundman 1913 in general)

$\Rightarrow \mathcal{A}(x)$ stays finite on such a solution !!!

Poincaré was already aware of this problem when he proposed the use of the minimization method to find (relative) periodic solutions of the planar 3-body problem

His solution: "cheat" by replacing the $\frac{1}{r}$ potential by $\frac{1}{r^2}$ (strong force) for which $\mathcal{A}$ is infinite if $\exists$ collisions.

SUR LES SOLUTIONS PÉRIODIQUES ET LE PRINCIPE DE MOINDRE ACTION

Comptes rendus de l'Académie des Sciences, t. 123, p. 915-918 (30 novembre 1896).

La théorie des solutions périodiques peut, dans certains cas, se rattacher au principe de moindre action.

Supposons trois corps se mouvant dans un plan et s'attirant en raison inverse du cube des distances ou d'une puissance plus élevée de ces distances; j'appelle a, b, c ces trois corps.

Proving that a minimizer is collision-free

(II_1) = empty condition

$\vdots$

$\Downarrow$

$(\mathrm{II}_{p-1}) = \{ \nexists$ m-body collisions for $m \leq p-1 \}$

$\Downarrow$ Montgomery / Venturelli

$(\mathrm{I}_p) = (\mathrm{II}_{p-1}) \cup \{$ p-body collisions are isolated $\}$

$\Downarrow$ Marchal (+ AC) + "BLOW UP" (Terracini / Venturelli) $\Rightarrow$ OPS $x(t)$ is a parabolic homothetic collision-ejection solution.

$(\mathrm{II}_p) = \{ \nexists$ m-body collisions for $m \leq p \}$

$\vdots$

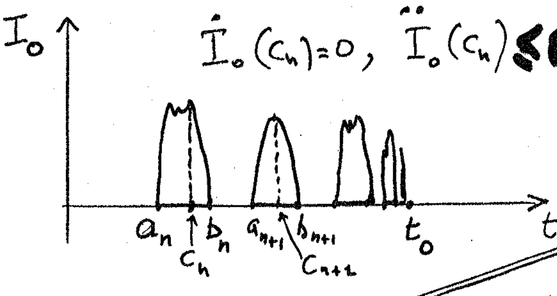
$\Downarrow$

5 (II_n) Q.E.D.

$(\mathrm{II}_{p-1}) \xRightarrow{\text{Montgomery Venturelli}} (\mathrm{I}_p)$: if $\nexists$ m-body collisions for $m \leq p-1$, p-body collisions are isolated.

$I_0 = \left\{ \begin{array}{l} \text{moment of inertia / its c. of mass} \\ \text{of the colliding cluster} \end{array} \right\}$ is C^2 because $\left\{ \begin{array}{l} \text{no subcluster collide} \\ \text{no collision with bodies of other clusters} \end{array} \right.$

$\dot{I}_0(c_n) = 0, \ \ddot{I}_0(c_n) \leq 0$



BUT Lagrange-Jacobi:

$$\ddot{I}_0 = 4H_0 + 2U_0 + \underbrace{\text{Regular}}_{\text{bounded}} \rightarrow +\infty$$

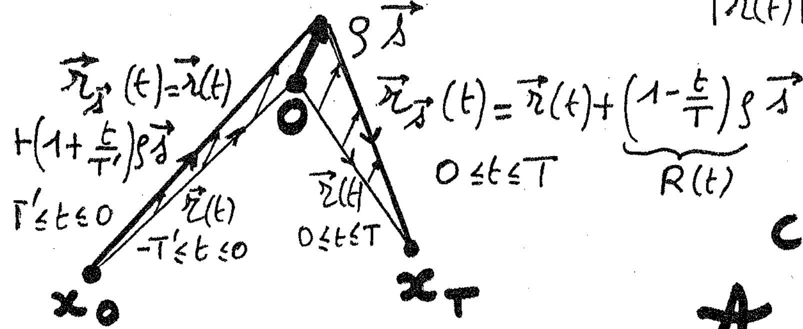
H_0: bounded; $U_0 \to +\infty$ (when $t \to t_0$)

Key point: $H_0 = \left\{ \begin{array}{l} \text{energy of the} \\ \text{colliding cluster} \end{array} \right\}$ is **a.c.** near t_0 (constant if $p = n$)

Proof: reparametrizations of $x(t)$.

$(\mathrm{II}_p) \Rightarrow (\mathrm{III}_p)$: the parabolic Kepler case in $\mathbb{R}^3$

$|\vec{r}(t)| = \gamma |t|^{2/3}$



Marchal's idea:

compute the mean action

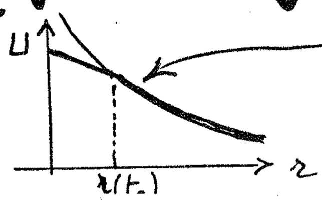
$$\mathcal{A}_m \equiv \int_{\vec{s}\in S^2} d\sigma\left(\mathcal{A}\left(\vec{r}_s(t)\right)\right)$$

$\Updownarrow$

replace moving body by moving sphere

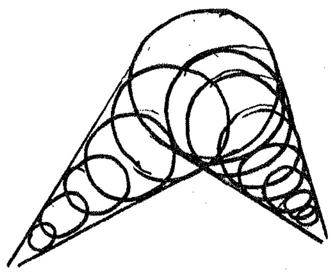
$\Updownarrow$

replace $\frac{1}{r}$ potential by truncated



$$\mathcal{A}_m - \mathcal{A} = 2\left(-\frac{2}{\gamma}|t_0|^{1/3} + O(|t_0|^{5/3})\right)$$

17 (Works also in $\mathbb{R}^2$: )



17 bis

$(I_p) \Rightarrow (II_p)$: general case via BLOW UP

Theorem (Terracini / Venturelli) If $\exists$ minimizer $x(t)$ of the fixed-ends problem for n bodies with an isolated collision of $p \leq n$ bodies, $\exists$ minimizer $\bar{x}(t)$ of the fixed-ends problem for p bodies, which is a parabolic homothetic collision-ejection solution

Idea of proof: $x_\lambda(t) = \lambda^{-2/3} x(\lambda t)$ (if collision at $t=0$)

Sundman's estimates $\Rightarrow \{x_\lambda\}_{\lambda \text{ close to } 0}$ bounded in H^1

$\Downarrow$

$\exists$ subsequence $x_{\lambda_n} \xrightarrow[\text{weakly}]{} \bar{x}$ with the said properties

(Note: if $p < n$, the $(n-p)$ non-colliding bodies are "sent to ∞").

18 Then $(I_p) \Rightarrow (II_p)$ as in Kepler case.

Applications to periodic solutions via minimization under symmetry constraints

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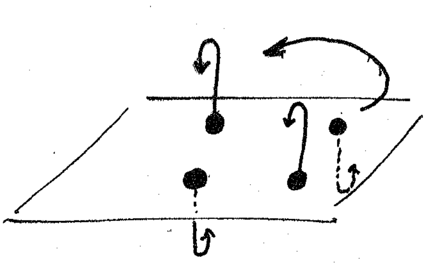
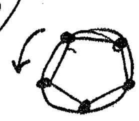
Idea: restrict loop to fundamental domain $[t_0, t_1]$

- **Generalized HIP-HOPS**

  $\forall m_1, \dots, m_n$, $\exists$ a "simple" **non-planar** periodic solution

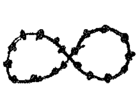


- **Choreographies** (equal masses)



  - $\mathbb{Z}/n$ - symmetry only ($\mathbb{R}^2$ or $\mathbb{R}^3$)
  - $(n=3, \mathbb{R}^2)$ D_6, D_3 or $\mathbb{Z}/6$ - symmetry

  (and continuation P_{12})

  - . . .  ? etc . . .

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Generalized HiP-HOPS

Italian symmetry: $x\left(t+\frac{T}{2}\right) = -x(t)$

$\forall t_0$, $x|_{[t_0, t_0+\frac{T}{2}]}$ minimises with fixed ends.

$\Longrightarrow$ a minimizer has no collision.

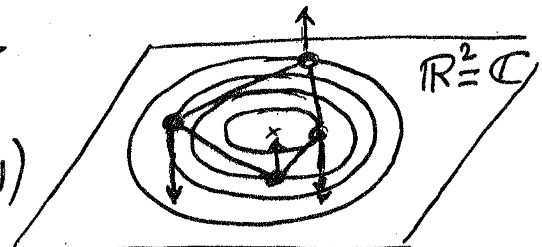
Theorem: $\forall m_1, \dots, m_n > 0$, a minimizer in $\mathbb{R}^3$ under italian symmetry is NEVER PLANAR

Idea of proof

① **Planar case**: Among minimizers, are the RELATIVE EQUILIBRIA

$$x(t) = x_0 e^{2\pi i t/T}$$

Such that x_0 = minimizer of $\tilde{U} = \sqrt{I}U$)



(A.C. & N. DESOLNEUX corrected thanks to V. COTI ZELATI).

② If $z_0 \perp$ plane of x_0

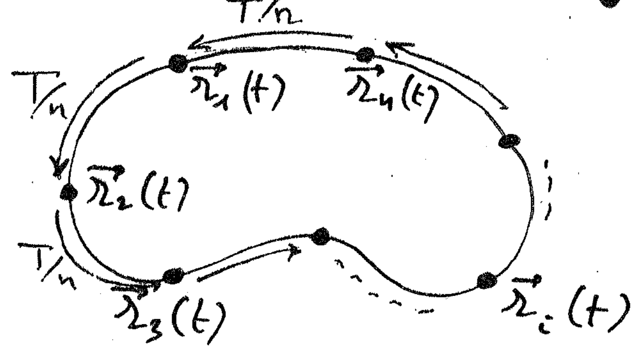
and $z(t) = z_0 \cos 2\pi t/T$

$$\boxed{\begin{aligned} &d^2\mathcal{A}(x(t))(z(t), z(t)). \\ &\overset{\shortparallel}{(\text{const.}>0)}\cdot \underbrace{d^2\tilde{U}(x_0)(z_0, z_0)} \end{aligned}}$$

③ (F. Pacella. R. Moeckel): $\exists z_0$ s.t. < 0

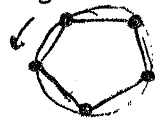
CHOREOGRAPHIES (n equal masses)

$\mathbb{Z}/n$ - symmetry



$\forall t_0,\ x|_{[t_0,\, t_0+T/n]}$
minimizes with fixed ends
$\Downarrow$
a minimizer has no collision.

? min = regular n-gon



YES if $n \leq 5$ in $\mathbb{R}^2$, $n \leq 3$ in $\mathbb{R}^3$

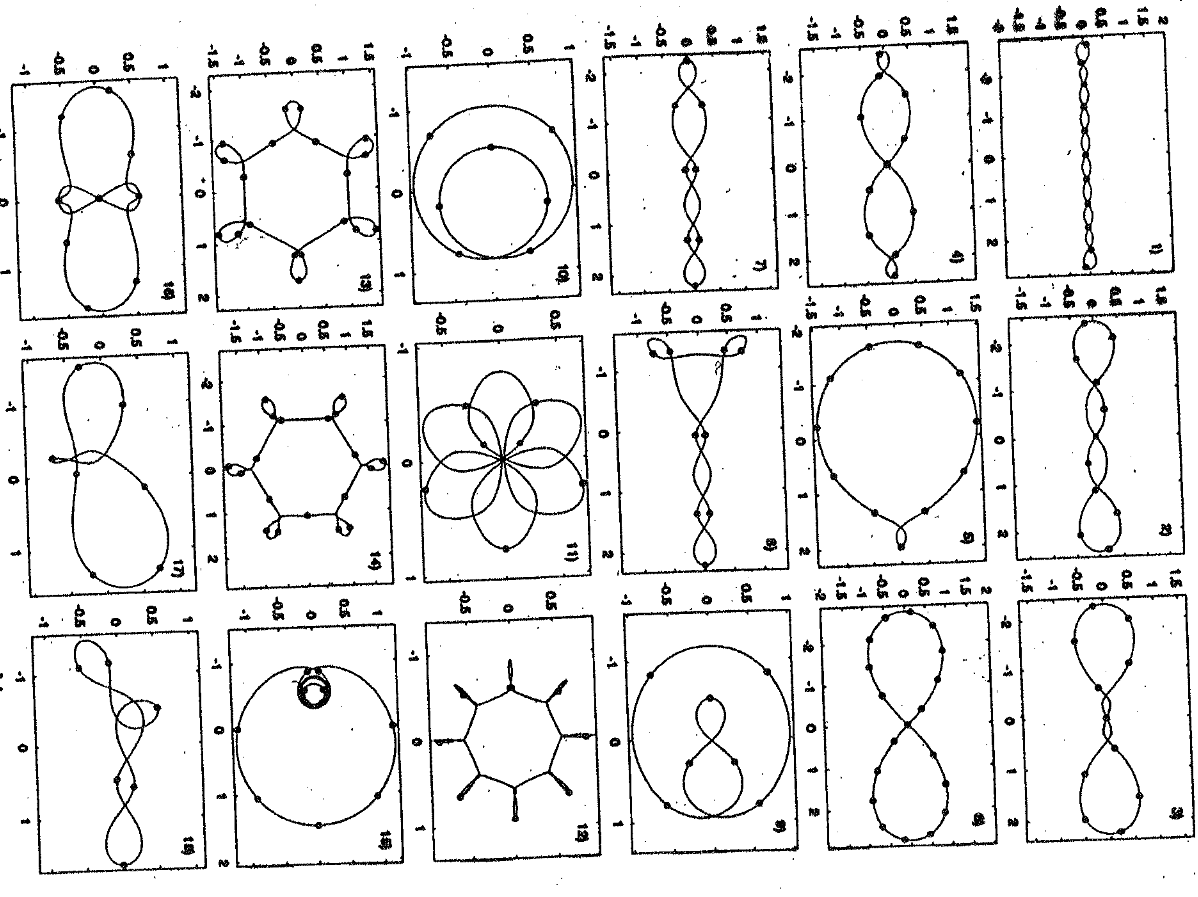
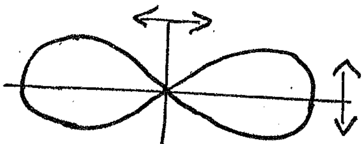


FIGURE 3. A sample of different choreographies.

EIGHT(S) (3 equal masses)

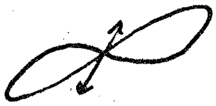
SYMMETRY GROUP OF THE "SPACE OF TRIANGLES" $D_6 = \{s, \sigma ;\ s^6 = 1, \sigma^2 = 1, s\sigma = \sigma s^{-1}\}$

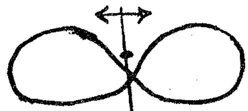


$x|[0, T/12]$ minimizes

PB AT THE ENDS

$D_3 = \{s^2, \sigma\}$ $\mathbb{Z}/6 = \{s\}$





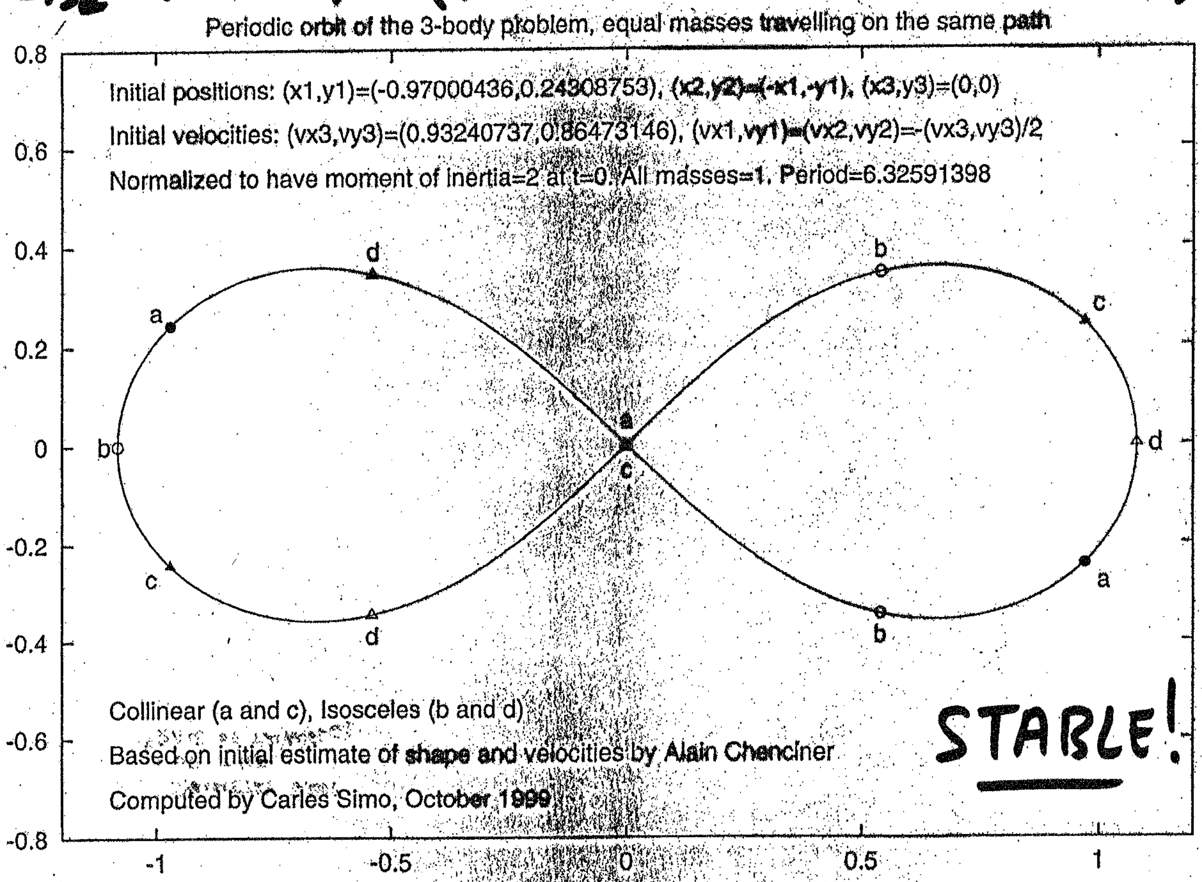
$x|[0, T/6]$ minimizes

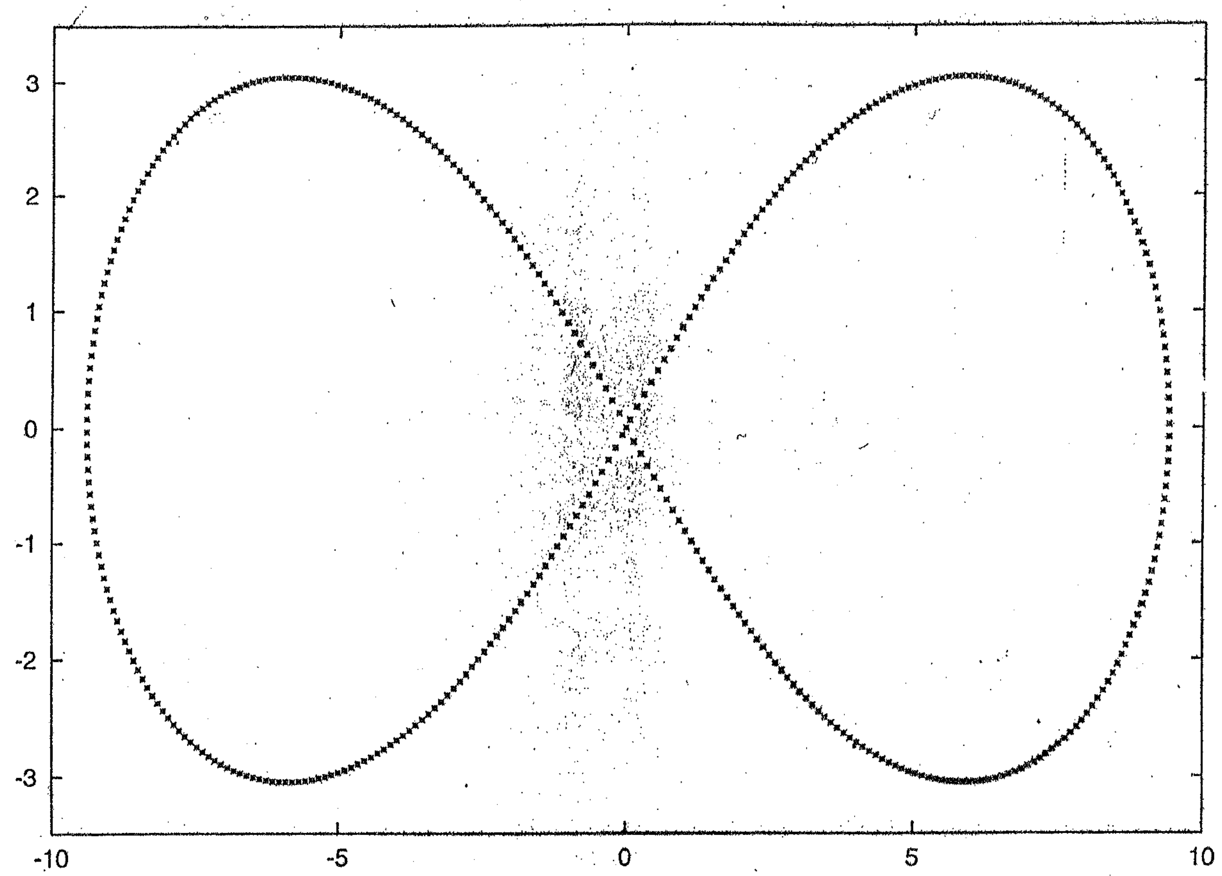
PB AT THE ENDS

$\forall t_0,\ x|[t_0, t_0 + T/6]$ minimizes

? all the same.

? unicity





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? min D_{798} COMPUTED BY C. SIMÓ