THE FLOW of the EQUAL-MASS SPATIAL 3-BODY PROBLEM in the NEIGHBORHOOD of the LAGRANGE EQUILATERAL RELATIVE EQUILIBRIUM

A. Chenciner (P VII & ASD)
(with J. Féjoz)

TO CARLES,

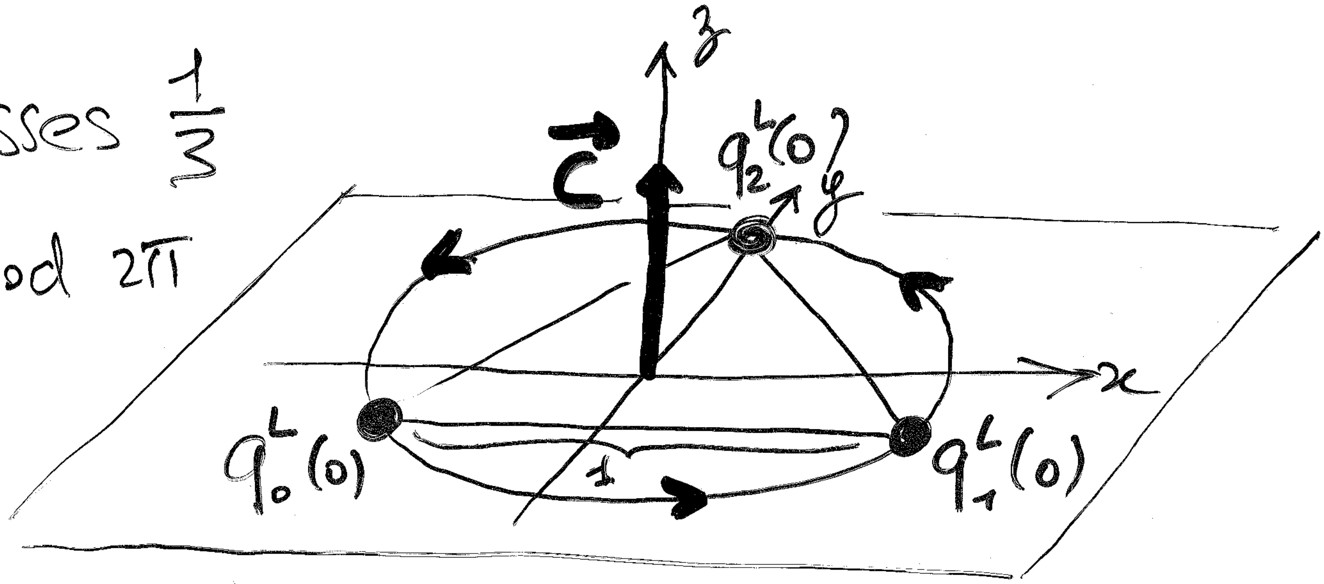
for his

32 000 000th minute

LAGRANGE RELATIVE EQUILIBRIUM

masses $\frac{1}{3}$

period 2π



$j = 0, 1, 2$

$$\ddot{q}_j = \sum_{k \neq j} \frac{1}{3} \frac{q_k - q_j}{\|q_k - q_j\|^3}$$

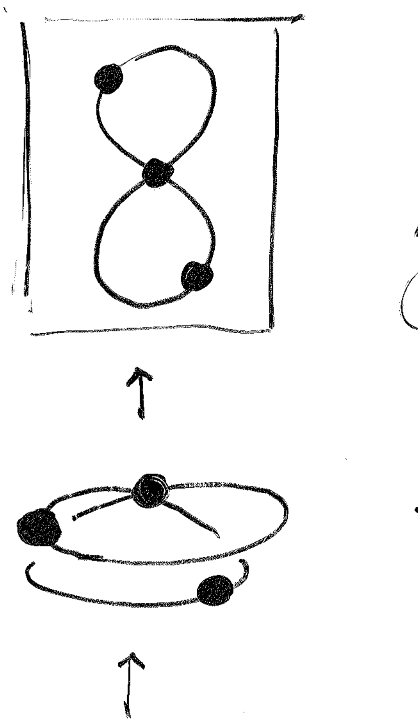
SCALING: $\lambda^{-2/3} q(\lambda t)$

Period $\longleftrightarrow$ Angular momentum

THEOREM:

Mod. | similarities
time shifts

$\exists$ 2 families of relative periodic solutions bifurcating from the Lagrange rel. equil.:

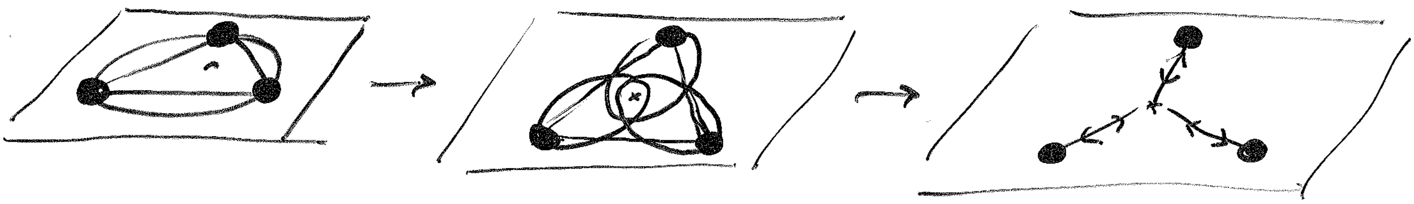


MARCHAL'S P12 FAMILY

(Choreography in rotating frame)

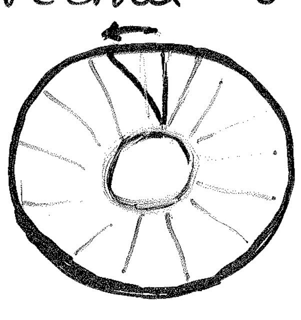
D6 - symmetric

(natural to fix the period!)



HOMOGRAPHIC FAMILY

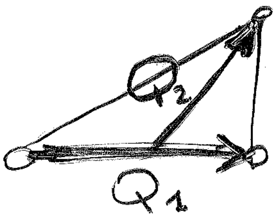
Moreover, restriction to center manifold reduces to



REDUCTION :

NOT FOR READING !!

1] translations : dim 18 $\longrightarrow$ dim 12



• Q_1, Q_2, P_1, P_2
Jacobi coord. (symplectic)

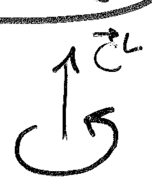
• $R_1, \Theta_1, Z_1, R_2, \Theta_2, Z_2, R_1', \Theta_1', Z_1', R_2', \Theta_2', Z_2'$

$Q_1 = \left[(1+R_1)e^{i\Theta_1}, Z_1\right]$, $Q_2 = \dots$

$P_1 = \left[\left(R_1' + i\frac{1/6+\Theta_1'}{1+R_1}\right)e^{i\Theta_1}, Z_1'\right]$, $P_2 = \dots$

polar coord. centered at $L(0)$

2] rotations : dim 12 $\longrightarrow$ dim 8

Fix $\vec{C} = \vec{C}^L$ and quotient by 
$= (0,0,1/3)$
(reduction of the node)

$\Rightarrow$ non symplectic coordinates

$\Theta_1, \Theta_1', R_1, R_1', R_2, R_2', Z_1, Z_1'$

 = equilibrium of reduced v.f.

REDUCTION:

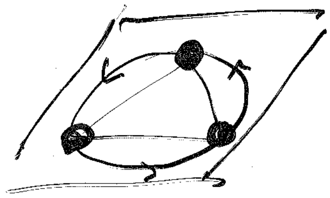
dim 18

$\downarrow$ translations

dim 12

$\downarrow$ rotations

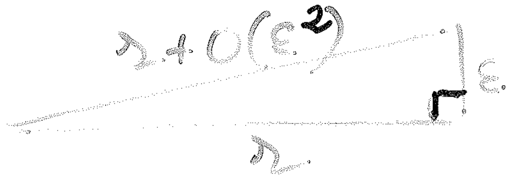
dim 8



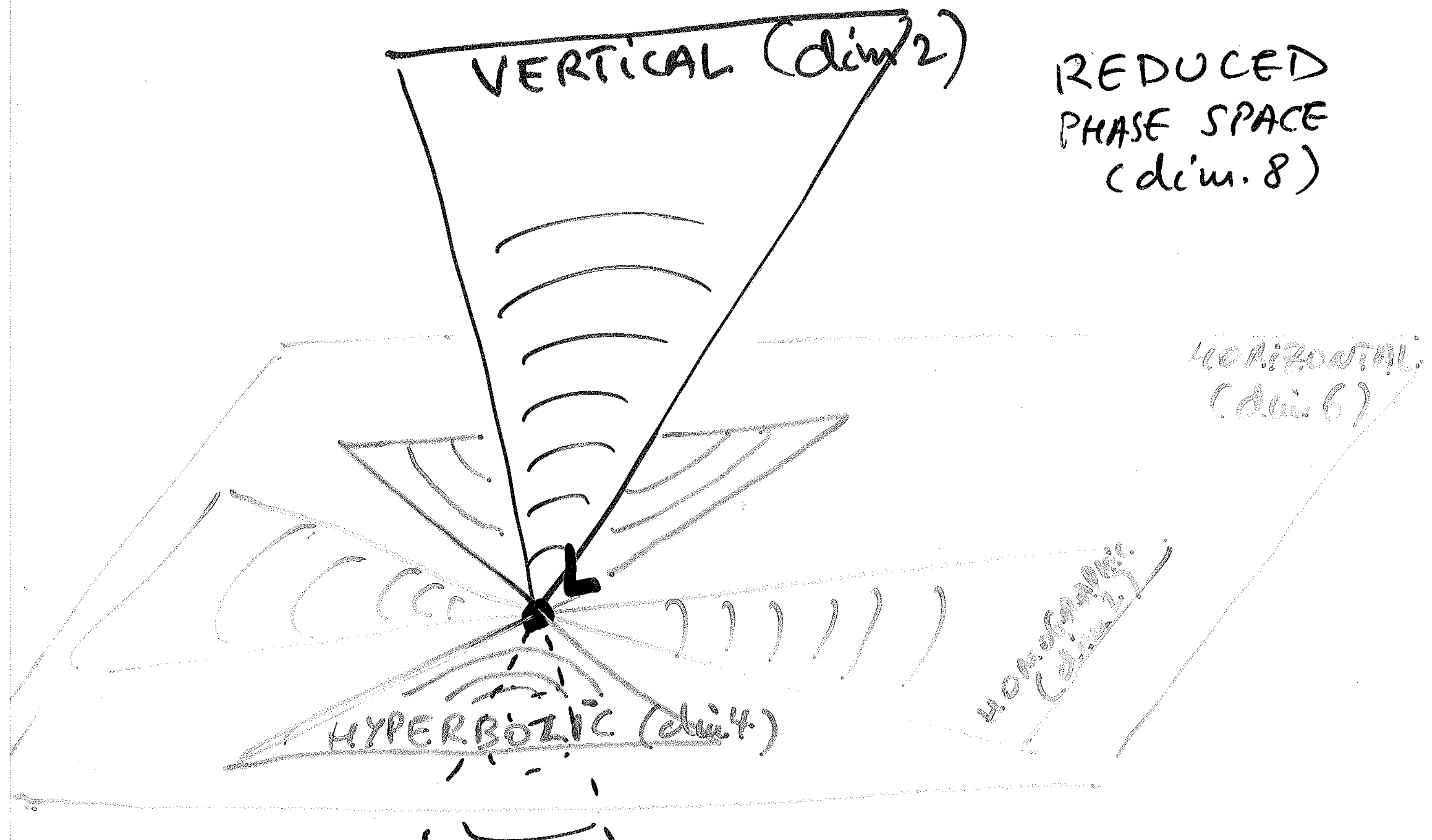
= EQUILIBRIUM of the REDUCED VECTOR-FIELD

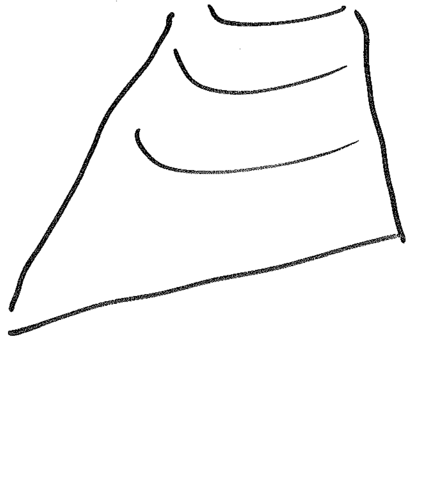
LINEAR ANALYSIS

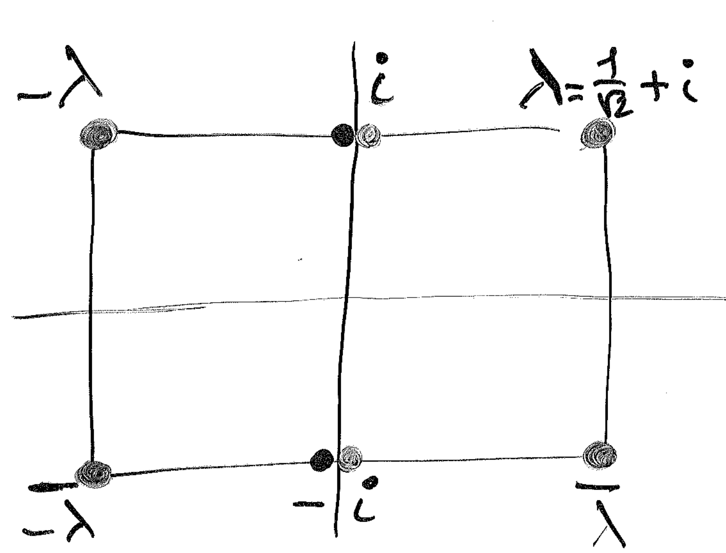
Pythagoras's thm ⇒ splitting of Variational equation



$$VE = HVE \oplus VVE$$







CENTER MANIFOLDS

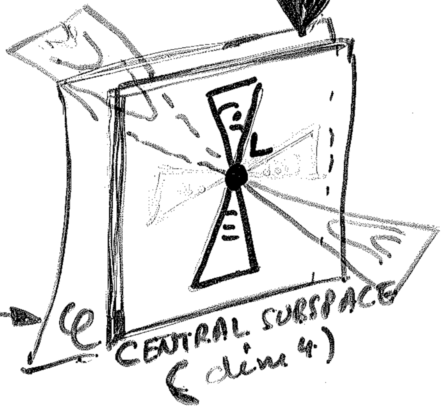
- **Lemma 1** (well-known ... to some people)

$\rho, \sigma \in \mathbb{R}^+$ $\boxed{\{q = \rho R q^\circ,\ p = \sigma S p^\circ\}}$ dim 8

$R, S \in SO(3)$

(q°, p°)
$\parallel$
$(q^L(t_0), p^L(t_0))$
(any t_0)

$\Downarrow$ reduction



Z NOT a CENTER MANIFOLD

Proof: $\begin{cases} H: & \text{Kepler in } \mathbb{C} \quad x(t) = \rho(t)\, x(0) \\ V: & \text{Pythagoras} \Rightarrow \text{rigid} \end{cases}$

- **Lemma 2** : (probably well known ... to the ~~above~~ same people) in the reduced space,

$\boxed{H|_{\text{center mnfld}}}$ has non deg. MINIMUM at L

(ρ, σ, R, S)

$\downarrow$ reduction

$R^{-1}S$, ρ, σ

with relation

$$\rho\sigma \left\| \sum_{i=0}^{2} q_i^\circ \wedge R^{-1}S\, p_i^\circ \right\| = \|\vec{C}^L\|$$

Proof: Computation:

a, b, c, d coord. in $\mathcal{C}$

$$R^{-1}S = \exp\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \rightsquigarrow \boxed{H|_{\mathcal{C}} = -\frac{1}{6} + \frac{1}{24}(a^2+b^2) + \frac{1}{6}(c^2+d^2) + \ldots}$$

$d = \sigma - 1$

$\Downarrow$

$H|$ ... centrale idem

$\Rightarrow$ (1,1) Resonant flow in $\mathbb{R}^4$ with S^3 energy surfaces

i.e. perturbations of Hopf flow

$\Downarrow$

$\rightsquigarrow$ Need higher order normal form to study Lyapunov families

(idem Hill's orbits in PR3BP, see Conley's thesis)

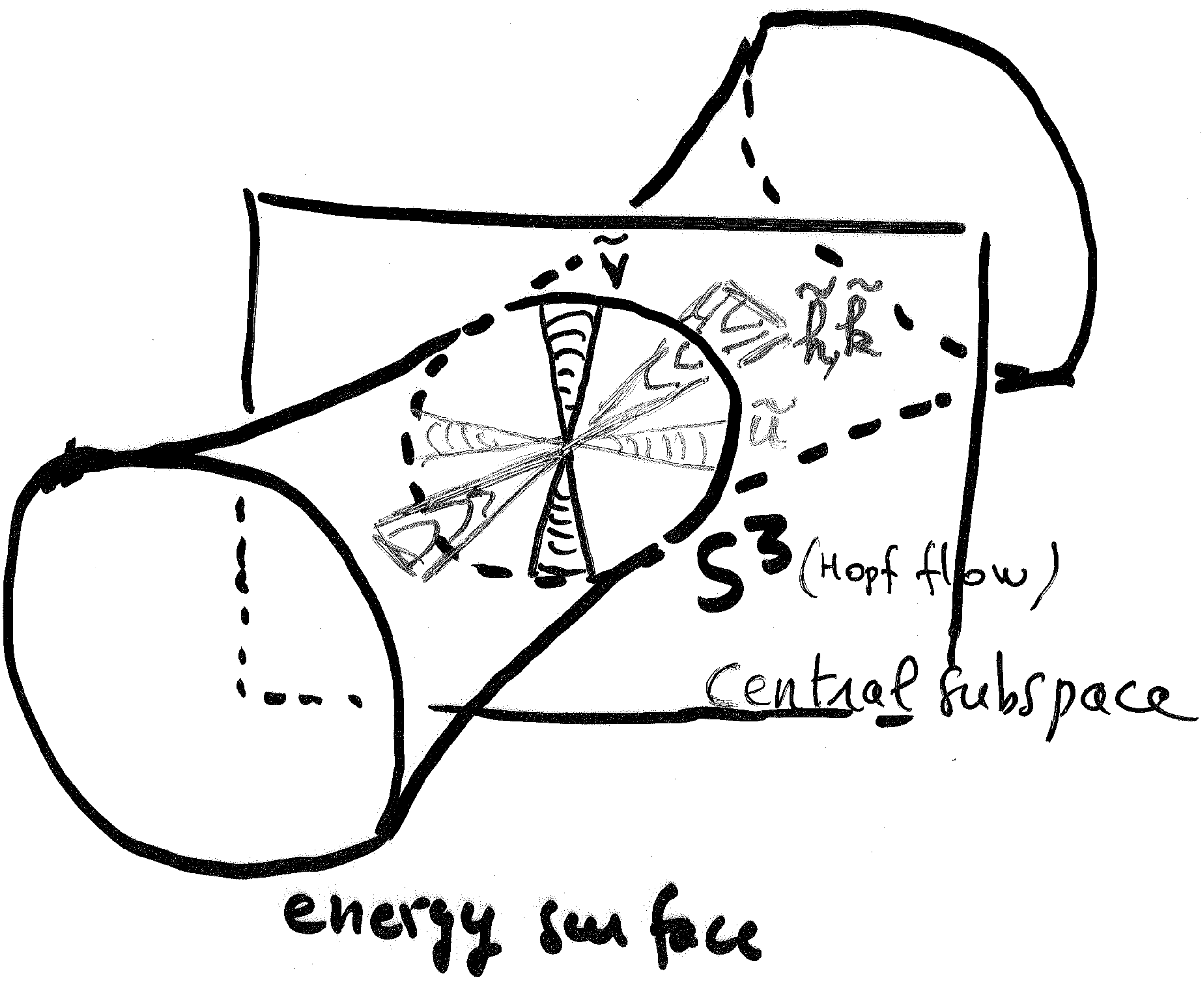
...

but with a grain of salt

- resonance with <u>hyperbolic part</u>
- degeneracy at <u>all orders</u>

SUMMARY: the LINEAR PICTURE

(reduced flow)



THIRD ORDER NORMAL FORM

Using ... Maple ... Sorry, Carles !
and TRIP (better ?)

$$\begin{cases} \dot{u} = iu\left[1+\boxed{\alpha}|u|^2+\boxed{\beta}|v|^2+\gamma hk+\bar{\gamma}\bar{h}\bar{k}\right]+O_5 \\ \qquad\quad\ \ \| \qquad\quad \neq \\ \dot{v} = iv\left[1+\boxed{a}|u|^2+\boxed{b}|v|^2+chk+\bar{c}\bar{h}\bar{k}\right]+A\bar{v}\bar{h}k+O_5 \\ \dot{h} = \lambda h\left[1+r|u|^2+s|v|^2+thk+t'\bar{h}\bar{k}\right]+Rv^2\bar{h}+O_5 \\ \dot{k} = -\lambda k\left[1+r|u|^2+s|v|^2+tk+t'\bar{h}\bar{k}\right]-R\bar{v}^2\bar{k}+O_5 \end{cases}$$

$\Leftarrow$ invariant under $(u,v,h,k)\overset{\mathcal{S}}{\mapsto}(u,-v,h,k)$

Restriction to a center manifold : $h = O_2$
(∃ one $\underline{\mathcal{S}}$- invariant) $\quad k = O_2$

$$\Downarrow$$

$$\begin{cases} \dot{u} = iu\left[1+\alpha|u|^2+\beta|v|^2\right]+O_5 \\ \dot{v} = iv\left[1+a|u|^2+b|v|^2\right]+O_5 \end{cases}$$

$\mathcal{S}$- invariant

i.p. $H = -\frac{1}{6}+\frac{|u|^2}{9}+\frac{|v|^2}{9}+O_4$

$\exists$ and local 1! of the "vertical Lyapunov family (1)

In center manifold: $(u,v) \to (u(T), v(T))$

$$\begin{cases} u(T) = u e^{iT}\left(1 + i[\alpha|u|^2 + \beta|v|^2]T\right) + O_5 \\ v(T) = v e^{iT}\left(1 + i[a|u|^2 + b|v|^2]T\right) + O_5 \end{cases}$$

$\exists H \Rightarrow (u,v)$ initial cond. of a per. sol. of period $T \simeq 2\pi$

$$\Updownarrow$$

$$\begin{cases} \operatorname{Arg} v(T) = \operatorname{Arg} v + 2\pi \\ u(T) = u \end{cases}$$

$$\Updownarrow$$

!

$$\begin{cases} T = 2\pi - \operatorname{Arg}\left[1 + i(a|u|^2 + b|v|^2) + \dfrac{O_5}{v}\right] \\ \quad = 2\pi\left(1 - a|u|^2 - b|v|^2\right) + \text{"}O_4\text{"} \\ 2\pi i u\left[(\alpha - a)|u|^2 + (\beta - b)|v|^2\right] = \text{"}O_4\text{"} \end{cases}$$

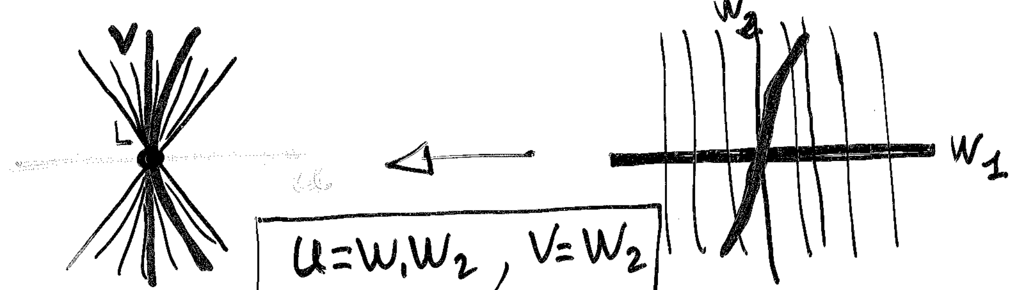
$= \text{"}O_4\text{"}$

still too naive!

∃ and local 1! of the "vertical" Lyapunov family (2)

Σ Idem Conley but center mnflds not analytic

Direct proof in $\mathbb{C}^m$ via complex blow up



$\boxed{u = w_1 w_2 \,,\ V = w_2}$

$\exists$ H i.p $\Rightarrow$ (u, v) initial condition of per. sol. of period $T \simeq 2\pi$ near $0\ (=L)$

$$\Updownarrow$$

$$\begin{cases} \operatorname{Arg} v(T) - \operatorname{Arg} v = 2\pi \\ u(T) = u \end{cases}$$

$$\Updownarrow$$

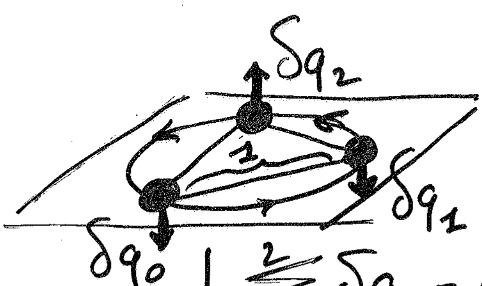
$$\begin{cases} T = 2\pi - 2\pi\left(b + a|w_1|^2\right)|w_2|^2 + O_3 \\ \left[2\pi i\left(\underbrace{(\beta - b)}_{\neq 0} + (\alpha - a)|w_1|^2\right)|w_2|^2 + O_3\right] w_1 + O_3 = 0 \end{cases}$$

[vanishes on $\underline{w_2 = 0}$ with 3 first derivatives]

$$\Downarrow$$

$w_1 = O_1(w_2, \bar{w}_2)$ C.Q.F.D.

THE D6 - SYMMETRY at the LEVEL OF V.V.E.



$$\boxed{\sum_{j=0}^{2} \delta q_j = 0}$$

$$(VVE)\ \begin{cases} \delta \ddot{q}_j = \dfrac{1}{3} \displaystyle\sum_{k \neq j} \dfrac{\delta q_k - \delta q_j}{\| q_k - q_j \|^3} \\ \quad = - \delta q_j \end{cases}$$

General solution $\boxed{\underbrace{\nu e^R(t+\psi)}_{\text{rotations}} + \underbrace{\mu e^P(t+\varphi)}_{P12}}$

$\nu, \mu, \varphi, \psi \in \mathbb{R}$

$$\underline{e}^R(t) = \begin{pmatrix} \mathrm{Re}\, \zeta e^{i(t+\frac{\pi}{2})} \\ \mathrm{Re}\, \zeta^2 e^{i(t+\frac{\pi}{2})} \\ \mathrm{Re}\, e^{i(t+\frac{\pi}{2})} \end{pmatrix}, \quad \underline{e}^P(t) = \begin{pmatrix} \mathrm{Re}\, \zeta^2 e^{i(t+\frac{\pi}{2})} \\ \mathrm{Re}\, \zeta e^{i(t+\frac{\pi}{2})} \\ \mathrm{Re}\, e^{i(t+\frac{\pi}{2})} \end{pmatrix}$$

$\Longrightarrow$ per. sol. (at 1st order) $\boxed{q_\mu^{P,R}(t) = q^L(t) + \mu e^{P,R}(t)}$

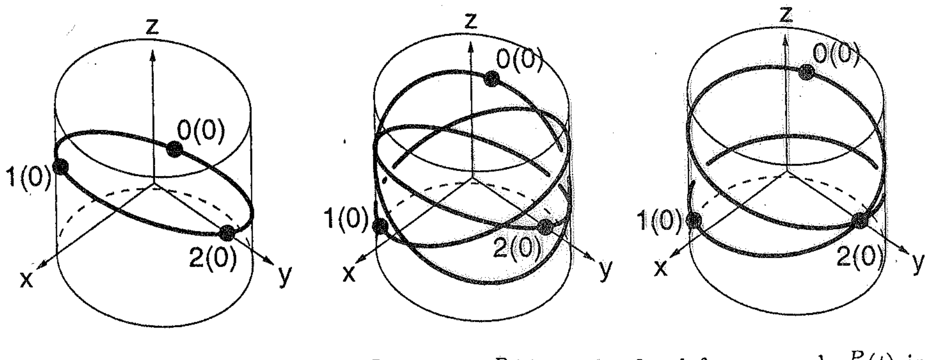


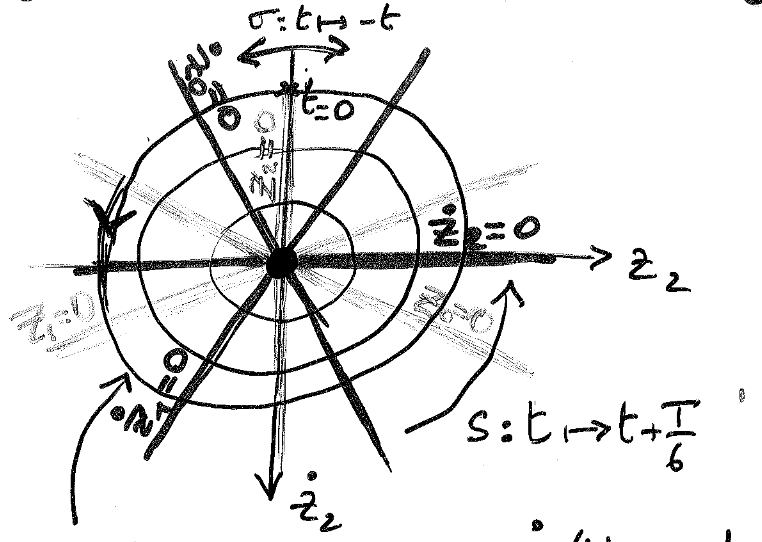
Figure 1: From left to right, $q^R(t)$ and $q^P(t)$ in the fixed frame, and $q^P(t)$ in the rotating frame. The positions of the bodies are indicated at time $t = 0$.

D6 - invariant

(D6 = symmetry group of the space of normalized oriented triangles

$\cong$ symmetry grp of the hexagon

D6 SYMMETRY at the REDUCED LEVEL



1) VVE

(sol. $\mu e^{?}(t)$)

$$D6 = \left\{ s, \sigma \;\middle|\; \begin{matrix} s^6 = 1, \sigma^2 = 1 \\ s\sigma = \sigma s^{-1} \end{matrix} \right\}$$

$z_2(t) = \mu \cos(t + \frac{\pi}{2}) = -\mu \sin t$, $\dot z_2(t) = -\mu \cos t$.

2) Vertical Lyapunov family :

1! locale $\Rightarrow$ idem for sol. $\ni$ $z_2 = 0$, $\dot z_2 < 0$ at $t = 0$

D6 SYMMETRY NON REDUCED

- Orbits in Lyap. family can be lifted continuously to period. orbits in rotating frame (rot ω 1! by continuity if uniform)

1! up to time shifts

- If moreover
  - body 2 $\in$ positive y semi-axis at $t=0$
  - $\omega(t) \cdot 1$

(as for Lagrange)

1! up to $\frac{T}{2}$ time shift

$\Longrightarrow$ D6 symmetric loop

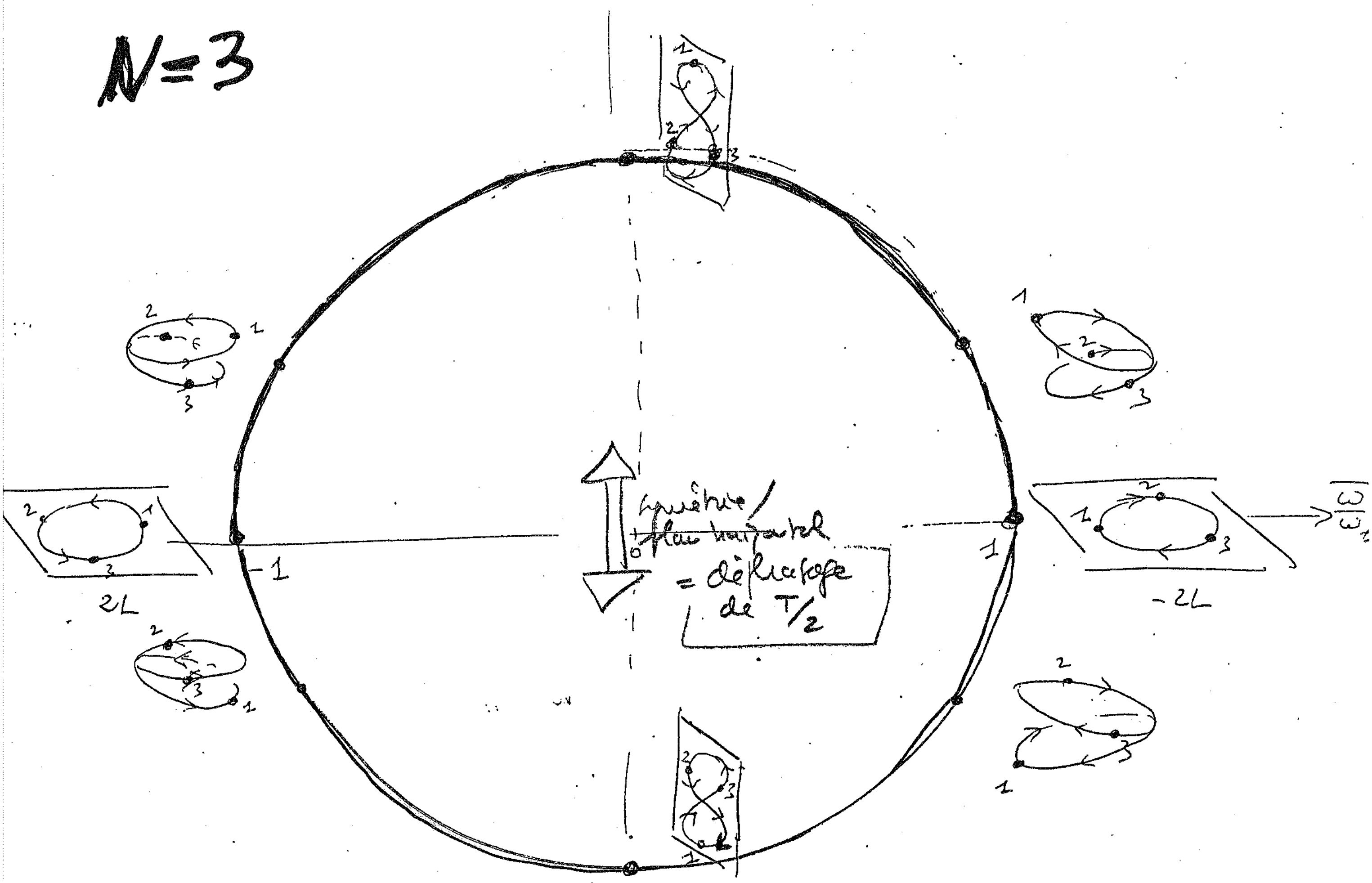
GLOBAL CONTINUATION

Marchal's P12 family

" C.F.M.'s Γ_1 family of "rotating Eights"

[redacted] TOPOLOGICAL AMUSEMENT

$N = 3$



The union of the 2 families of tori ($\vec{C}\uparrow$ and $\vec{C}\downarrow$) is a lens space

$L(4,1) \cong T_1 P_2(R)$.

To get the full invariant subset in the phase space make rotations and scaling act on this lens space $\Rightarrow$ 6 dim. invariant submanifold.

$\exists$ ONLY 2 LYAPUNOV FAMILIES, HOMOGRAPHIC & P12

1) The persistant resonance:

$a = \alpha$ and higher order analogues
(coeff $v|u|^{2n}$ in $\dot v$ = coeff. $u|u|^{2n}$ in $\dot u$)

$\Longrightarrow$ cannot use blow up $u = w_1,\ v = w_1 w_2$ to study neighb. of homographic family.

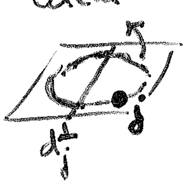
Explanation | VVE along homographic solution $q(t)$

idem Vuelles

$q(t) = \rho(t)\hat q \qquad \ddot\rho = -\dfrac{c\rho}{|\rho|^3}$, $\hat q$ c.c. in $\mathbb{R}^2 \equiv \mathbb{C}$

(VVE) $\boxed{\ddot z = -\dfrac{1}{r(t)^3} z}$ $\quad z = (z_0, z_1, z_2),\ \sum_{j=0}^{2} z_j = 0$

idem



Compare Danby ...

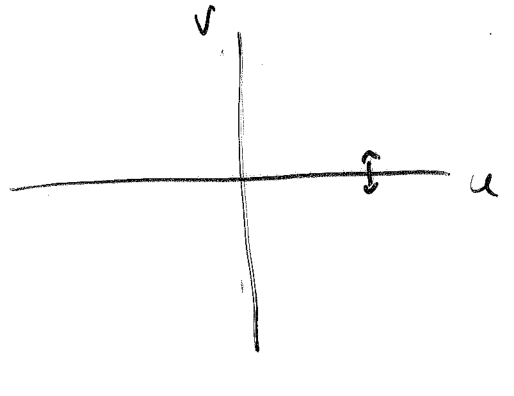
Solutions $z_j(t) = \langle q_j(t), d_j \rangle,\ j = 0, 1, 2,$

with $\sum \langle q_j(t_0), d_j \rangle = 0;\ \sum \langle \dot q_j(t_0), d_j \rangle = 0$

$\Downarrow$

$\boxed{\text{all have same period as } q(t)}$

After reduction:

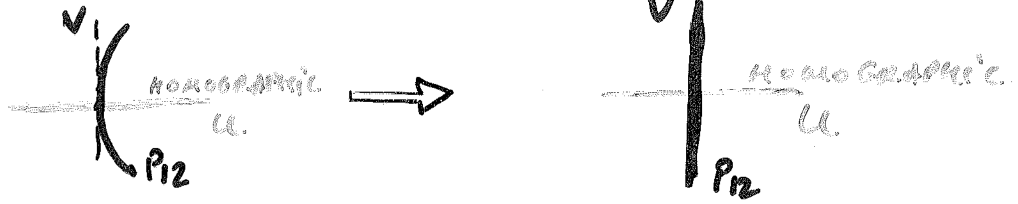


$\dot u = iu\, f(|u|^2) + \dots$
$\dot v = iv\, g(|u|^2) + \dots$

$f \equiv g$

2) The annulus map :

(i)



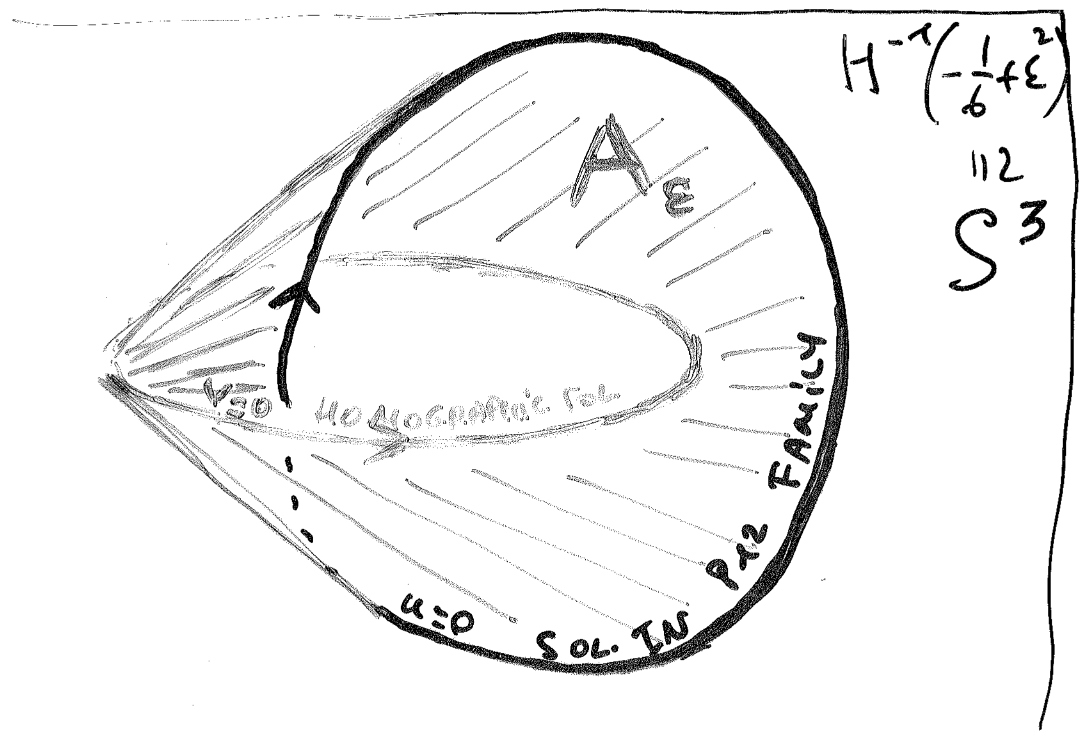
Change coord. so that $\begin{cases} v = 0 & \text{HOMOGRAPHIC} \\ u = 0 & P_{12} \end{cases}$

(ii) Polar coordinates

$$u = r_1 e^{i\theta_1}, \quad v = r_2 e^{i\theta_2}$$

(iii) Define A_ε by

$$(A_\varepsilon) \begin{cases} H = -\frac{1}{6} + \varepsilon^2 \\ \theta_1 + \theta_2 = 0 \pmod{2\pi} \end{cases}$$

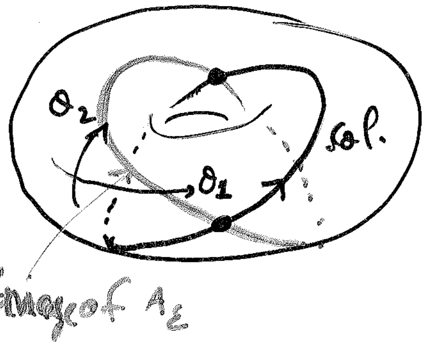


(iv) Second return map $P_\varepsilon : A_\varepsilon \to A_\varepsilon$

$(\dot\theta_1, \dot\theta_2)$ close to $(1,1)$

$\Downarrow$

a periodic sol. of period close to 2π crosses TWICE A_ε



return time T_ε def. by

$$4\pi = (\theta_1 + \theta_2)(T_\varepsilon)$$

$P_\varepsilon :$ $(r_1, r_2, \theta_1, \theta_2) \mapsto (r_1(T_\varepsilon), r_2(T_\varepsilon), \theta_1(T_\varepsilon), \theta_2(T_\varepsilon))$

$$\begin{aligned}
r_1(T_\varepsilon) &= r_1(1 + O_4)\\
r_2(T_\varepsilon) &= r_2(1 + O_4)\\
\theta_1(T_\varepsilon) &= \theta_1 + 2\pi\left[1 + \cancel{\tfrac{\alpha - a}{2}} r_1^2 + \tfrac{\beta - b}{2} r_2^2\right] + O_4\\
\theta_2(T_\varepsilon) &= \theta_2 + 2\pi\left[1 - \cancel{\tfrac{\alpha - a}{2}} r_1^2 - \tfrac{\beta - b}{2} r_2^2\right] + O_4
\end{aligned}$$

- Lemma: If $\varepsilon > 0$ small,

$$\theta_1(T_\varepsilon) = \theta_1 + 2\pi \quad (*)$$

$$\Longrightarrow r_2 = 0$$

Hence, NO OTHER LYAPUNOV FAMILY!

Proof: $(*)$ $\quad 2\pi\left(\frac{\beta - b}{2}\right) r_2^2 + \boxed{O_4} = 0$

$\boxed{O_4}$ is $r_2 O_3$ because $P_\varepsilon|_{\substack{\text{Homoc.}\\(r_2=0)}} = \mathrm{Id}$

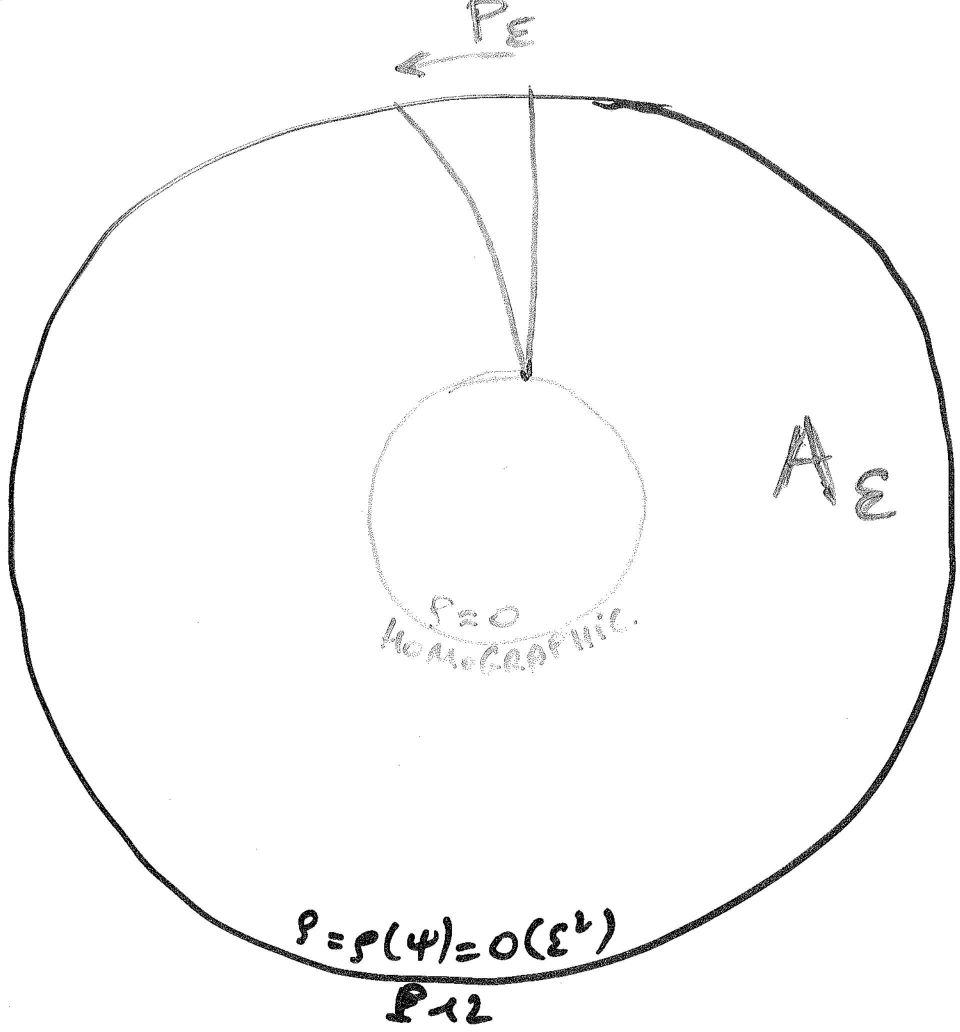
indeed $r_2^2 O_2$ because invariance under $(r_1, r_2, \theta_1, \theta_2) \mapsto (r_1, -r_2, \theta_1, \theta_2)$

(resonance VVE homog.)

$$\Rightarrow (*) \quad r_2^2\left[\underbrace{(\beta - b)}_{\neq 0}\pi + O(\varepsilon^2)\right] = 0$$

CQFD

FINALLY ...



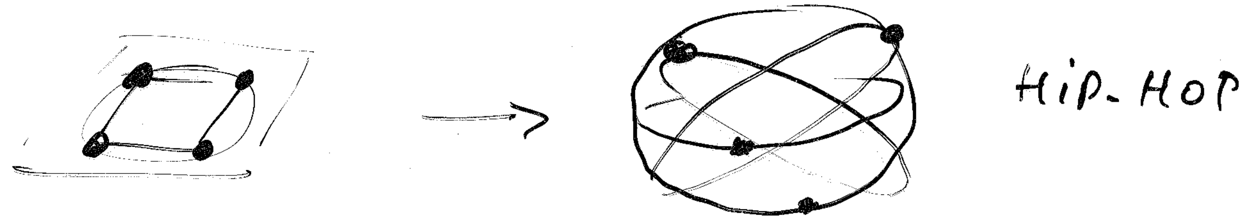
$$\rho = r_2^2 \, , \quad \psi = \theta_1$$

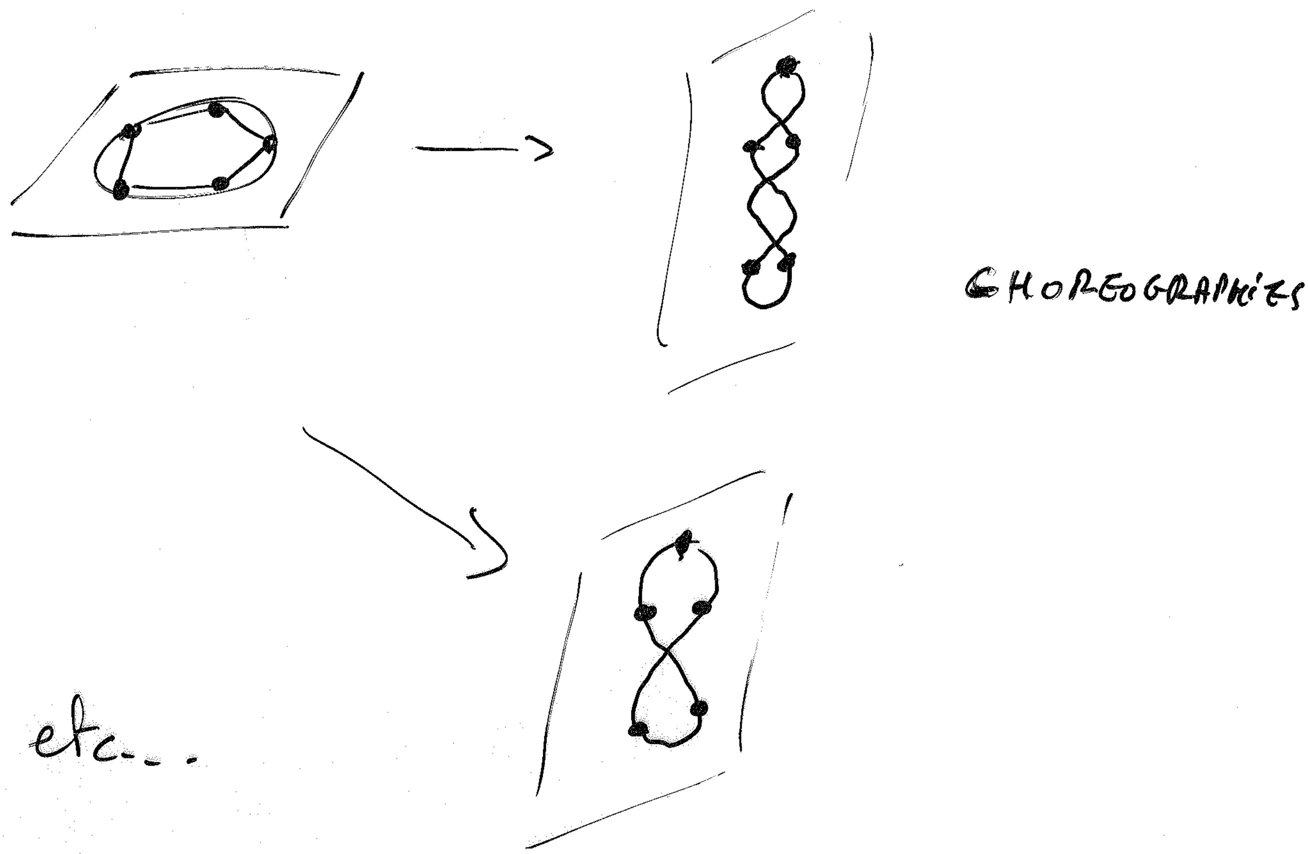
$$(\rho, \psi) \mapsto \left(\rho(1 + O(\varepsilon^4)), \psi + \left[\pi\left(\frac{\beta - b}{2} \right) + O(\varepsilon^2) \right] \rho + O(\rho^2) \right)$$

$$\left(\begin{aligned} \beta &= -1 \\ b &= -\frac{21}{19} \end{aligned} \right) \qquad \boxed{\beta - b = \frac{2}{19} > 0}$$

MORE BODIES ....

Global continuation of vertical Lyapunov families associated to horiz. relative equilibria leads to interesting solutions :





etc...

Orbite 50 choreo3.gnu