天津南开大学

Action minimization and global continuation of Lyapunov families stemming from relative equilibria.

Alain Chenciner

巴黎七大，

巴黎天文台

COLLABORATION WITH Jacques FÉJOZ

N-body problem in $\mathbb{R}^3$

$$\ddot{\vec{r}}_i = \sum_{j \neq i} \frac{m_j(\vec{r}_j - \vec{r}_i)}{\|\vec{r}_j - \vec{r}_i\|^3}$$

$$i = 1, \ldots, N$$

SYMMETRIES

- **ISOMETRIES of $\mathbb{R}^3$**

$$\Downarrow$$

Relative equilibrium solutions
(necessarily planar) ← easy

- **SCALING :**

if $x(t) = (\vec{r}_1(t), \dots, \vec{r}_N(t))$ solution,
$\forall \lambda > 0$, $\lambda^{-2/3} x(\lambda t)$ solution

$$\Downarrow$$

Homographic solutions
(necessarily planar) ← less easy

- **PERMUTATIONS OF EQUAL MASSES**

$$\Downarrow$$

Solutions minimizing action

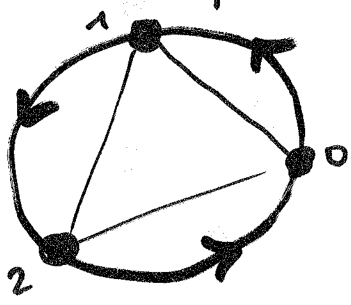
$$\int_0^T \left(\underbrace{\frac{1}{2} \sum_i m_i \left|\dot{\vec{r}}_i(t)\right|^2}_{K} + \underbrace{\sum_{i<j} \frac{m_i m_j}{\|\vec{r}_i(t) - \vec{r}_j(t)\|}}_{U} \right) dt$$

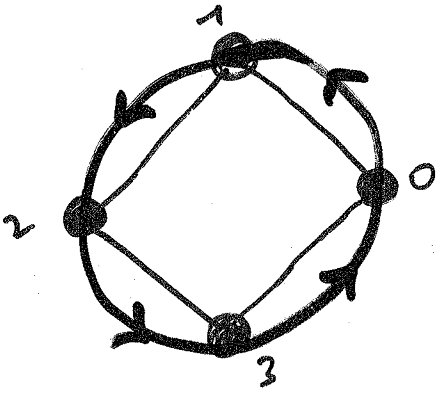
under symmetry constraints

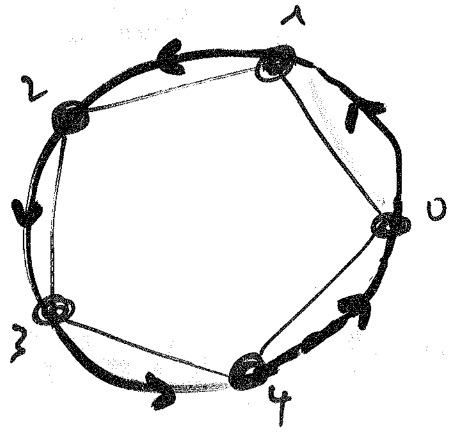
<u>FACT</u> : $\exists$ strong relation between

Simplest R.E.

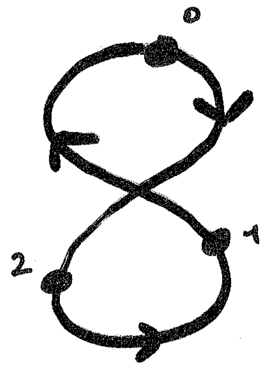
Regular N-gon with N equal masses



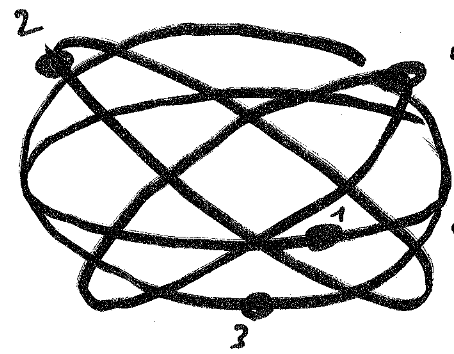




Simplest action minimizing solutions:

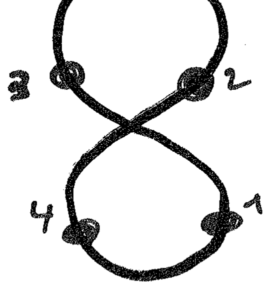


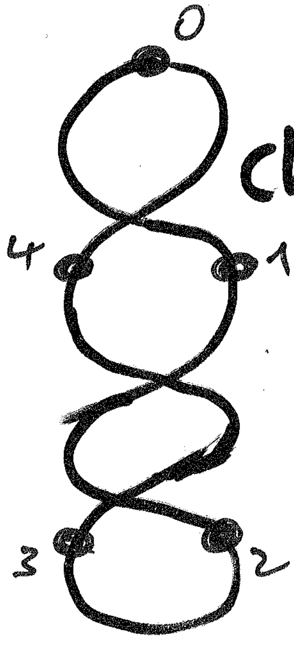
Eight



Hip-Hop

Eight

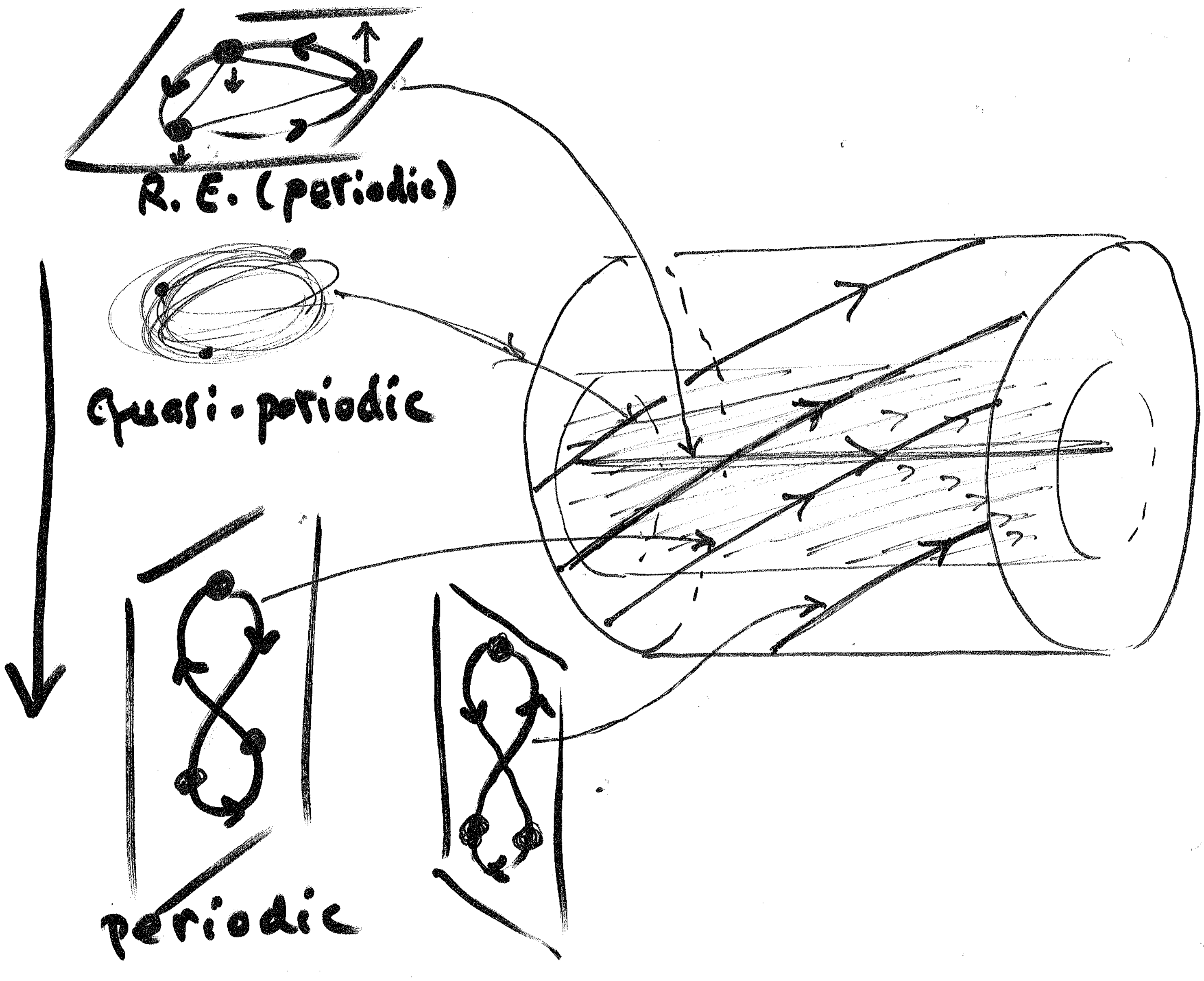




Chain

ORIGIN: (first discovered by C. Marchal for 3 bodies)

Vertical Lyapunov families stemming from horizontal Relative equilibrium



E. MARCHAL
The three-body problem.
1990

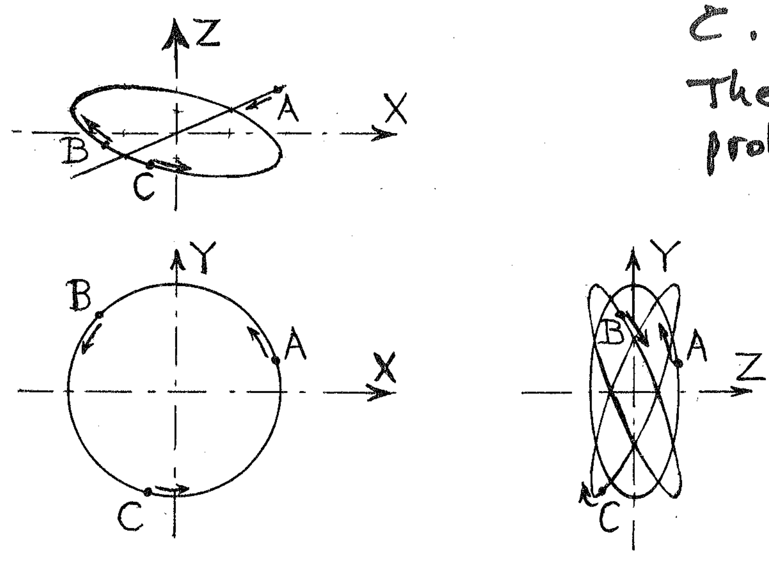


Fig. 78. Symmetric-periodic orbits with twelve space-time symmetries per period.

The three main projections of the orbits of A,B and C for small values of c_1 and t.

These almost circular orbits rotate very slowly about the Z-axis because of the small "angle of rotation".

For large c_1 the series (779)-(795) are of course less accurate and can no longer be used. However the periodic orbits of the family retain their twelve space-time symmetries per period and it would be interesting to look numerically for their evolution up to termination.

Notice that this termination is sometimes surprising as we will see in Section 10.9.1 for the family of retrograde pseudo-circular orbits that ends into rectilinear orbits !

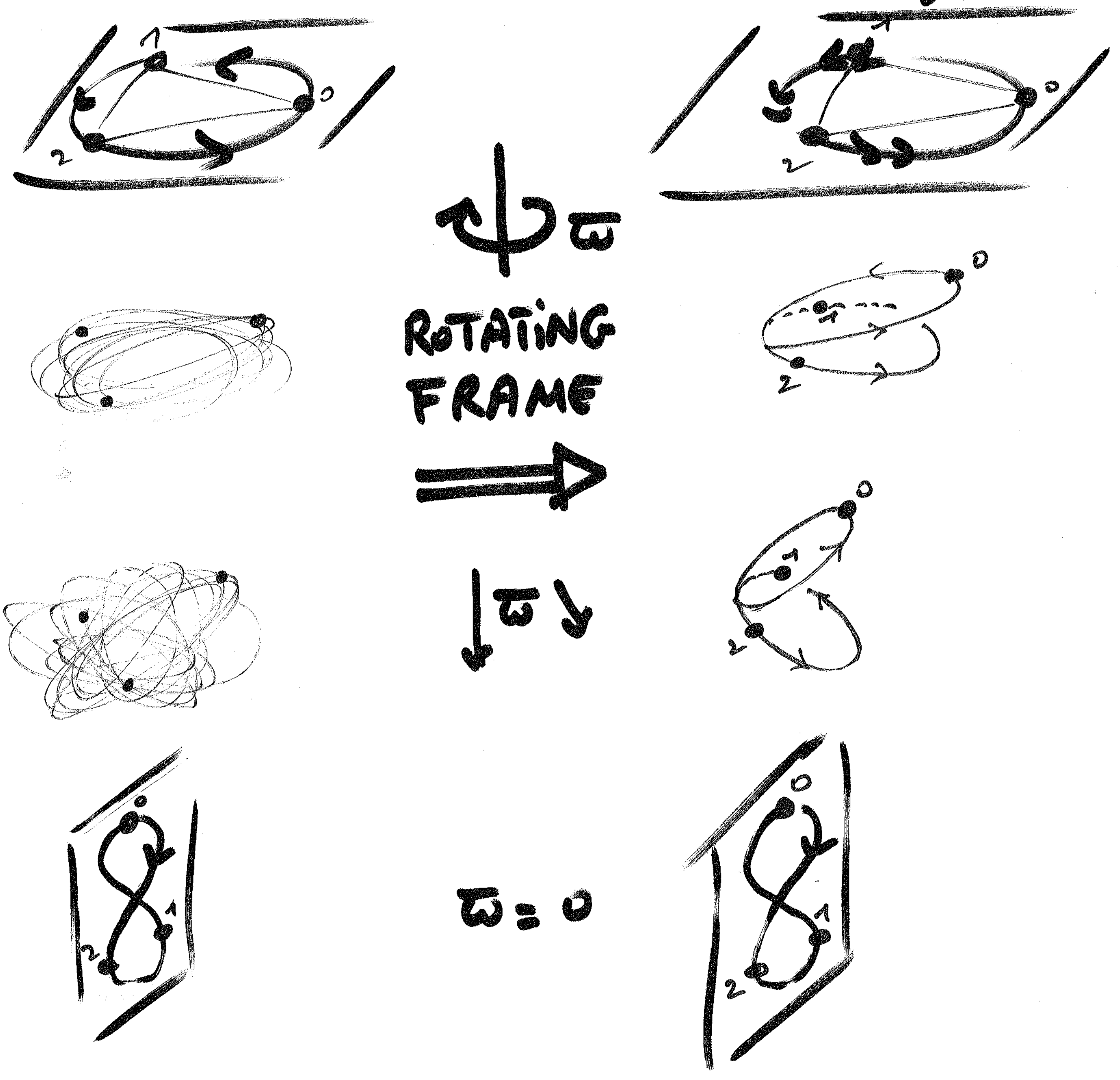
10.8.3 The Halo orbits about the collinear Lagrangian points.

The Halo orbits are a family of simple periodic orbits of the circular restricted three-body problem. these orbits remain in the vicinity of a collinear Lagrangian point and are among the most useful orbits for many types of missions (Fig. 79). They have already been presented in Section 9.1 and in Fig. 24.

KEY : Start with very symmetric R.E. (ex regular N-gon, N equal masses):

Quasi-periodic in inertial frame

Periodic with huge symmetry in rotating frame.



(In this case: D_6 symmetry)

3 STEPS

– INFINITESIMAL : VVE , symmetries of solutions in rotating frames.

= LOCAL : Bifurcation of vertical Lyapunov families: existence, unicity

≡ GLOBAL : Continuation of local families via (local) minimization under symmetry constraints.

– The vertical variational equation (VVE)

Pythagoras $\Longrightarrow$ splitting of VE

$$\text{HVE}\quad \ddot{\vec{h}}_i = \sum_{j\neq i} m_j \frac{\vec{h}_j - \vec{h}_i}{\|\vec{r}_j - \vec{r}_i\|^3} - 3\sum_{j\neq i} m_j \frac{\langle \vec{r}_j - \vec{r}_i, \vec{h}_j - \vec{h}_i\rangle}{\|\vec{r}_j - \vec{r}_i\|^5}(\vec{r}_j - \vec{r}_i)$$

$$\text{VVE}\quad \ddot{z}_i = \sum_{j\neq i} m_j \frac{z_j - z_i}{\|r_j - r_i\|^3}$$

constant for a R.E.

VVE of R.E. $\quad \ddot{Z} = W Z$

(Symmetric for the "mass scalar product"
< 0 after quotient by translations
($\Leftrightarrow$ fix $\sum m_i z_i = 0$))

$\Rightarrow$ $\exists$ basis of eigenvectors $Z_1, \dots, Z_{N-1}$
with eigenvalues $-\omega(1)^2 \dots\dots -\omega(N-1)^2$

General solution:

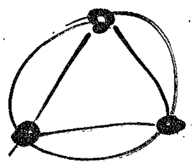
$$Z(t) = \sum_{k=1}^{N-1} \mathrm{Re}\Big(\underbrace{\alpha_k Z_k}_{=W_k} e^{i\omega(k)}\Big)$$

W_k complex eigenvector of W with eigenvalue $-\omega(k)^2$

Solutions of VVE (regular N-gon)

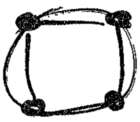
After reduction of translations, phase space of VVE has dim $2(N-1)$

(OPS $\sum_{i=0}^{N-1} z_i = 0$ and $\sum_{i=0}^{N-1} \dot{z}_i = 0$)



$N = 2m+1$

Phase space $= \bigoplus_{k=1}^{m}$ of m 4-dim spaces



$N = 2m$

Phase space $= \bigoplus_{k=1}^{m}$ of $\begin{cases} m-1 & \text{4-dim space} \\ 1 & \text{2-dim space} \end{cases}$

if $N=2m+1$
$$\boxed{\lambda_k = -2\sum_{j=1}^{m} \frac{1}{\rho_j^3}\left(1-\cos\frac{2\pi jk}{N}\right) = -\omega_k^2}$$

if $N=2m+2$, idem $-\frac{1}{\rho_{m+1}^3}\left(1-(-1)^k\right)$.

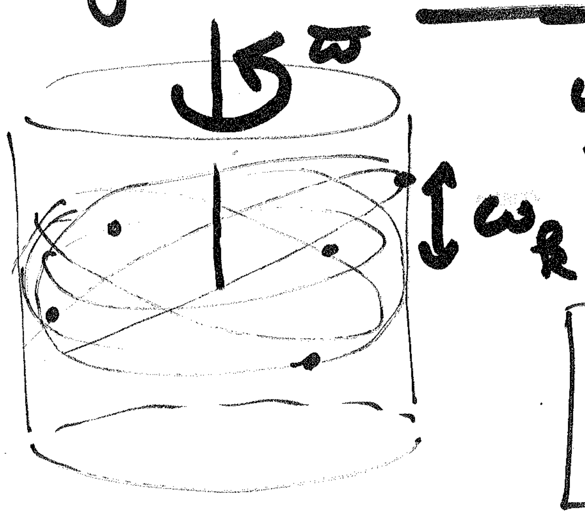
$$\left(\rho_j = |\zeta^j - 1| \ , \ \zeta = e^{\frac{2\pi i}{N}}\right)$$

• Proposition: $\omega_1 < \omega_2 < \dots < \omega_k < \dots$

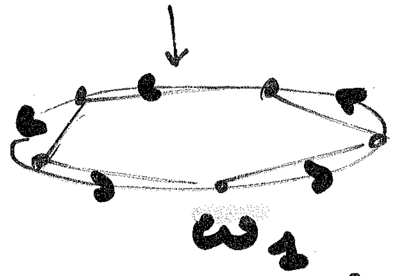
I $\left(\mathrm{Re}(e^{i\omega_k t}), \mathrm{Re}(\zeta^k e^{i\omega_k t}), \dots, \mathrm{Re}(\zeta^{k(N-1)} e^{i\omega_k t})\right)$

II $\left(\mathrm{Re}(e^{i\omega_k t}), \mathrm{Re}(\bar{\zeta}^k e^{i\omega_k t}), \dots, \mathrm{Re}(\bar{\zeta}^{k(N-1)} e^{i\omega_k t})\right)$

Symmetries of 1st order solutions in rotating frame



$$\boxed{\varpi = \omega_1 - \gamma\,\omega_k}$$



$$S_\gamma(N,k,\eta) : \Big(\underbrace{\underbrace{\zeta^j e^{i\gamma\omega_k t}}_{\text{HORIZONTAL}} ,\ \underbrace{\mathrm{Re}\, \zeta^{\overset{\pm 1}{\eta} k j} e^{i\omega_k t}}_{\text{VERTICAL}}}_{j^{\text{th}}\ \text{body}} \Big) \qquad j = 0,\dots,N-1 \ (\text{or } j \in \mathbb{Z}/N\mathbb{Z})$$

$\dfrac{2\pi s}{\omega_k}$-periodic $\iff \gamma = \dfrac{r}{s}$ (dense).

OPS $= 1$ (time scale)

$$\boxed{\vec{r}_j(t) = \rho\, \vec{r}_{\xi(j+\delta)}\big(\xi(t-\theta)\big)}$$

$\xi = \pm 1 \ (\in \mathbb{F}_2)$, $\delta \in \mathbb{Z}/N\mathbb{Z}$, $\theta \in \mathbb{R}/1\mathbb{Z}$

$$\rho(H,V) = \left(e^{2\pi i \alpha}\, \overline{H}^{\xi},\ e^{i\pi\beta} V\right)$$

$\beta \in \mathbb{Z}/2\mathbb{Z}$, $\overline{H}^{\xi} = H$ if $\xi = +1$, $\overline{H}$ if $\xi = -1$

with
$$\begin{cases} \alpha = \dfrac{r}{s}\theta - \dfrac{\delta}{N} \pmod 1 \\[2mm] \theta = \dfrac{\beta}{2} + k\eta\dfrac{\delta}{N} \pmod 1 \quad (\Rightarrow s \text{ values}) \end{cases}$$

Identification of the symmetry group $G_{\frac{r}{s}}(N,k,\hbar)$

It is the subgroup of

$$\left(\underset{\theta}{\mathbb{R}/s\mathbb{Z}} \times \underset{\delta}{\mathbb{Z}/N\mathbb{Z}} \times \underset{\beta}{\mathbb{Z}/2\mathbb{Z}}\right) \rtimes \underset{\xi}{\mathbb{F}_2}$$

[group law $(\theta',\delta',\beta',\xi')\cdot(\theta,\delta,\beta,\xi) = (\theta'+\xi'\theta,\ \delta'+\xi'\delta,\ \beta'+\beta,\ \xi'\xi)$]

defined by the equation

$$\boxed{\theta = \frac{\beta}{2} + k\hbar\frac{\delta}{N} \quad (\mathrm{mod}\ 1)}$$

Example: $(s=1)$

$$G_r(N,k,\hbar) \cong D_N \times \mathbb{Z}/2\mathbb{Z}$$

$(\cong D_{2N}$ if N odd$)$

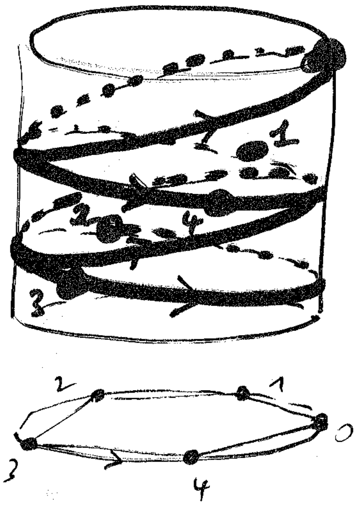
Examples : (1) choreographies

$N = 2n+1$

$\frac{r}{s} = N-1$

$k = 1$

$\eta = -1$



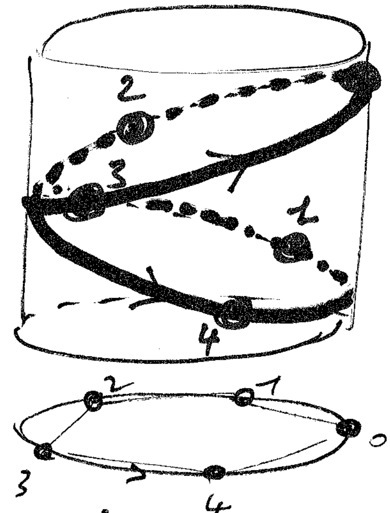
CHAINS

$N = 2n+1$

$\frac{r}{s} = 2$

$k = n$

$\eta = -1$



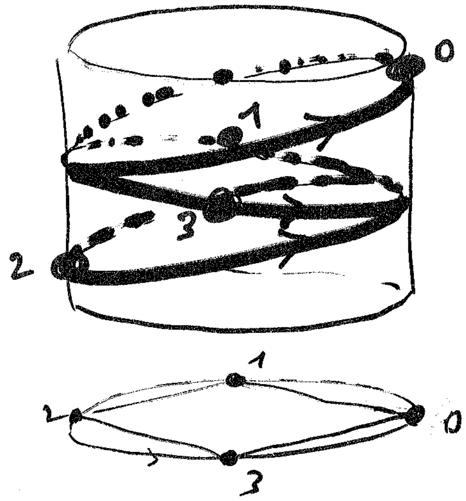
EIGHTS

$N = 4$

$\frac{r}{s} = 3$

$k = 1$

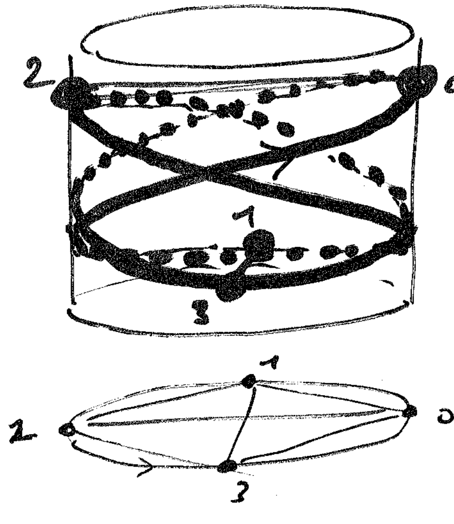
$\eta = -1$



$N = 4$

$\frac{r}{s} = \frac{3}{2}$

$k\eta = 2$

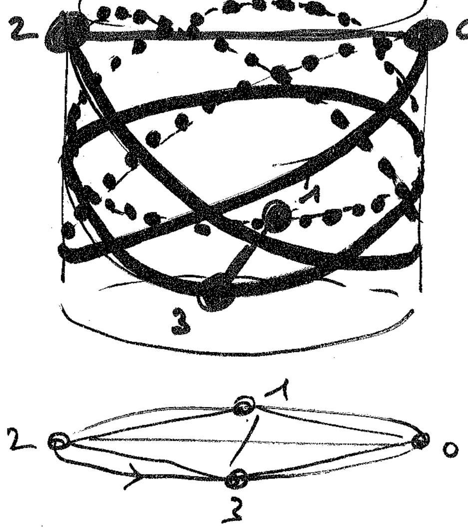


(2) Hip-Hops

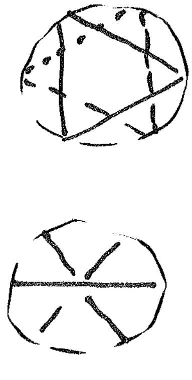
$N = 2n$

$\frac{r}{s} = 1$

$k\eta = n$



etc ...



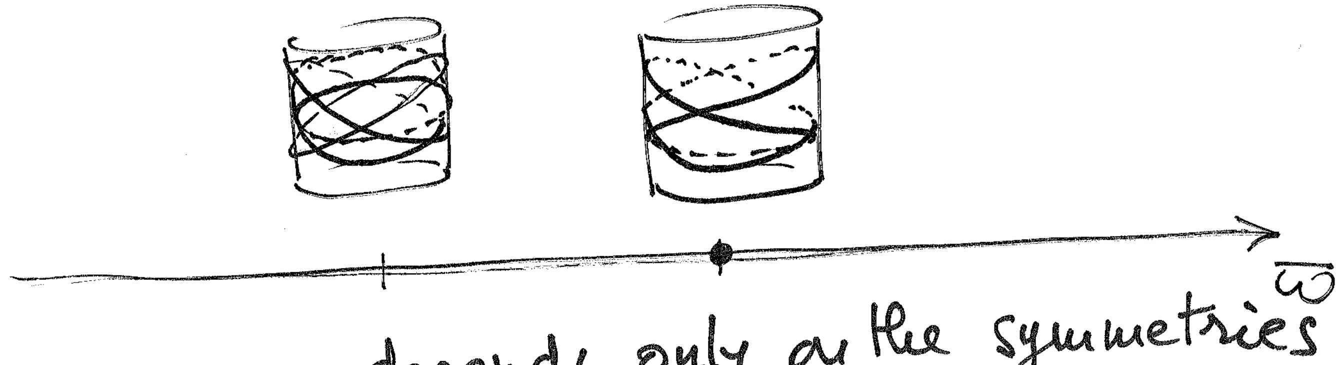
...

Remark : N, k, h given,

$$\left\{ \overline{\omega} = \omega_1 - \frac{r}{s}\omega_k \ , \ S_{\frac{r}{s}}(N,k,h) \text{ is a simple choreography} \right\}$$

(1! curve)

is **dense** in $\mathbb{R}$



depends only on the symmetries

Proof : $\Leftrightarrow \exists\, g \in G_{\frac{r}{s}}(N,k,h)$,

$\zeta = 1, \ \delta \neq 0, \ \alpha = 0, \ \beta = 0$

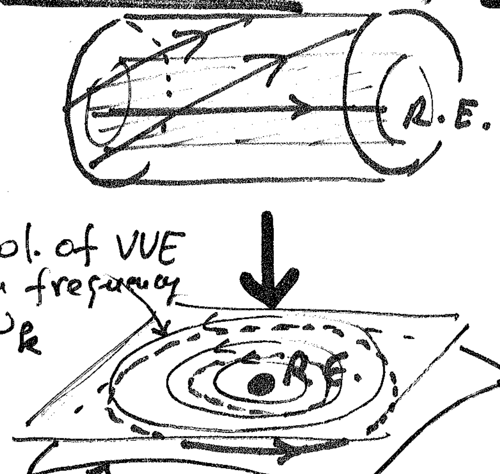
$(\delta, N) = 1$ (simple)

$$\Leftrightarrow \begin{cases} (r,s) = 1 \\ s - khr = 0 \ (N) \end{cases}$$

then easy ... 3 lines.

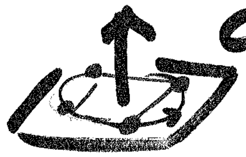
= Local Lyapunov families

Reduction of isometries :



Lyapunov periodic solutions of frequency close to ω_k (invariant surface in reduced phase space)

- fix center of mass / angular momentum



- quotient by $SO(2)$

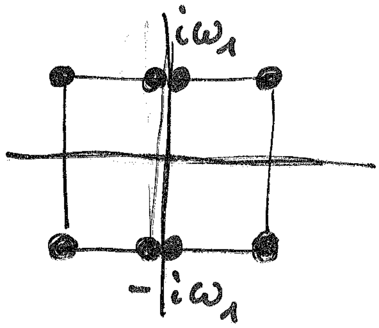
eliminates a pair of $\pm i\omega_1$ (one H, one V)

dim $6N-10$

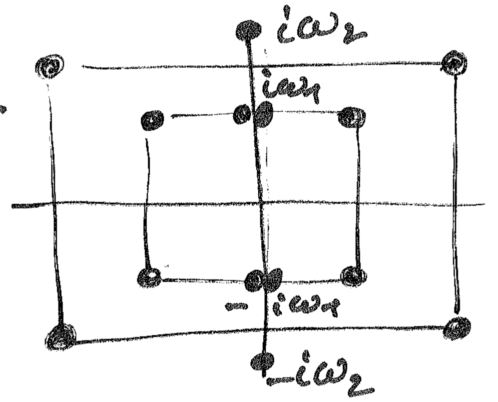
$4N-6$ HORIZONTAL $2N-4$ VERTICAL

Problem: Spectrum very resonant!

$N=3$



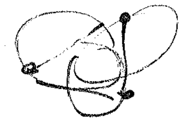
$N=4$



For $N \geq 6$, there appears new horizontal purely imaginary eigenvalues!

$N = 3$ (A.C. & J. Féjoz 2006
ω_1 See also Marchal's book 1990)

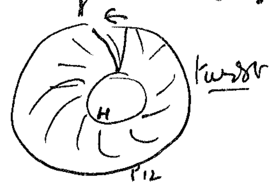
Using normal forms



$\Downarrow$

∃ exactly 2 Lyapunov families (same frequency ω_1)

- 1 HORIZONTAL : Homographic family
- 1 VERTICAL : P_{12} family, D_6 symmetric, containing the 8.

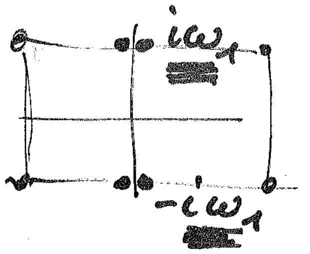


$N = 2n$
ω_n (E. Barrabés, JM Cors, C Pinyol, J. Soler 2006 see also K. Meyer & D. Schmidt 1993 (with 1 central mass).

∃ and 1! of the corresponding Lyapunov family (Hip-Hop).

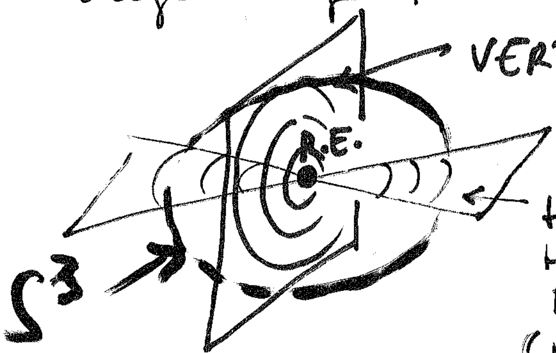
CASES FOR WHICH A PROOF IS AVAILABLE

$N = 3$ (A.C. & J. Féjoz) 2006 : see also Marchal's book



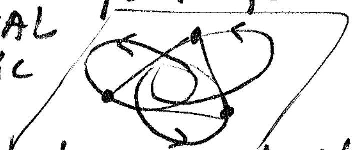
in Center manifold of dim 4, energy surface $\cong S^3$

Normal forms have resonance at any order along homographic family. Still one can prove $\exists$ exactly 2 Lyapunov families originating from the Lagrange equilateral R.E.:

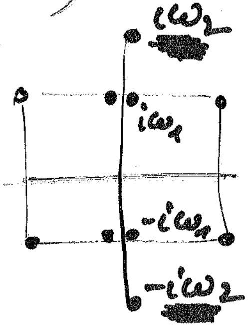


VERTICAL FAMILY WITH D6 SYMMETRY (Marchal's P12 family) quasi-periodic before reduction

HORIZONTAL HOMOGRAPHIC FAMILY (periodic before reduction)



$N = 2n$, $k = n$ (Hip-Hop symmetry) (E. Barrabés, J.M. Cors, C. Pinyol, J. Soler) 2006



$N=4$

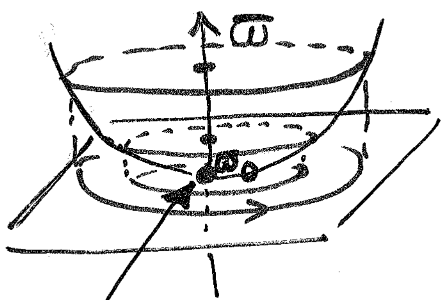
These authors prove the existence of the Lyapunov families by analytic continuation. See also similar case by K. Meyer & D. Schmidt (1993) when $\exists$ central mass (model for Saturn braided rings).

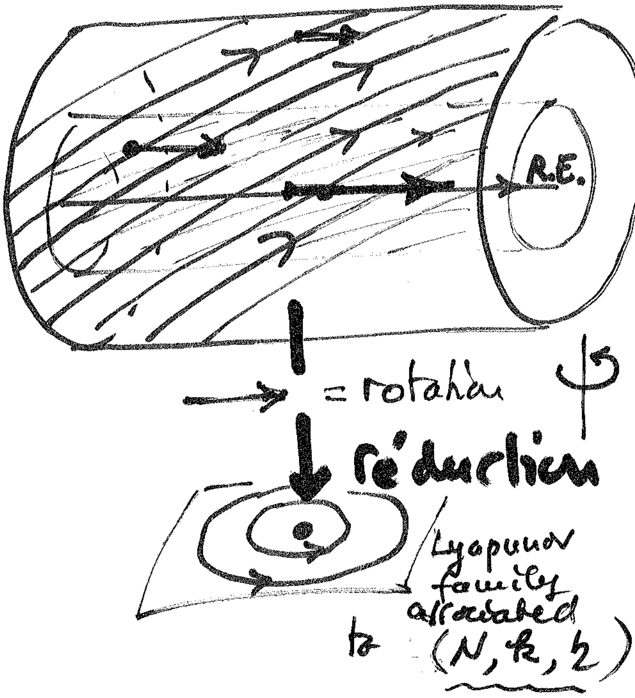
Global continuation via minimization under symmetry constraints

Periodic solution of the reduced problem

$\Updownarrow$

Periodic solution in rotating frame



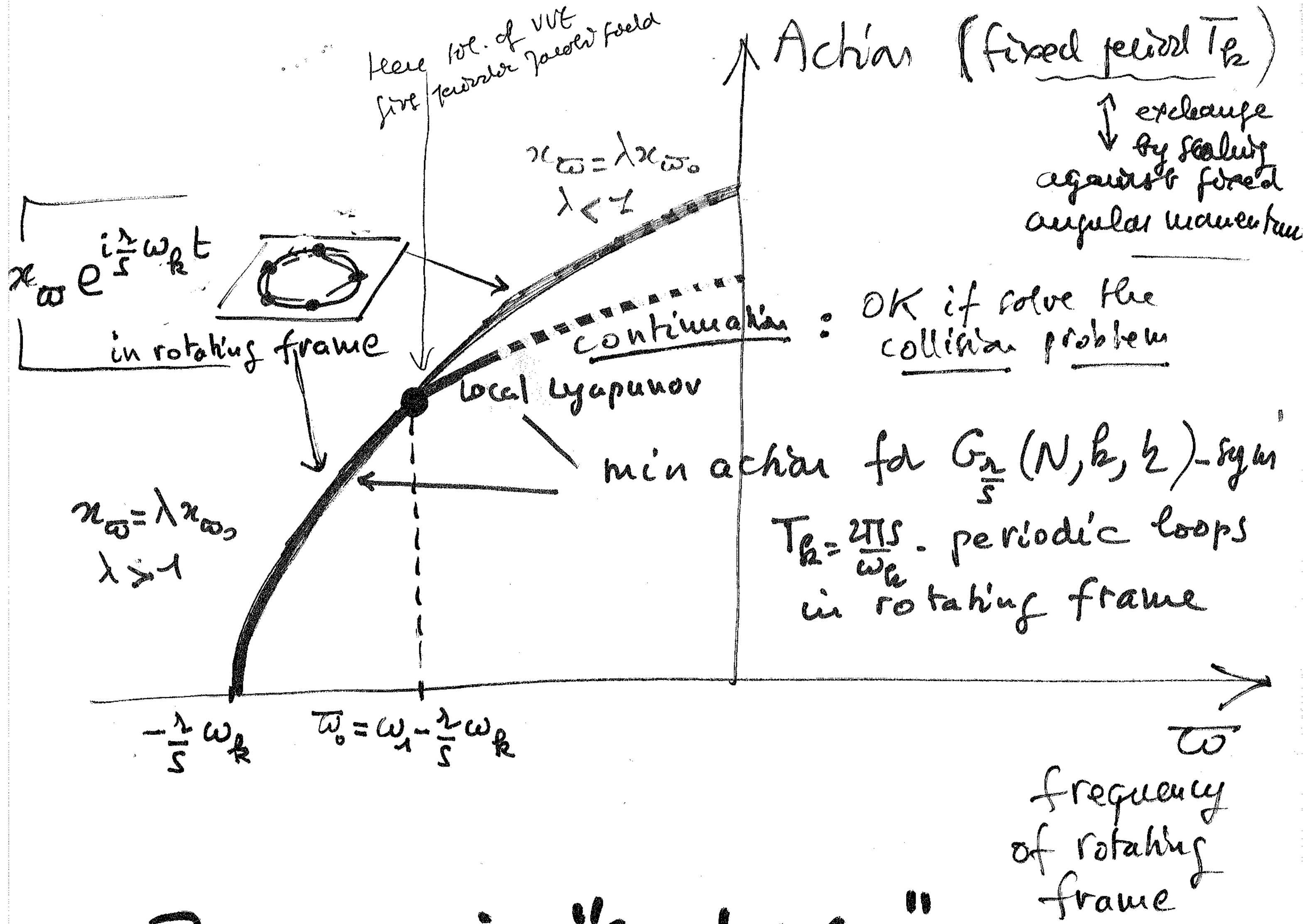


Necessarily $\exists\ \frac{r}{s} \in \mathbb{Q}$, $\quad \bar\omega_0 = \omega_1 - \frac{r}{s}\omega_k$

If 1! $\Rightarrow$ family of $G_{\frac{r}{s}}(N, k, l)$-symmetric (in rotating frame) $\frac{2\pi s}{\omega_k}$-periodic solutions parametrized by $\bar\omega$.

Look for such a family as a family of (local) action minimizers

Why is this reasonable?



Because in "good cases" one has the above situation.

not minimizing : Z eigenvector of $W(x_{\varpi_0}) \to -\omega_k^2$

$$d^2A\Big(\underset{\lambda x_{\varpi_0}}{\underset{\parallel}{x_\varpi}}\, e^{i\frac{r}{s}\omega_k t}\Big)(Z,Z) = \underbrace{\int_0^{T_k}\tfrac{1}{2}\|\dot Z\|^2\,dt}_{+\,\omega_k^2\int_0^{T_k}\|Z\|^2\,dt} + \underbrace{\int_0^{T_k} d^2U(x_\varpi)(Z,Z)\,dt}_{-\frac{1}{\lambda^2}\,\omega_k^2\int_0^{T_k}\|Z\|^2\,dt}$$

$$>0 \iff \lambda > 1$$

$$<0 \iff \lambda < 1$$

Remark :

$G_{\frac{r}{s}}(N,k,h)$ symmetry in rotating frame with frequency ϖ

$$\Downarrow \quad \alpha \Rightarrow \alpha + \varpi \theta$$

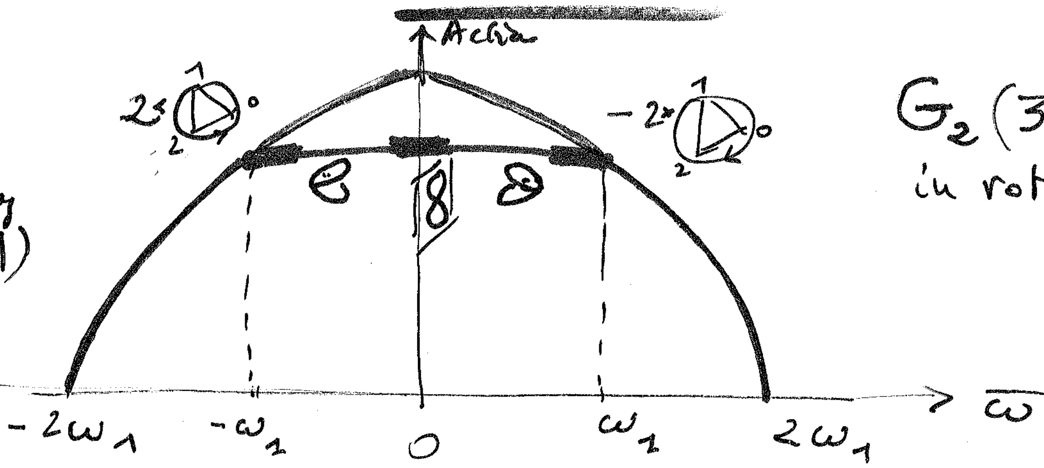
$G_{\frac{r}{s}+\varpi}(N,k,h)$ symmetry in inertial frame.

Corollary : choreographies dense in (vertical) Lyapunov families

EXAMPLES

$N = 3$

P_{12} family (Marchal)

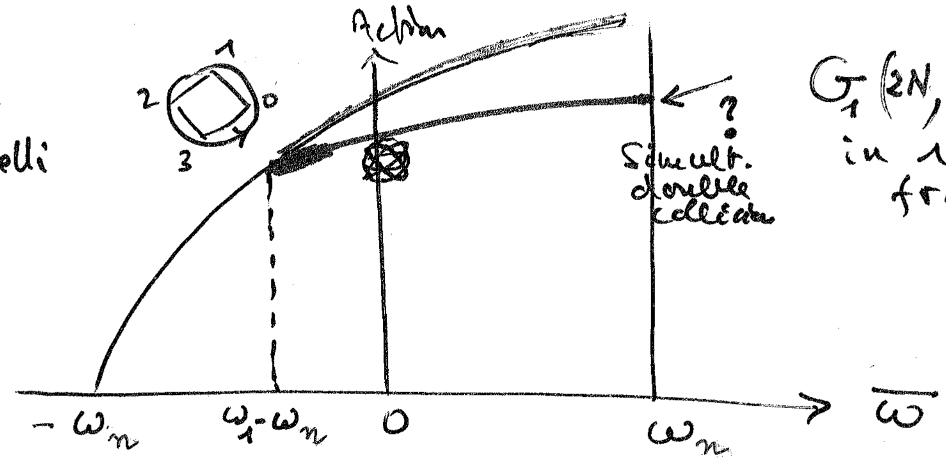


$G_2(3,1,-1)$ in rotating frame

$N = 2N$

Hip-Hop family

Terracini-Venturelli



$G_1(2N, n, 1)$ in rotating frame

$N = 5$

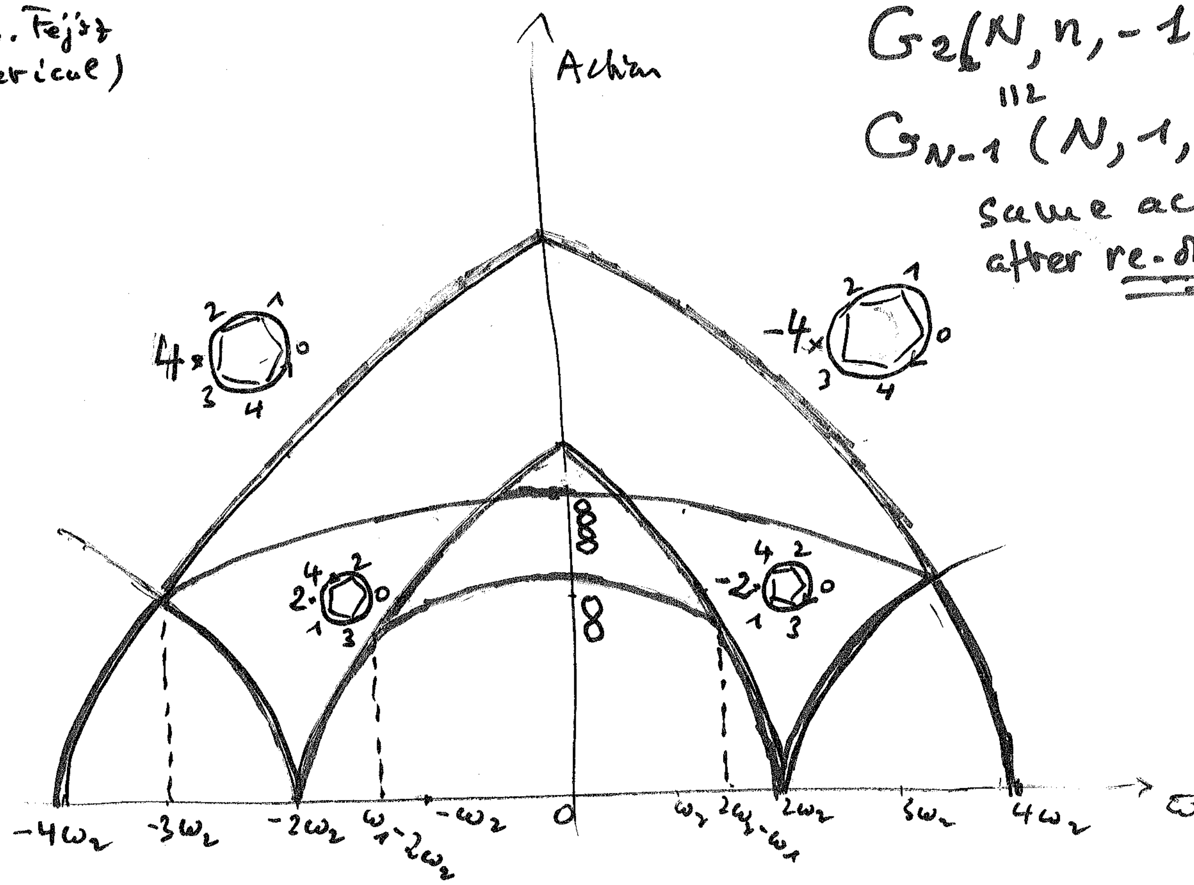
Chenciner-Féjoz (numerical)

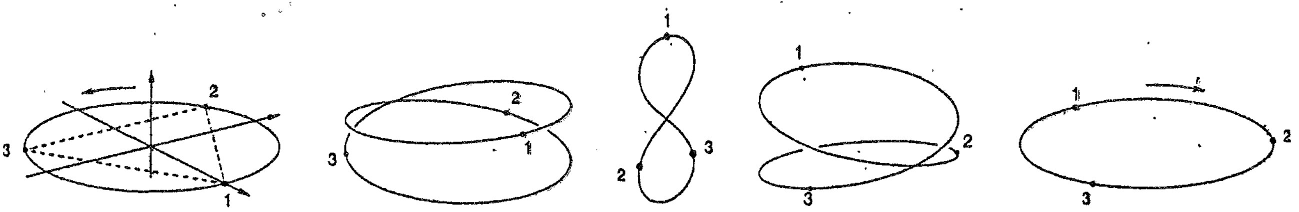
$G_2(N, n, -1)$

$\cong$

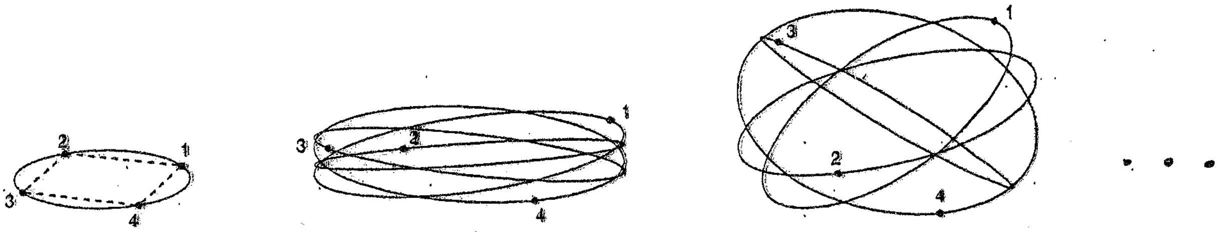
$G_{N-1}(N, 1, -1)$

same action after re-ordering

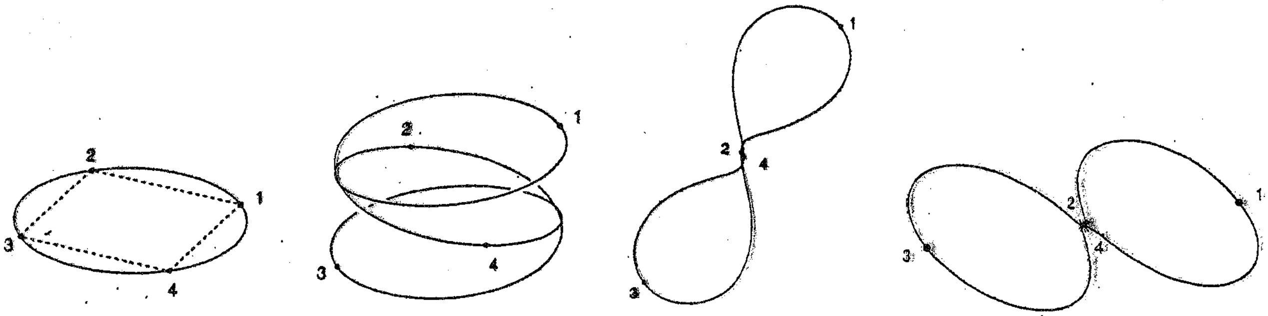




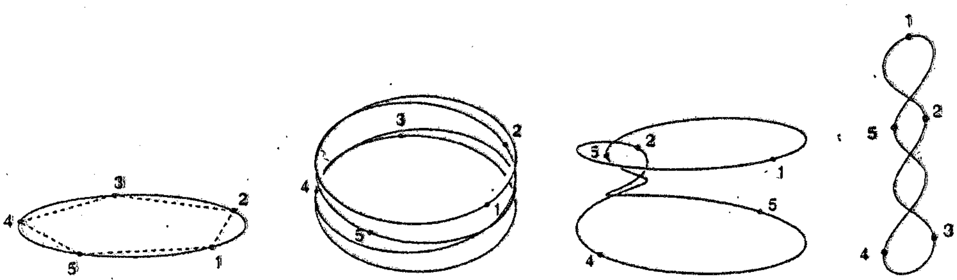
FROM EQUILATERAL TO EIGHT : THE P12 FAMILY



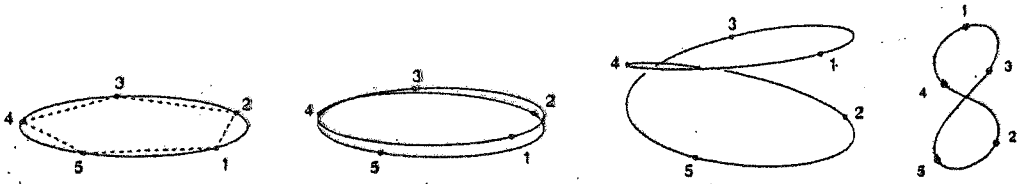
THE HIP-HOP FAMILY



4 BODIES : THE FIRST VERTICAL FREQUENCY



FROM PENTAGON TO 4-CHAIN



FROM PENTAGON TO EIGHT