THE ANGULAR MOMENTUM OF RELATIVE EQUILIBRIA IN HIGHER DIMENSIONS

Alain Chenciner (IMCCE)

BON ANNIVERSAIRE

HAMSYS 2010
MEXICO

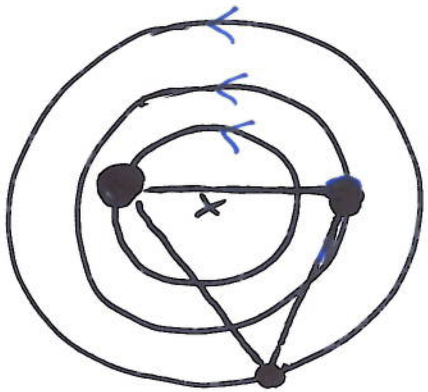
ERNESTO

RELATIVE EQUILIBRIA in $\mathbb{R}^3$

॥

RIGID BODY MOTIONS OF THE N-BODY PROBLEM

PLANAR $\quad \mathbb{R}^2 \equiv \mathbb{C}$

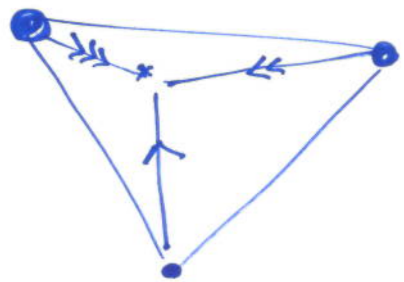


Lagrange 1772

PERIODIC $\quad \vec{r}_j(t) = e^{i\omega t}\vec{r}_j(0)$

CENTRAL $\quad$ forces // form

$$\nabla_\mu U(x) \,//\, x$$



RELATIVE EQUILIBRIA IN $\mathbb{R}^N$

Albouy - Chenciner 1998

EVEN DIM $\quad \mathbb{R}^{2p} \cong \underset{\underset{\mathbb{C}^{k_1}}{\cong}}{\mathbb{R}^{2k_1}} \oplus \cdots \oplus \underset{\underset{\mathbb{C}^{k_r}}{\cong}}{\mathbb{R}^{2k_r}}$

choice of complex structures

QUASI-PERIODIC $\quad \vec{r}_j(t) = \left(e^{J_1 \omega_1 t} \vec{r}_{j1}(0), \ldots, e^{J_r \omega_r t} \vec{r}_{jr}(0) \right)$

BALANCED $\quad \nabla_\mu U(x) = \mathcal{S} x$

forces = sym ∘ form

$\mathcal{S} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ symmetric

$\Longrightarrow$ enough to study the most degenerate case of CENTRAL CONFIGURATIONS

$$\mathbb{R}^{2p} \underset{\substack{\text{choice of} \\ \text{a complex structure } J}}{\cong} \mathbb{C}^{p}$$

$$\vec{r}_j(t) = e^{J\omega t}\,\vec{r}_j(0)$$

$\boxed{\text{PERIODIC MOTION}}$

ANGULAR MOMENTUM

BIVECTOR $\mathcal{L} = \sum_i m_i \vec{r}_i \wedge \dot{\vec{r}}_i$

$\Updownarrow$ euclid. structure

ANTISYMMETRIC $\mathcal{L} : \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$

$$\boxed{\mathcal{L} = \omega(S_0 J + J S_0)}$$

INERTIA $S_0 = \begin{array}{cc} l & \\ \left(\sum_i m_i r_{ik} r_{il} \right) & k \end{array}$

Choice of orth. basis $\Rightarrow$ $S_0 = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{2p} \end{pmatrix}$

COMPLEX STRUCTURE

$J \in SO(2p)$

$J^2 = -Id$

$J = R^{-1} \underbrace{\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}}_{J_0} R$

STRUCTURE OF $\frac{1}{\omega}\mathcal{C}$

$$\frac{1}{\omega}\mathcal{C} = S_0 J + J S_0 \quad (\mathbb{C}^p, J) \to (\mathbb{C}^p, J) \text{ skew-hermitian}$$

$$C = R\mathcal{C}R^{-1} = S J_0 + J_0 S \quad (\mathbb{C}^p, J_0) \to (\mathbb{C}^p, J_0) \text{ skew-hermitian}$$

$$\Sigma = J_0^{-1} C = J_0^{-1} S J_0 + S \quad (\mathbb{C}^p, J_0) \to (\mathbb{C}^p, J_0) \text{ hermitian}$$

$$\boxed{\begin{array}{l} J = R^{-1} J_0 R \\ S = R S_0 R^{-1} \end{array}} \quad \text{if } R = \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{s}_j \\ \vdots \end{pmatrix} \qquad \boxed{\Sigma_{jk} = \langle \vec{r}_j + i\vec{s}_j, \overline{\vec{r}_k + i\vec{s}_k} \rangle_{S_0}}$$

SPECTRUM

	réel	complexe
$\frac{1}{\omega}\mathcal{C}, C$	$\pm i\nu_1, \dots, \pm i\nu_p$	$i\nu_1, \dots, i\nu_p$
Σ	$\nu_1, \nu_1, \dots, \nu_p, \nu_p$	$\nu_1, \dots, \nu_p$

THE FREQUENCY MAP

$$SO(2p)/U(p) \longrightarrow W_p^+$$

Complex structures   positive Weyl chamber

$$J \longmapsto (\nu_1 \geq \nu_2 \geq \dots \geq \nu_p \geq 0)$$

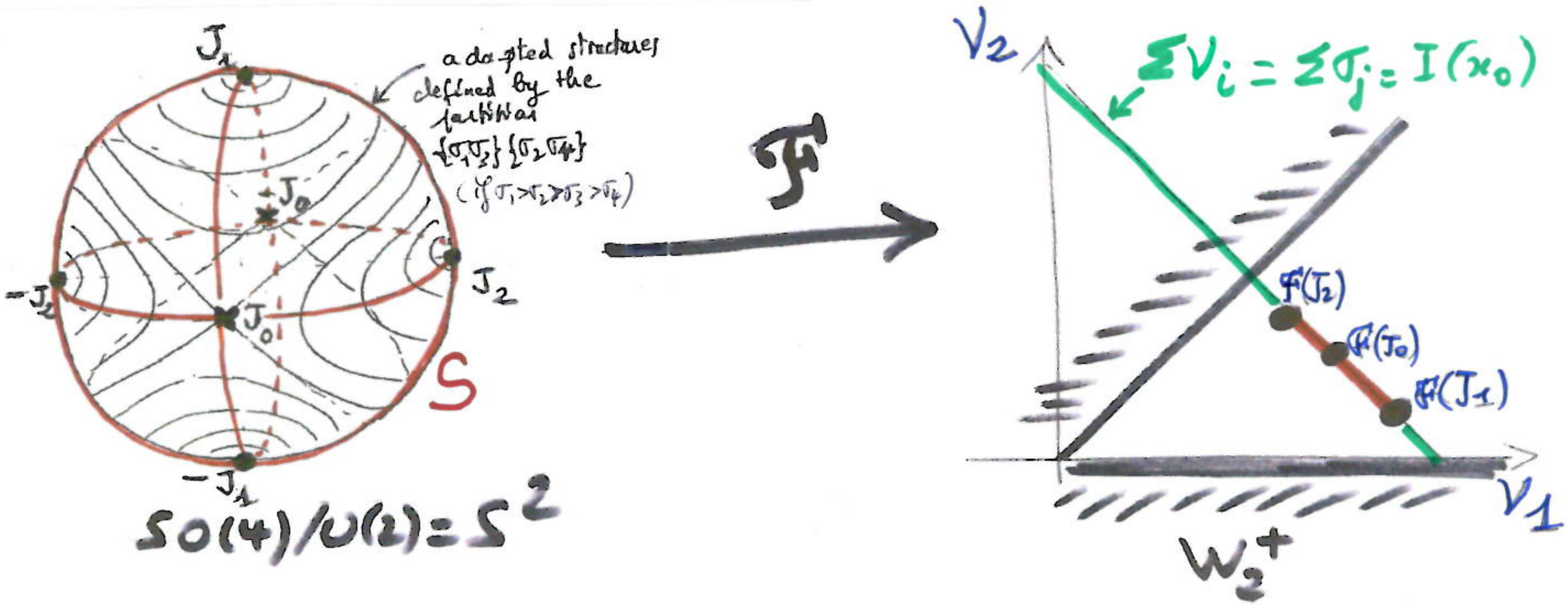
$$J = R^{-1} J_0 R \qquad (\nu_1 \geq \dots \geq \nu_p \geq 0) = \operatorname{Spec}\left(J_0^{-1} R S_0 R^{-1} J_0 + R S_0 R^{-1} \right)$$

$$(\mathbb{C}^p, J_0) \longrightarrow (\mathbb{C}^p, J_0)$$

Problem : $\boxed{\text{determine } \operatorname{Im} \mathcal{F} \text{ for a given } S_0 .}$

Obvious relation: $\displaystyle\sum_{i=1}^{p} \nu_i = \sum_{j=1}^{2p} \sigma_j = I(x_0)$

THE CASE OF $\mathbb{R}^4$



A [Red circles = <u>adapted</u> complex structures
$$R=\begin{pmatrix}\rho & 0\\ 0 & \mathrm{Id}\end{pmatrix}P\ ,\quad J=P^{-1}\begin{pmatrix}0 & -\rho^{-1}\\ \rho & 0\end{pmatrix}P\ ,\quad \rho\in O(p),\ P \text{ signed permutation}$$

B [Black dots = <u>basic</u> complex structures
$$R=P,\qquad J=P^{-1}J_0P$$

Proposition : The critical points of $\mathcal{F}$ are exactly the basic complex structures $\pm J_0, \pm J_1, \pm J_2$

$\mathbb{R}^4$

Proof : explicit computation with spherical coordinates of the critical pts of $J \longmapsto v_1, v_2$.

Corollary : in the case of $\mathbb{R}^4$,
$$\mathrm{Im}\,\mathcal{F} = \mathcal{F}(\mathcal{A})$$

In fact, we have even that $\exists$ 1 of the 3 red circles, S, such that $\mathrm{Im}\,\mathcal{F} = \mathcal{F}(S)$

$$S_0 = \begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_{2p}\end{pmatrix}$$

HIGHER DIMENSIONS

$$J_{\rho,P} = P^{-1}\begin{pmatrix}0 & -\rho^{-1}\\ \rho & 0\end{pmatrix}P = R_{\rho,P}^{-1} J_0 R_{\rho,P}, \qquad R_{\rho,P} = \begin{pmatrix}\rho & 0\\ 0 & Id\end{pmatrix}P$$

$\rho \in SO(p)$, $P \in SO(2p)$ signed permutation

$\mathcal{B}$ $\boxed{\text{BASIC Complex structures}}$ $\subset$ $\mathcal{A}$ $\boxed{\text{ADAPTED Complex structures}}$

$J_{Id,P}$

fixed by group of order 2^p of involutions $J \mapsto \eta D'' J D''$,

where $P D'' P^{-1} = \mathrm{diag}(\tilde{v}_1, \dots, \tilde{v}_{2p})$,

with $\eta \tilde{v}_i \tilde{v}_{p+i} = +1$, $i = 1, \dots, p$.

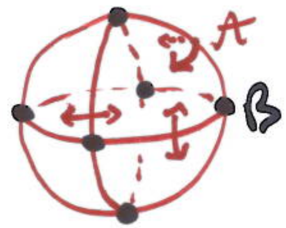
$\eta = \pm 1$

$J_{\rho,P}$

fixed by involution $J \longmapsto -D''_P J D''_P$,

$$D''_P = P\begin{pmatrix}Id_{p\times p} & 0\\ 0 & -Id_{p\times p}\end{pmatrix}P^{-1}$$

ex $p=2$:



PARTITIONS OF THE INERTIA SPECTRUM

$$\vec{P}^{1}(\vec{e}_i) = \varepsilon_i \vec{e}_{\pi(i)}$$

$$\begin{array}{ccccc} 1 & 2 & \cdots & \cdots & 2p \\ & & \downarrow \pi & & \\ \pi(1) & \pi(2) & \cdots & \cdots & \pi(2p) \end{array}$$

$J_{Id,p}$ BASIC

$$\{\sigma_{\pi(1)}, \sigma_{\pi(p+1)}\}$$
$$\{\sigma_{\pi(2)}, \sigma_{\pi(p+2)}\}$$
$$\vdots$$
$$\{\sigma_{\pi(p)}, \sigma_{\pi(2p)}\}$$

$J_{g,p}$ ADAPTED

$$\{\sigma_{\pi(1)}, \cdots, \sigma_{\pi(p)}\} = \sigma_-^{\pi}$$

$$\{\sigma_{\pi(p+1)}, \cdots, \sigma_{\pi(2p)}\} = \sigma_+^{\pi}$$

STRUCTURE OF $\mathcal{F}(\mathcal{A})$ (1)

$$\mathcal{A} = \bigcup_{\underline{p}} \mathcal{A}_{\underline{p}} \ , \ \mathcal{A}_{\underline{p}} = \bigcup_{\rho} J_{\rho,\underline{p}} \ , \ J_{\rho,\underline{p}} = R_{\rho,\underline{p}}^{-1} J_0 R_{\rho,\underline{p}}$$

$$S_0 = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{2p} \end{pmatrix} \ , \ S_{\rho,\underline{p}} = R_{\rho,\underline{p}} S_0 R_{\rho,\underline{p}}^{-1}$$

$$\Sigma_{\rho,\underline{p}} = J_0^{-1} S_{\rho,\underline{p}} J_0 + S_{\rho,\underline{p}} = \begin{pmatrix} 0 & -(\rho\sigma_-^{\pi}\rho^{-1} + \sigma_+^{\pi}) \\ (\rho\sigma_-^{\pi}\rho^{-1} + \sigma_+^{\pi}) & 0 \end{pmatrix} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$$

$\sigma_-^{\pi}, \sigma_+^{\pi}$ diagonales

$$\Updownarrow$$

$$\boxed{\Sigma_{\rho,\underline{p}} = \rho\,\sigma_-^{\pi}\rho^{-1} + \sigma_+^{\pi} : \mathbb{C}^p \to \mathbb{C}^p}$$

real !

STRUCTURE OF $F(\mathcal{A})$ (2)

... Hence

$$\mathcal{F}(\mathcal{A}_P) = \left\{ \begin{array}{l} (\nu_1 \geq \dots \geq \nu_p) = \text{spectrum of } A+B, \\ A, B : \mathbb{R}^p \to \mathbb{R}^p \text{ symmetric}, \\ \quad \operatorname{Spec} A = \sigma_-^\pi \ , \ \operatorname{Spec} B = \sigma_+^\pi \end{array} \right\}$$

$$\mathcal{F}(\mathcal{A}) = \left\{ \begin{array}{l} (\nu_1 \geq \dots \geq \nu_p) = \text{spectrum of } A+B, \\ A, B : \mathbb{R}^p \longrightarrow \mathbb{R}^p \text{ symmetric}, \\ \quad \operatorname{Spec} A \amalg \operatorname{Spec} B = \{\sigma_1, \dots, \sigma_{2p}\} \end{array} \right\}$$

Of course,

$$\mathcal{F}(J_{Id,p}) = \left(\sigma_{\pi(1)} + \sigma_{\pi(p+1)}, \dots, \sigma_{\pi(p)} + \sigma_{\pi(2p)} \right),$$

up to reordering.

OH! BUT THIS IS HORN'S PROBLEM!

Theorem 1 | Horn, Guillemin-Sternberg, Kirwan

$\forall P$, $\mathcal{F}^{-1}(\mathcal{A}_P)$ = convex polytope

= image of the moment map of the diagonal action of $U(p)$ on the product $\mathcal{O}_{\sigma_-^\pi} \times \mathcal{O}_{\sigma_+^\pi}$ of the classes of conjugation under $U(p)$ of hermitian matrices, identified with coadjoint orbits.

Theorem 2 | Klyachko, Knutson & Tao

The faces of this polytope are defined by Horn's inequalities (Horn's conjecture).

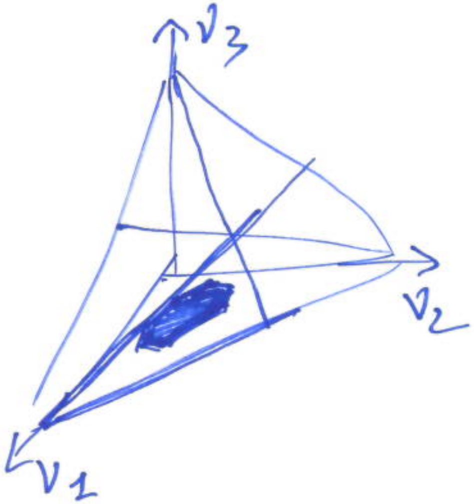
Theorem 3 | Fomin, Fulton, Li, Poon

If $\underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2p}}$ and $P_0 \to \left(\begin{array}{l} \sigma_-^\pi = \{\sigma_1, \sigma_3, \dots, \sigma_{2p-1}\} \\ \sigma_+^\pi = \{\sigma_2, \sigma_4, \dots, \sigma_{2p}\} \end{array} \right.$

$$\mathcal{F}^{-1}(\mathcal{A}) = \mathcal{F}^{-1}(\mathcal{A}_{P_0})$$

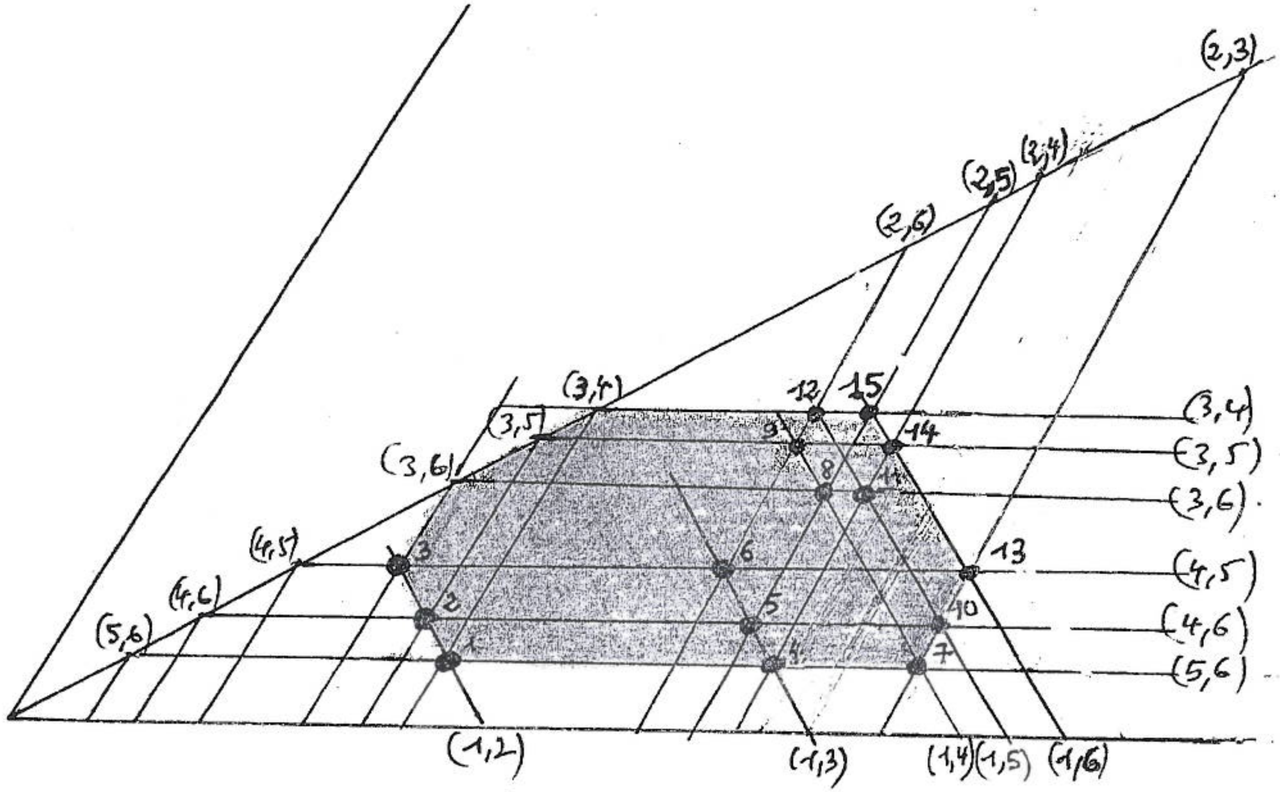
(ex $p=2$  $\mathcal{A}_{P_0}$)

AN EXAMPLE IN $\mathbb{R}^6$



$$\sigma_1+\sigma_2>\sigma_1+\sigma_3>\sigma_1+\sigma_4>\sigma_1+\sigma_5>\sigma_1+\sigma_6>$$
$$\sigma_2+\sigma_3>\sigma_2+\sigma_4>\sigma_2+\sigma_5>\sigma_2+\sigma_6>$$
$$\sigma_3+\sigma_4>\sigma_3+\sigma_5>\sigma_3+\sigma_6>$$
$$\sigma_4+\sigma_5>\sigma_4+\sigma_6>$$
$$\sigma_5+\sigma_6.$$

$$\mathcal{F}(\mathcal{A}_{P_0}) = \mathcal{F}(\mathcal{A})$$



A QUESTION (Conjecture?)

$$\boxed{\operatorname{Im} \mathcal{F} = \mathcal{F}(\mathcal{A})}$$

$= \operatorname{conv} \mathcal{F}(\mathcal{B})$ after symmetrization?

? ? ?

$\Longleftrightarrow$? does every level surface of $\mathcal{F}$ possess a fixed pt of one of the involutions?

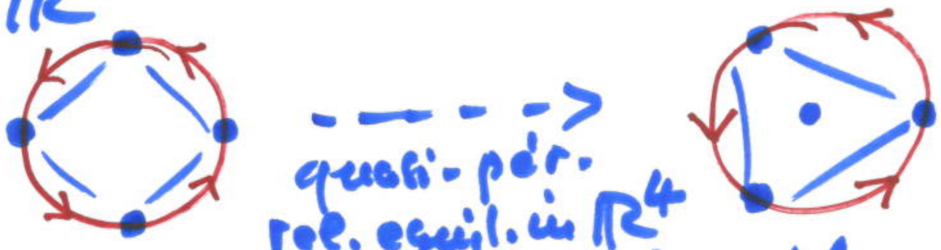
ANOTHER QUESTION

Is $\mathfrak{F}$ a moment map?

$\mathbb{Z}$ nothing to do with the symplectic structure of the phase space, as the motion takes place in a Lagrangian submanifold (fiber of the cotangent bundle).

FINALLY, WHY $\mathbb{R}^{N>3}$?

- Reduction of symmetries is much more transparent (Albouy, Chenciner)
- $\exists$ of natural quasi-periodic families in $\mathbb{R}^N$ between periodic solutions in $\mathbb{R}^2$ or $\mathbb{R}^3$

ex : 

- Mathematics is even more fun than astronomy