ANGULAR MOMENTUM

and

HORN'S PROBLEM

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Dedicated to the memory of Jean Marie Souriau,
with admiration
Why study the N-body problem in an euclidean space of dimension $\geq 4$?
1. Reduction of symmetries is made transparent:

J.L. LAGRANGE, 1772 "Essai sur le problème des 3 corps"

10 reduced equations
3 first integrals
{ Energy
  2 invariants of the angular momentum
  vectors in $\mathbb{R}^4$

Indeed, lying in $\mathbb{R}^3$ is a constraint (degeneracy of the angular momentum vectors)

\[ \Rightarrow \text{study } \mathbb{R}^{2(n-1)} \]
Relativity equilibria are richer:

**Periodic (central configuration)**

\[ x(t) = e^{\text{J} \omega t} x(0) \]

\( \omega \), anisotropic non-degenerate on \( E^{2p} \)

\( \text{J complex structure on } E^{2p} \)

(hermitian)

\( \sum m \mathbf{r}_i (\mathbf{H} = 0) \)

**Quasi-periodic (balanced configuration)**

\[ x(t) = e^{\Omega t} x(0) \]

Ex: 3 equal masses

Equilateral triangle in \( E^2 \)

Isosceles triangle in \( E^4 \)

\text{Bifurcations?}

A. Albouy & A. C. "Le problème des } \text{N} \text{ corps et les distances mutuelles}"

Inventiones 1998
Matrix notations (very useful!)

Configuration
\[ X = \begin{pmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{pmatrix} \]
Coordinates of the \( \vec{x}_i \) in an orthogonal basis of \( \mathbb{E}^{d=2p} \)

Configuration of velocities
\[ Y = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \]

Masses
\[ \mu = \begin{pmatrix} m_1 & \cdots & 0 \\ 0 & \cdots & m_n \end{pmatrix} \]

Inertia matrix
\[ S = X \mu^T X = \begin{pmatrix} d_{ij} \end{pmatrix}, \quad d_{ij} = \sum_k m_k \vec{x}_i \vec{x}_j \vec{x}_k \]

Angular momentum
\[ \vec{L} = -X \mu^T \vec{y} + Y \mu^T \vec{x} = \begin{pmatrix} 0 & \vec{c}_{ij} \\ -\vec{c}_{ij} & 0 \end{pmatrix}, \quad \vec{c}_{ij} = \sum_k m_k \left( -\vec{x}_i \vec{x}_j \vec{v}_k + \vec{x}_j \vec{x}_i \vec{v}_k \right) \]
Central configurations

Critical points of

\[ U \mid (I = \text{cst}) \]

\[ I = \text{trace } S = \sum_{i=1}^{3} \frac{1}{m_i} \mathbf{x}_i \mathbf{x}_i^T \]

= moment of inertia w.r.t. the center of mass

\[ U = \sum_{i=1}^{3} \frac{w_i}{| \mathbf{x}_i - \mathbf{x}_j |} \]

= Lagrange potential function

(= - potential energy)

Balanced configurations

Critical points of

\[ U \mid \text{Spectre } (S = \text{cst}) \]

\[ S = X \mu X \text{ inertia matrix} \]
ANGULAR MOMENTUM OF A RELATIVE EQUILIBRIUM

\[ Y = \pi X \Rightarrow \mathcal{E} = S \Omega + \Omega S = \frac{S_0 \Omega + \Omega S_0}{J} \]

\( S_0 = \text{matrix of inertia of } x(0) \)

FOR A CENTRAL CONFIGURATION:

\[ Y = \omega J X \Rightarrow \mathcal{E} = \omega \left( S_0 J + J S_0 \right) \]

J-skew hermitian

(Spectrum \( \pm iv_1, \ldots, \pm iv_p \))

More conveniently, we can write

\[ \frac{1}{\omega} J^{-2} \mathcal{E} = J S_0 J + S_0 \]

J-hermitian

\( \{ \text{Spectrum } v_1, \ldots, v_p \text{ as J-complex matrix} \}

\( \{ v_i, v_i, \ldots, v_i \text{ as real matrix} \} \)
FREQUENCY MAP

\[ \mathcal{F} \]

\[ \text{given } S_0 \text{ is symmetric, } \]

\[ U(\mathbb{P}) \xrightarrow{\mathcal{F}} \mathcal{W}_p^+ \subset \mathbb{R}^p \]

\[ V_1 \geq \ldots \geq V_p \]

ordered spectrum of the J-hermitian matrix

\[ J^+ S_0 J + S_0 \]

Problems

1) Is \( \text{Im } \mathcal{F} \)?

2) Points in \( \text{Im } \mathcal{F} \) corresponding to bifurcations c.c \( \rightarrow \) b.c, p.c \( \rightarrow \) q.p.c.
Theorem: \( \text{Im } \tilde{F} \) is a convex polytope contained in the \((p-1)\) dimensional intersection of \( W_p^+ \) with \( \frac{\pi}{\lambda} \sum_{i=1}^{p} V_i = \text{trace } S_0 \).

Ref: • A.C., The angular momentum of a relative equilibrium, arXiv: 1102.0025
• A.C. & Hugo Jiménez-Pérez, Angular momentum and Horn's problem, arXiv 1110.5030

The bifurcation values are contained in the faces of subpolytopes of \( \text{Im } \tilde{F} \).
Structure of the proof

1. One produces to convex polytopes $P_1, P_2$, both contained in the intersection of $W_0^+$ with $\Sigma_{i=1}^{r} S_i$ such that

$$P_1 \subset \text{Im} F \subset P_2$$

related to a Horn problem in dim $p$

related to a Horn problem in dim $2p$

2. One notices that as a result of Fanin-Fulton-Koontz, itself based on a deep combinatorial lemma of Carse and Leclerc, implies that

$$P_1 = P_2$$
THE POLYTOPE $\mathcal{P}_1$

From now on, choose a basis of $\mathbb{E}^{2p}$ such that

\[ S_0 = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \ddots & \sigma_{2p} \\ \end{pmatrix}, \quad \sigma_1 \geq \cdots \geq \sigma_{2p}. \]

Definition: Given a (signed) permutation $2p \times 2p$ matrix $P$, a $P$-adapted complex structure $J$ is a complex structure of the form

\[ J = J_{g,P} = P^{-1} \begin{pmatrix} 0 & -g^2 \\ g & 0 \end{pmatrix} P, \quad g \in SO(p). \]

$J_{g,P}$ is "adapted" to symmetries of the inertia ellipsoid defined by $S$. 
To \( P \) correspond a partition \( \Pi \)

\[
\{ \sigma_{i}^{\pm} \} = \{ \sigma_{1}, \ldots, \sigma_{2p} \}
\]

of the spectrum of \( S_{0} \) into two equal parts, defined by

\[
P S_{0} P^{-1} = \begin{pmatrix}
\sigma_{\pi(1)} & \sigma_{-} & \sigma_{\pi(p+1)} \\
\sigma_{\pi(p)} & O & \sigma_{-} \\
\sigma_{\pi(2p)} & O & \sigma_{+}
\end{pmatrix}
\]

(We name also \( \sigma_{-}, \sigma_{+} \) the diagonal \( p \times p \) matrices)
Two characterizations of the P-adopted structures

1. They are the J's such that

\[ \langle \overrightarrow{e}_{\pi(1)}, \ldots, \overrightarrow{e}_{\pi(p)} \rangle \xrightarrow{J} \langle \overrightarrow{e}_{\pi(p+1)}, \ldots, \overrightarrow{e}_{\pi(2p)} \rangle, \]

where \( \{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_{2p}\} \) is the chosen eigenbasis of \( S_0 \).

2. They are the J's invariant under the involution

\[ J \mapsto \ -D_p J D_p \ , \]

where \( D_p = P (\text{Id} \ 0) \text{Id} P^{-1} \).

Notice that \( F'(-D_p J D_p) = F'(J) \)
(obvious) FACT:
\[ J_{p,p}^{-1} S_0 J_{p,p} + S_0 = \begin{pmatrix} \sigma_- + \bar{\sigma}_+ \sigma_+ & 0 \\ 0 & \sigma_- \bar{\sigma}_+ + \sigma_+ \end{pmatrix} \]

\[ \Sigma^1 (P\text{-adapted structure}) \]
= \{ \text{possible ordered spectra of real symmetric matrices} \}
= \{ c = a + b, \text{ with } \text{sp}(a) = \sigma_-, \text{sp}(b) = \sigma_+ \}

= \textbf{Horn polytope in dimension } p

= \text{intersection with the principal Weyl chamber of the image of the moment map of the diagonal action of } U(p) \text{ on the product of the coadjoint orbits } U(p) \times \mathbb{O}_{-1} \times \mathbb{O}_{\sigma_+} \]
Definition: \[ S_1 = \bigcup_{P} \mathcal{F}(P, \text{ adopted structure}) \]

\[ \{ \text{possible ordered spectra of real symmetric matrices} \} \]
\[ C = a + b, \text{ with } \text{sp}(a) \cup \text{sp}(b) = \{ \sigma_1, \ldots, \sigma_{2p} \} \]

Proposition (Fomin, Fulton, L., by)

\[ S_2 = \mathcal{F}(P_0, \text{ adopted structure}), \]

where \( P_0 \) corresponds to the partition

\[ \nabla_- = \{ \sigma_1, \sigma_2, \ldots, \sigma_{2p-2} \}, \quad \nabla_+ = \{ \sigma_2, \sigma_4, \ldots, \sigma_{2p} \} \]

Hence \( S_2 \) is a convex polytope.
Example: $p = 2, \quad U(2) \backslash SO(4) \cong S^2$

\[ J_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
Definition: \( \mathcal{P} = \left\{ \text{possible ordered spectra of real } 2p \times 2p \text{ symmetric matrices } C = A + B, \text{ with} \right\} \\
\text{sp}(A) = \text{sp}(B) = \text{sp}(S_0) = \{0_{2}, \ldots, 0_{2p}\} \\
(\text{How } p \text{ in dim. } 2p) \\

Recall that \( C = J^{-1} S_0 J + S_0 \) is J-hermitian

Lemma: The symmetric matrix \( C \) is J-hermitian for some \( J \) iff \( \text{sp}(C) = \{v_1, v_1, \ldots, v_p, v_p\} \)

Notation: \( \Delta \subset \mathcal{W}_{2p}^+ = \{(\xi_1, \ldots, \xi_{2p}), \xi_1 = \xi_2, \xi_3 = \xi_4, \ldots, \xi_{2p-1} = \xi_{2p}\} \)

Definition: \( \mathcal{P}_2 = \{(v_1, \ldots, v_p) \in \mathcal{W}_p^+, (v_1, v_1, \ldots, v_p, v_p) \in \mathcal{P}\} \)

Obviously \( \mathcal{P}_1 \subset \text{Im } J \subset \mathcal{P}_2 \)
**Theorem:** \( \text{Im } \delta \subseteq \text{Im } \omega \)

\[ \mathcal{W}_p^+ \supseteq \mathcal{F}'(P_0\text{-adapted structures}) \xrightarrow{\delta} \Delta \equiv \mathcal{W}_p^+ \]

\( \omega \) orthogonal projection

\[ \delta (v_1, \ldots, v_p) = (v_1, v_1, \ldots, v_p, v_p) \]

\[ \omega (\gamma_1, \ldots, \gamma_{2p}) = \left( \frac{\gamma_1 + \gamma_2}{2}, \ldots, \frac{\gamma_{2p-2} + \gamma_{2p}}{2} \right) \]

**Corollary:** \( \mathcal{P}_1 = \text{Im } \mathcal{F}' = \mathcal{P}_2 \)
For $p = 2$:

The subset of $P_0$-adapted structures meets every level hypersurface of $\mathcal{F}$.
**Proof:** it follows from

1. The description of the Horn polytope by the so-called Horn's inequalities (Klyachko, Knutson & Tao):

   \[ \exists A, B, C = A + B \text{ n x n Hermitian (resp. real symmetric) matrices s.t.} \]

   \[ \mathcal{Sp} A = \{ \alpha_1, \ldots, \alpha_n \}, \quad \mathcal{Sp} B = \{ \beta_1, \ldots, \beta_n \}, \quad \mathcal{Sp} C = \{ \gamma_1, \ldots, \gamma_n \} \text{ iff} \]

   \[ \sum_{k=1}^{n} \delta_k \geq \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j \]

   \[ \sum_{k \in K} \delta_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \]

   where \( I = \{ i_1, \ldots, i_k \} \), \( J = \{ j_1, \ldots, j_k \} \), \( K = \{ k_1, \ldots, k_k \} \)

   and \( T^n_i \) is defined recursively.

   **Equivalent characterization:** \[ \mathcal{C} \alpha \beta > 0 \]

   Littlewood-Richardson coefficient
A combinatorial lemma (Fomin, Falta, Li, Poon), based on a result of Carre & Leclerc on the domino decompositions of Young diagrams:

\[(I, J, K) \in T^p_2 \Rightarrow (I_2, J_2, K_2) \in T^{2p}_{2n}\]

where

\[I_2 = J_2 = \{2i_1-1, \ldots, 2i_2-1\} \cup \{2j_1, \ldots, 2j_2\}\]

\[K_2 = \{2k_1-1, \ldots, 2k_{2n}-1\} \cup \{2k_1, \ldots, 2k_{2n}\}\]

Hence, if \(\sigma \in S\), with \(\alpha = \beta = \sigma = \{\sigma_1 > \ldots > \sigma_{2p}\}\),

\[\forall (I, J, K) \in T^p_2, \text{ one has}\]

\[\sum_{h' \in K_2} \frac{1}{\sum_{i' \in I_2} + \sum_{j' \in J_2}}, \text{ that is}\]

\[\sum_{h' \in K_2} (\sigma_{2h-1} + \sigma_{2h}) \leq 2 \left( \sum_{i' \in I_2} \frac{1}{\sigma_{2i-1}} + \sum_{j' \in J_2} \right), \text{ i.e. } \sum_{i' \in I_2} = S_{2n}^{\sigma}(h) \in S_4\]

cqfd
$P_1: 25000$ random $P_0$-adopted structures
Orthogonal projection of $g_{5\Delta}^2$ on $\Delta = 50000$ random points

$p = 3$
**Bifurcations**: they occur necessarily for values of the angular momentum whose spectrum lies on the boundary of the subpolytopes $\mathcal{T}_d$ ($P$-adapted structures)

- **Example**: 3 bodies, $p = 2$:
  
  $S_0 = \text{diag}(0, 0, 0, 0)$
  
  ![Bifurcation Vertex](image)

- Some of the 20 subpolytopes for $P = 3$:

  ![Diagram](image)
APPENDIX: definition of $T^n_2$

$U^n_2 = \{ (I, J, K), \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} \frac{k(k+1)}{2} \}$

$T^n_2 = U^n_2$

$T^n_2 = \{ (I, J, K) \in U^n_2, \forall k < 2, \forall (F, G, H) \in T^n_2, \}$

$\sum_{f \in F} f + \sum_{g \in G} g \leq \sum_{k \in K} \frac{k(k+1)}{2}$

Simplest case: $T^n_1 = U^n_1$: Weyl's $\leq$ (Mathi Annalen)

71 (1912) 441-479

\[ \gamma_i + \gamma_j + 1 \leq \alpha_i + 1 + \beta_j + 1 \]

Easy proof

Easy to see that

Weyl's $\leq$ for $\gamma_1$ and $\beta_2$ are the same.