ANGULAR MOMENTUM and HORN'S PROBLEM

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Dedicated to the memory of
Jean-Marie SOURIAU,
with admiration

Why study the N-body problem in an euclidean space of dimension $\geqslant 4$

?

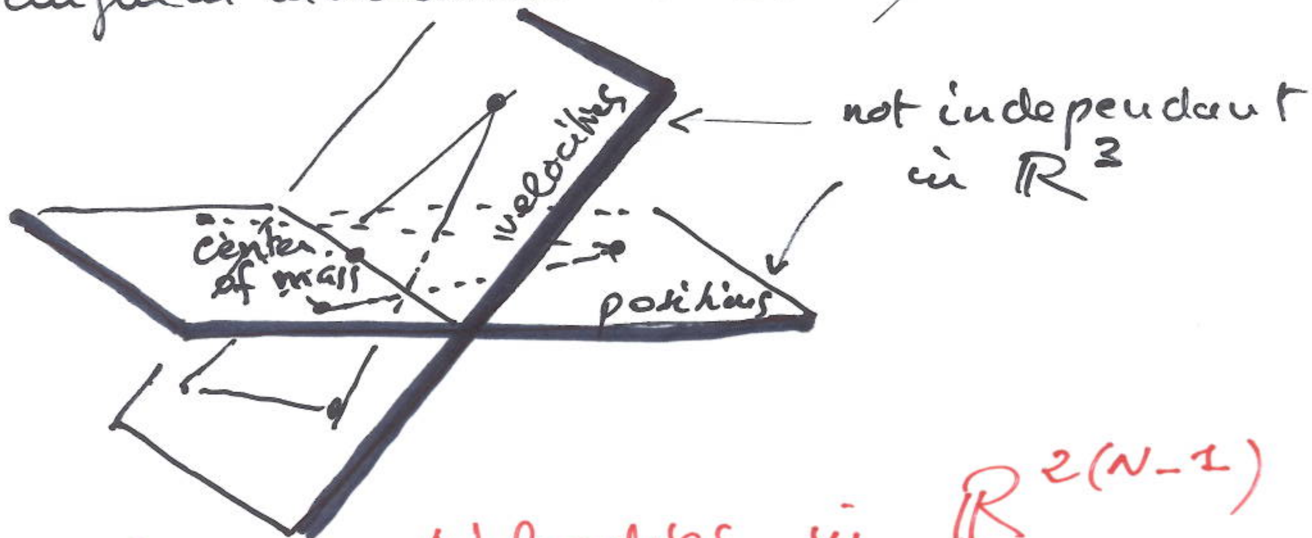
(1) Reduction of symmetries is made transparent :

J.L. LAGRANGE, 1772 "Essai sur le problème des 3 corps"

10 reduced equations

3 first integrals $\begin{cases} \text{Energy} \\ \text{2 invariants of the angular momentum bivector in } \mathbb{R}^4 \end{cases}$

Indeed, lying in $\mathbb{R}^3$ is a constraint (degeneracy of the angular momentum bivector)



$\Longrightarrow$ Study N-bodies in $\mathbb{R}^{2(N-1)}$

(2) Relative equilibria are richer :

Periodic (central configuration)

$$x(t) = e^{J\omega t}\,x(0)$$

$$\| \qquad\qquad \|$$

$$\left(\vec{r}_1(t),\dots,\vec{r}_N(t)\right) \quad \left(e^{J\omega t}\vec{r}_1(0),\dots,e^{J\omega t}\vec{r}_N(0)\right)$$

$$\sum_k m_k \vec{r}_k(t) \equiv 0$$

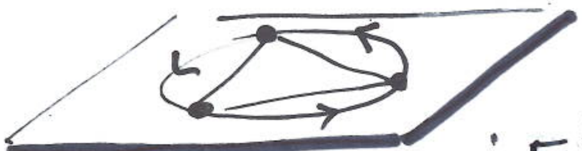
J complex structure on E^{2p} (hermitian)

Quasi-periodic (balanced configuration)

$$x(t) = e^{\Omega t}\,x(0)$$

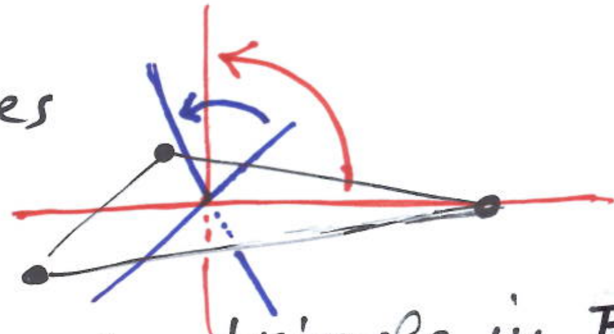
Ω, antisymmetric non degenerate on E^{2p}

ex : 3 equal masses



Equilateral triangle in E^2

? BIFURCATIONS ?



$\forall$ isosceles triangle in E^4

A. Albouy & A.C. "Le problème des N corps et les distances mutuelles" Inventiones 1998

Matrix notations (very useful!)

Configuration $X = \begin{array}{c} \begin{array}{ccc} \vec{r}_1 & & \vec{r}_N \end{array} \\ \begin{pmatrix} | & \cdots & | \end{pmatrix} \end{array}$ ← coordinates of the $\vec{r}_i$ in an orthonormal basis of $E^{d=2p}$

Configuration of velocities $Y = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & & \vec{v}_N \end{array} \\ \begin{pmatrix} | & \cdots & | \end{pmatrix} \end{array}$

$$\boxed{\begin{array}{l} \sum m_k \vec{r}_k = 0 \\ \sum m_k \vec{v}_k = 0 \end{array}}$$

Masses $\mu = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}$

$\vec{r}_k = (x_{1k}, \dots, x_{dk})$

$\vec{v}_k = (y_{1k}, \dots, y_{dk})$

Inertia matrix $\boxed{S = X \mu\, {}^tX} = \begin{pmatrix} & s_{ij} \\ s_{ij} & \end{pmatrix}$, $s_{ij} = \sum_k m_k x_{ik} x_{jk}$

Angular momentum

$\boxed{\mathcal{C} = -X\mu\, {}^tY + Y\mu\, {}^tX} = \begin{pmatrix} 0 & c_{ij} \\ -c_{ij} & 0 \end{pmatrix}$, $c_{ij} = \sum_k m_k \left(-x_{ik} v_{jk} + x_{jk} v_{ik}\right)$

Central configurations

$\shortparallel$

critical points of

$$U\big|_{(I = cst)} \,,$$

$$I = \operatorname{trace} S = \sum_k m_k \|\vec{r}_k\|^2$$

= moment of inertia w.r.t. the center of mass

Balanced configurations

$\shortparallel$

critical points of

$$U\big|_{(\text{Spectre de } S = cst)}$$

$$S = X \mu \, {}^t X = \text{inertia matrix}$$

$$U = \sum_{i<j} \frac{m_i m_j}{\|\vec{r}_i - \vec{r}_j\|} = \text{Lagrange potential function}$$

$$(= - \text{potential energy})$$

ANGULAR MOMENTUM OF A RELATIVE EQUILIBRIUM

$$Y = \Omega X \implies \mathcal{C} = S\Omega + \Omega S = \boxed{S_0\Omega + \Omega S_0}$$

(S_0 = matrix of inertia of $x(0)$)

FOR A CENTRAL CONFIGURATION :

$$Y = \omega J X \implies \boxed{\mathcal{C} = \omega(\underbrace{S_0 J + J S_0}_{})}$$

J- skew hermitian
(Spectrum $\pm i\nu_1, \dots, \pm i\nu_p$)

More conveniently, we consider

$$\boxed{\frac{1}{\omega} J^{-1}\mathcal{C} = J^{-1} S_0 J + S_0}$$ J-hermitian

{ Spectrum $\nu_1, \dots, \nu_p$ as J-complex matrix
$\nu_1, \nu_1, \dots, \nu_p, \nu_p$ as real matrix

FREQUENCY MAP

given S_0 $2p\times 2p$ symmetric,

$$\underset{\substack{\text{(oriented)}\\ \text{complex}\\ \text{structures}\\ \text{on } E^{2p}}}{U(p)\backslash SO(2p)} \xrightarrow{\mathcal{F}} W_p^+ \subset \mathbb{R}^p$$

$$\begin{aligned} J &= R^{-1} J_0 R \longmapsto (\nu_1 \geq \dots \geq \nu_p) \\ &= (UR)^{-1} J_0 (UR) \end{aligned}$$

ordered spectrum of the J-hermitian matrix $J^{-1} S_0 J + S_0$

$$J_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

Problems

1) $\mathrm{Im}\,\mathcal{F}$?

2) Points in $\mathrm{Im}\,\mathcal{F}$ corresponding to bifurcations $\underset{\text{per.}}{C.C} \to \underset{\text{q. per.}}{B.C}$

Theorem: $\mathrm{Im}\,\mathcal{J}'$ is a convex polytope contained in the $(p-1)$ dimensional intersection of W_p^+ with $\sum_{i=1}^{p} \nu_i = \mathrm{trace}\, S_0$

Ref: • A. C., The angular momentum of a relative equilibrium, arXiv: 1102.0025

• A.C. & Hugo Jiménez-Pérez, Angular momentum and Horn's problem, arXiv 1110.5030

• The bifurcations values are contained in the faces of sub polytopes of $\mathrm{Im}\,\mathcal{J}'$.

Structure of the proof

① One produces to convex polytopes $\mathcal{P}_1, \mathcal{P}_2$, both contained in the intersection of W_p^+ with $\sum_i v_i = \mathrm{trace}\, S_0$, such that

$$\boxed{\mathcal{P}_1 \subset \mathrm{Im}\,\mathcal{F} \subset \mathcal{P}_2}$$

related to a Horn problem in dim p

related to a Horn problem in dim $2p$

② One notices that a result of Fomin-Fulton-Li-Poon, itself based on a deep combinatorial lemma of Carré and Leclerc, implies that

$$\boxed{\mathcal{P}_1 = \mathcal{P}_2}$$

THE POLYTOPE $\mathcal{P}_1$

From now on, choose a basis of E^{2p} such that

$$S_0 = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{2p} \end{pmatrix}, \quad \sigma_1 \geq \cdots \geq \sigma_{2p} .$$

Definition: given a (signed) permutation $2p \times 2p$ matrix P, a P-adapted complex structure J is a complex structure of the form

$$\boxed{J = J_{\rho,P} = P^{-1} \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix} P \quad , \rho \in SO(p)}$$

$J_{\rho,P}$ is "adapted" to symmetries of the inertia ellipsoid defined by S.

To P corresponds a partition π

$$\boxed{\sigma_- \amalg \sigma_+ = \{\sigma_1, \dots, \sigma_{2p}\}}$$

of the spectrum of S_0 into two equal parts, defined by

$$P S_0 P^{-1} = \left(\begin{array}{c|c} \boxed{\begin{matrix} \sigma_{\pi(1)} & & \\ & \ddots & \\ & & \sigma_{\pi(p)} \end{matrix}}\ \sigma_- & 0 \\ \hline 0 & \boxed{\begin{matrix} \sigma_{\pi(p+1)} & & \\ & \ddots & \\ & & \sigma_{\pi(2p)} \end{matrix}}\ \sigma_+ \end{array}\right)$$

(We name also σ_-, σ_+ the diagonal $p \times p$ matrices)

Two characterizations of the P-adapted structures

① They are the J's such that

$$\langle \vec{e}_{\pi(1)}, \dots, \vec{e}_{\pi(p)} \rangle \xrightarrow{J} \langle \vec{e}_{\pi(p+1)}, \dots, \vec{e}_{\pi(2p)} \rangle,$$

where $\{\vec{e}_1, \dots, \vec{e}_{2p}\}$ is the chosen eigenbasis of S_0.

② They are the J's invariant under the involution

$$J \longmapsto -D_P J D_P ,$$

where $D_P = P \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} P^{-1}$.

Notice that $\mathcal{F}^{1}(-D_P J D_P) = \mathcal{F}^{1}(J)$

(obvious) FACT :

$$\boxed{J_{\rho,p}^{-1} S_0 J_{\rho,p} + S_0 = \begin{pmatrix} \sigma_- + \rho^{-1}\sigma_+\rho & 0 \\ 0 & \rho\sigma_-\rho^{-1} + \sigma_+ \end{pmatrix}}$$

$$\Downarrow$$

$\mathcal{F}^{-1}$ (P-adapted structures)

$= \left\{ \begin{array}{l} \text{possible ordered spectra of real symmetric matrices} \\ c = a + b, \text{ with } sp(a) = \sigma_-, \ sp(b) = \sigma_+ \end{array} \right\}$

$=$ Horn polytope in dimension p (idem with hermitian matrices)

= intersection with the principal Weyl chamber of the image of the moment map of the diagonal action of $U(p)$ on the product of the coadjoint orbits $\mathcal{O}_{\sigma_-} \times \mathcal{O}_{\sigma_+}$

• Definition : $\boxed{\mathcal{S}_1 = \bigcup_P \mathcal{F}(P\text{-adapted structures})}$

$= \left\{ \begin{array}{l} \text{possible ordered spectra of real symmetric matrices} \\ c = a + b, \text{ with } sp(a) \cup sp(b) = \{\sigma_1, \ldots, \sigma_{2p}\} \end{array} \right\}$

• Proposition (Fomin, Fulton, Li, Poon)

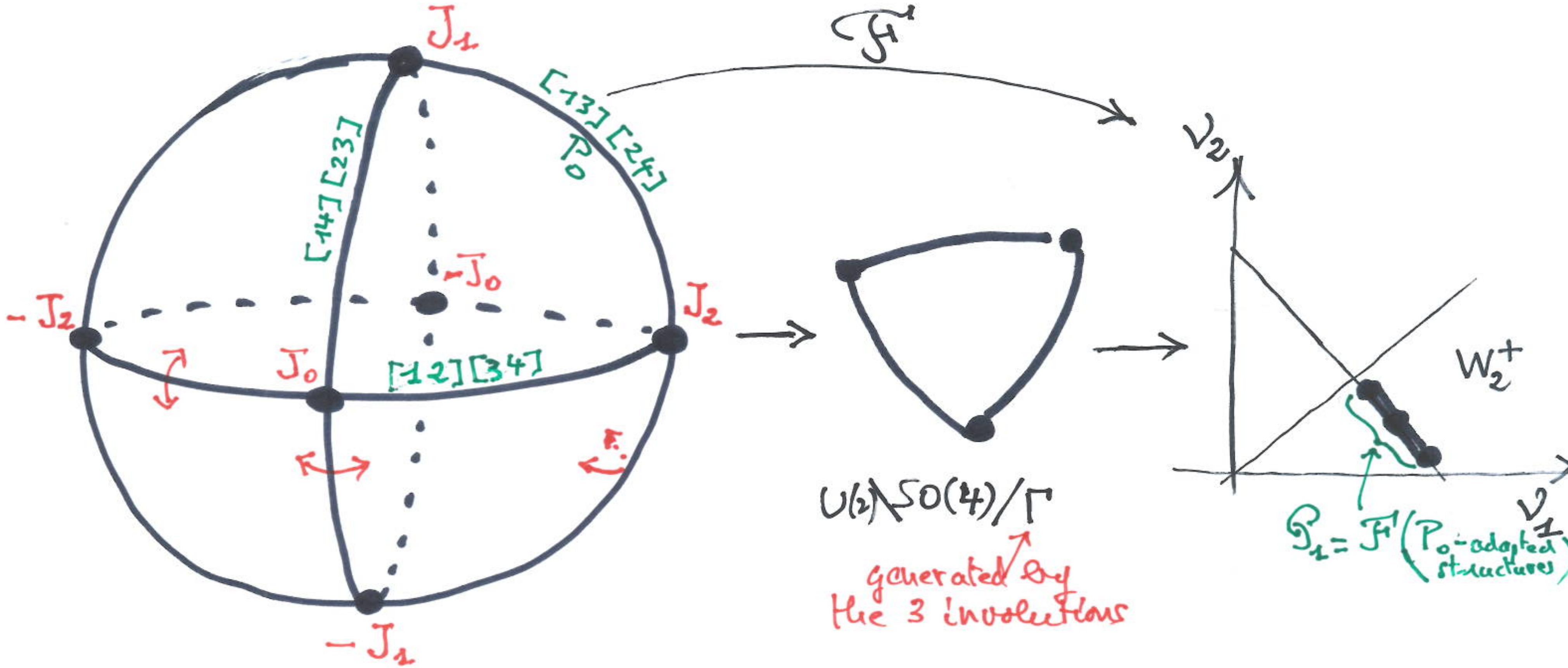
$$\mathcal{S}_1 = \mathcal{F}(P_0\text{-adapted structures}),$$

where P_0 corresponds to the partition

$$\sigma_- = \{\sigma_1, \sigma_3, \ldots, \sigma_{2p-1}\}, \quad \sigma_+ = \{\sigma_2, \sigma_4, \ldots, \sigma_{2p}\}.$$

Hence $\mathcal{S}_1$ is a convex polytope.

Example: $p = 2$, $U(2)\backslash SO(4) \cong S^2$



$$J_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

• **Definition** : $\mathcal{P} = \left\{ \begin{array}{l} \text{possible ordered spectra of real } 2p\times 2p \\ \text{symmetric matrices } C = A + B \text{, with} \\ \quad sp(A) = sp(B) = sp(S_0) = \{\sigma_1, \dots, \sigma_{2p}\} \end{array} \right\}$

(Horn pb in dim. $2p$)

Recall that $C = J^{-1} S_0 J + S_0$ is J-hermitian

Lemma : The symmetric matrix C is J-hermitian for some J iff $sp(C) = \{\nu_1, \nu_1, \dots, \nu_p, \nu_p\}$

Notation : $\Delta \subset W_{2p}^{+} = \{(\gamma_1 \geq \dots \geq \gamma_{2p}),\ \gamma_1 = \gamma_2,\ \gamma_3 = \gamma_4, \dots, \gamma_{2p-1} = \gamma_{2p}\}$

• **Definition** : $\mathcal{P}_2 = \{(\nu_1, \dots, \nu_p) \in W_p^{+},\ (\nu_1, \nu_1, \dots, \nu_p, \nu_p) \in \mathcal{P}\}$

Obviously $\mathcal{P}_1 \subset \mathrm{Im}\, \mathcal{J} \subset \mathcal{P}_2$

Theorem : $\operatorname{Im} \delta \supseteq \operatorname{Im} \varpi$

$$\mathcal{P} \subset W_{2p}^{+}$$

$$\varpi \Big\downarrow \text{orthogonal projection}$$

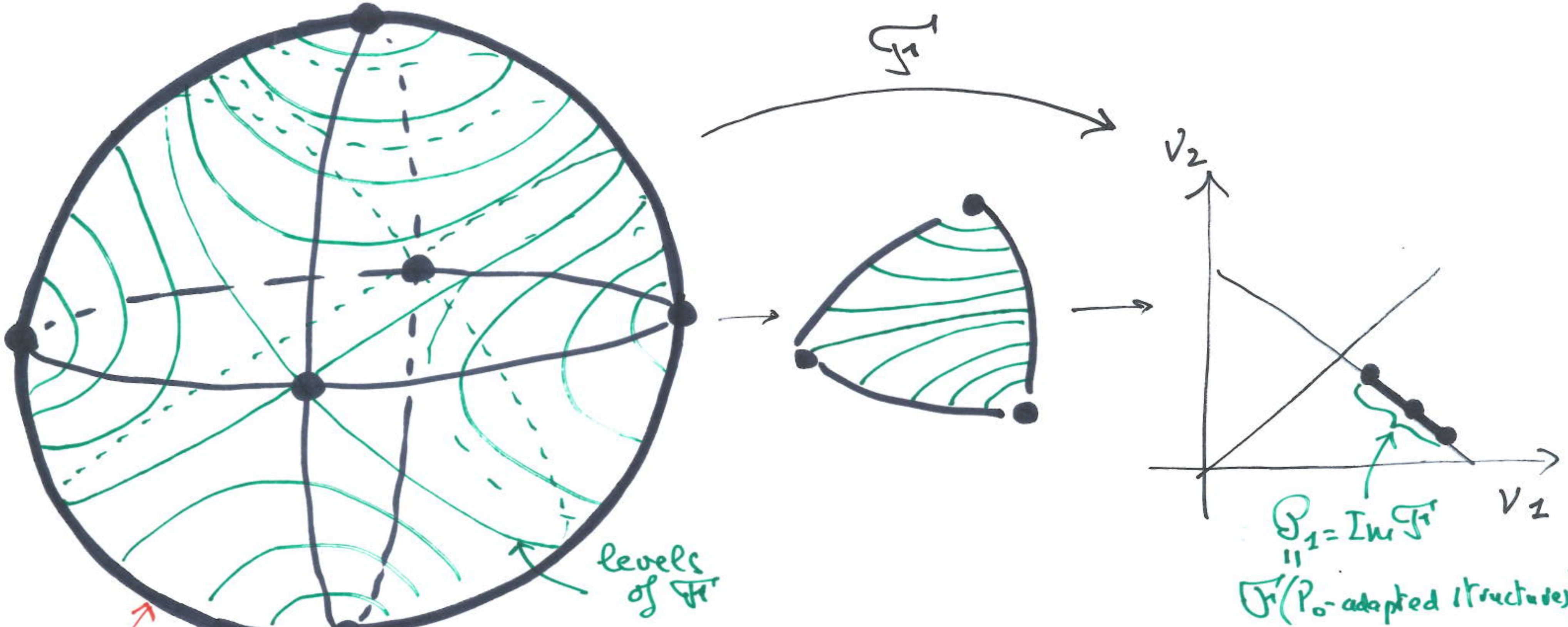
$$W_p^{+} \supset \mathcal{F}^{-1}(P_0\text{-adapted structures}) \xrightarrow[\delta]{} \Delta \equiv W_p^{+}$$

$$\delta(\nu_1, \dots, \nu_p) = (\nu_1, \nu_1, \dots, \nu_p, \nu_p)$$

$$\varpi(\gamma_1, \dots, \gamma_{2p}) = \left(\frac{\gamma_1+\gamma_2}{2}, \dots, \frac{\gamma_{2p-1}+\gamma_{2p}}{2}\right)$$

Corollary : $\mathcal{P}_1 = \operatorname{Im} \mathcal{F}^{-1} = \mathcal{P}_2$

For $p = 2$:



The subset of P_0-adapted structures meets every level hypersurface of $\mathcal{F}^1$

Proof : it follows from

(1) The description of the Horn polytope by the so-called Horn's inequalities (Klyachko, Knutson & Tao):

$\exists A, B, C = A+B$ $n \times n$ hermitian (resp. real symmetric) matrices s.t. $\operatorname{sp} A = \{\alpha_1 \geq \dots \geq \alpha_n\}$, $\operatorname{sp} B = \{\beta_1 \geq \dots \geq \beta_n\}$, $\operatorname{sp} C = \{\gamma_1 \geq \dots \geq \gamma_n\}$ iff :

$$\sum_{k=1}^{n} \gamma_k = \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j \quad \text{and}$$

$$\boxed{\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j} \quad \text{for all } r < n, \text{ all } (I,J,K) \in T_r^n,$$

where $I = \{i_1 < \dots < i_r\}$, $J = \{j_1 < \dots < j_r\}$, $K = \{k_1 < \dots < k_r\}$

and T_r^n is defined recursively.

Equivalent characterization : $c_{\alpha\beta}^{\gamma} > 0$

Littlewood-Richardson coefficient

(2) A combinatorial lemma (Fomin, Fulton, Li, Poon), based on a result of Carré & Leclerc on the domino decompositions of Young diagrams:

$$\boxed{(I,J,K)\in T_r^p \Longrightarrow (I_2,J_2,K_2)\in T_{2r}^{2p}},$$

where

$$\left.\begin{aligned} I_2 = J_2 &= \{2i_1-1,\dots,2i_r-1\}\cup\{2j_1,\dots,2j_r\} \\ K_2 &= \{2k_1-1,\dots,2k_r-1\}\cup\{2k_1,\dots,2k_r\}\end{aligned}\right)\text{reordered}$$

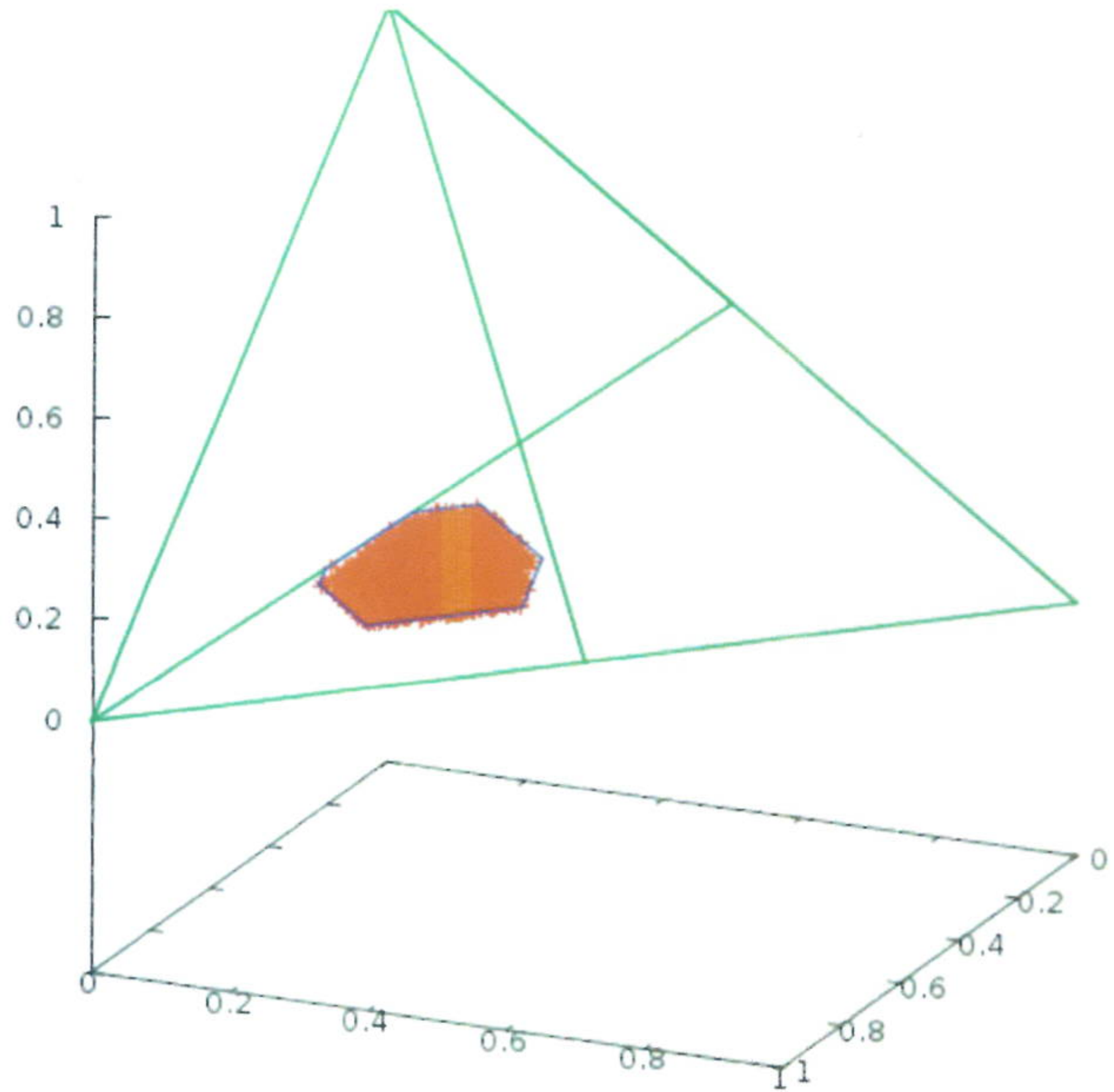
Hence, if $\gamma\in\mathcal{P}$, with $\alpha=\beta=\sigma=\{\sigma_1\geq\dots\geq\sigma_{2p}\}$, $\forall (I,J,K)\in T_r^p$, one has

$$\sum_{k'\in K_2}\gamma_{k'} \leq \sum_{i'\in I_2}\sigma_{i'} + \sum_{j'\in J_2}\sigma_{j'}, \text{ that is}$$

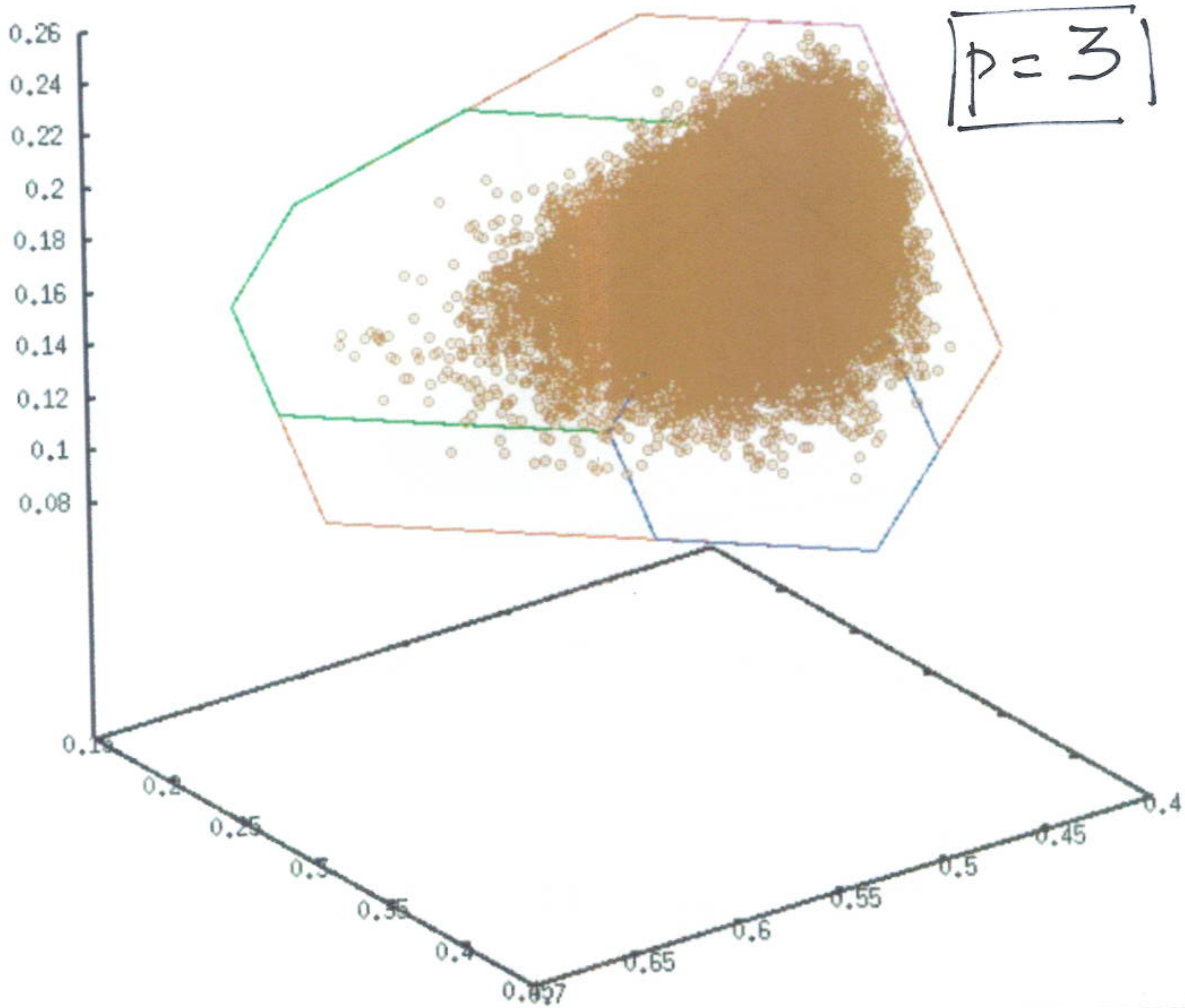
$$\sum_{k\in K}(\gamma_{2k-1}+\gamma_{2k}) \leq 2\Big(\sum_{i\in I}\sigma_{2i-1} + \sum_{j\in J}\sigma_{2j}\Big), \text{ i.e. } \underline{\delta_{\overline{\omega}}^{-1}(\gamma)\in\mathcal{P}_1}$$

cqfd

$\boxed{p = 3}$



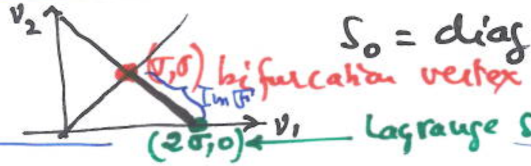
$\mathcal{P}_1$: 25000 random P_0-adapted structures



Orthogonal projection of $\mathcal{S}$ on Δ : 50000 random points
5 dim 2 dim

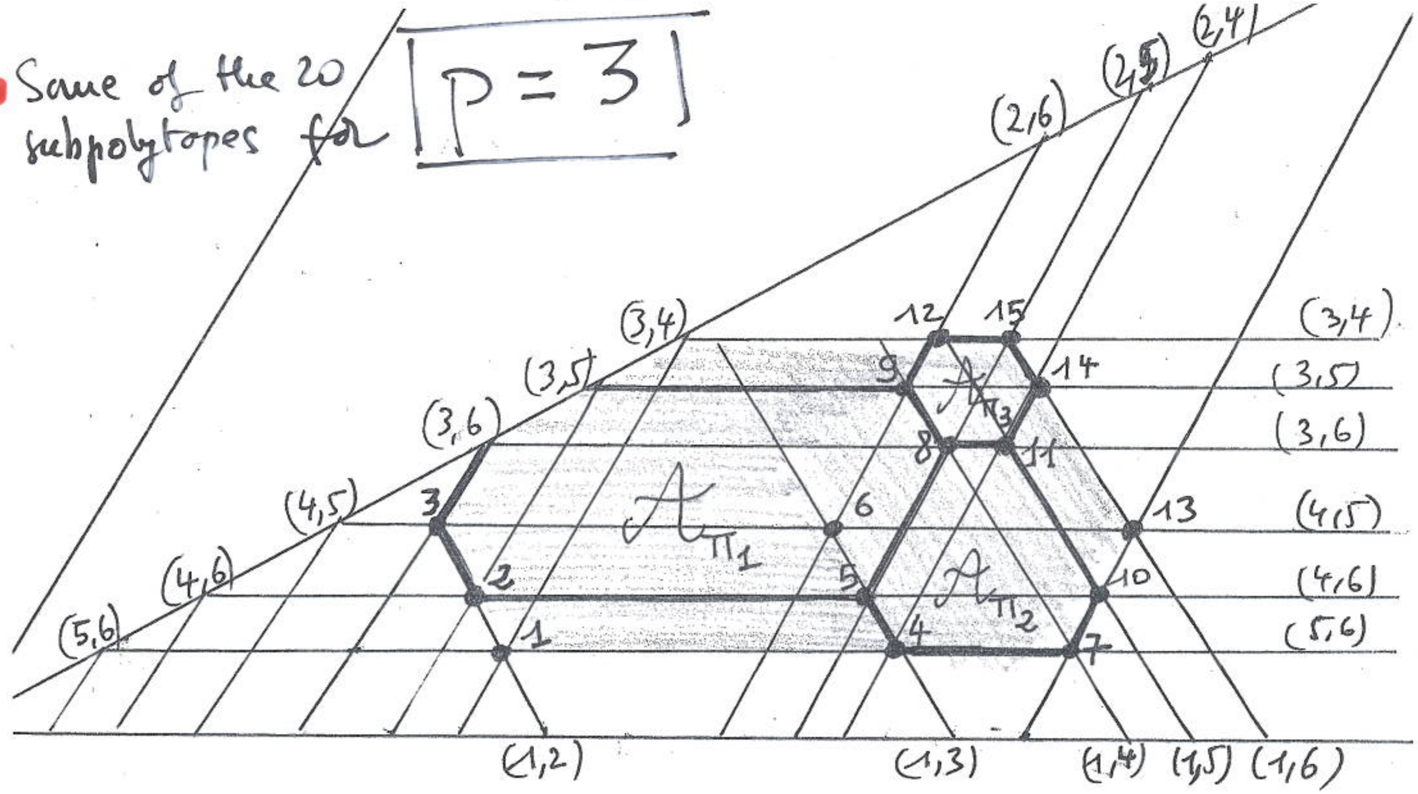
BIFURCATIONS : they occur necessarily for values of the angular momentum whose spectrum lies on the boundaries of the subpolytopes $\mathcal{F}$ (P-adapted structures)

- ex: 3 bodies, $p = 2$:



$S_0 = \mathrm{diag}(\sigma,\sigma,0,0)$

- Some of the 20 subpolytopes for $\boxed{p=3}$



APPENDICE : definition of T_r^n

$$U_r^n = \left\{ (I,J,K),\ \sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \frac{r(r+1)}{2} \right\}$$

$$T_1^n = U_1^n$$

$$T_r^n = \left\{ \begin{array}{c} (I,J,K) \in U_r^n,\ \forall s<r,\ \forall (F,G,H) \in T_s^r, \\ \sum_{f\in F} i_f + \sum_{g\in G} j_g \leq \sum_{h\in H} k_h + \frac{s(s+1)}{2} \end{array} \right\}$$

Simplest case : $T_1^n = U_1^n$: Weyl's $\leq$ (Math Annalen 71 (1912) 441-479)

$$\boxed{\gamma_{i+j+1} \leq \alpha_{i+1} + \beta_{j+1}}$$

- Easy proof
- Easy to see that Weyl's $\leq$ for γ_1 and γ_2 are the same.

- ex $n=2$