NON AVOIDED CROSSINGS

OR

WHY ASKING JACQUES LASKAR TO MAKE A COMPUTATION MAY LEAD TO UNEXPECTED RESULTS

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IMCCE & PARIS 7

Objet :	Solution
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Date :	Mercredi 11 septembre 2013 22h23

Bonsoir Alain,

Sous toutes reserves car je n'ai pas vérifie. Tout peut changer demain.

J'ai montré que les deux équations sont proportionnelles. (si $m1-m3$ et $m2-m4$ non nuls).

Donc on a une seule eq du 3 eme degré avec 3 solutions dont une seule réelle qui est < 0.

EN plus, l'equation finale nest pas trop compliquée. Elle me plait meme assez :

$$4 (m1*m2*m3*m4) + 3(m1*m2*m3+m1*m2*m4+m1*m3*m4+m2*m3*m4) X$$
$$+ 2(m1*m2+m1*m3+m1*m4+m2*m3+m2*m4+m3*m4) X^2 + (m1+m2+m3+m4) X^3 =0$$

Maintenant je fais le rapport que je devais faire avant ce soir
et je rentre.

A demain,

Jacques

CENTRAL CONFIGURATIONS : 3 bodies

LAGRANGE 1772

$$\ddot{x} = (\ddot{\vec{r}}_1, \ddot{\vec{r}}_2, \ddot{\vec{r}}_3) \,/\!/\, x = (\vec{r}_1 - \vec{r}_G, \vec{r}_2 - \vec{r}_G, \vec{r}_3 - \vec{r}_G)$$

$$\Updownarrow$$

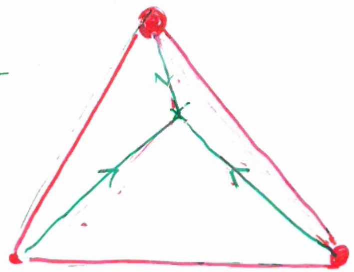
$$x \text{ critical point of } U\big|_{I = \text{cste}} ,$$

$$U(x) = \sum_{i<j} \frac{m_i m_j}{\|\vec{r}_i - \vec{r}_j\|} , \quad I = \sum_i m_i \|\vec{r}_i - \vec{r}_G\|^2$$

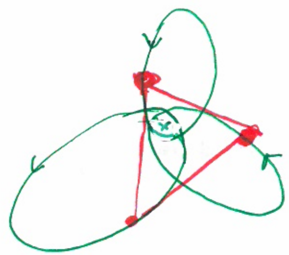
$$= \frac{1}{\sum m_i} \sum_{i<j} m_i m_j \|\vec{r}_i - \vec{r}_j\|^2$$

$\Downarrow$

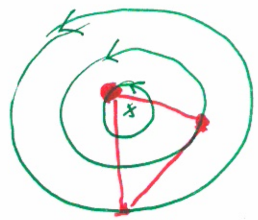
$\exists$ HOMOGRAPHIC (= KEPLER-LIKE) MOTIONS



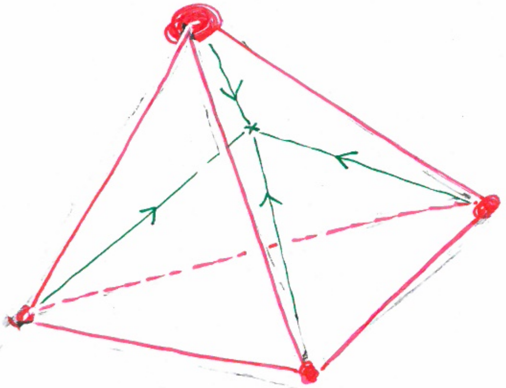
$e = 1$



$0 < e < 1$



$e = 0$ (RELATIVE EQUILIBRIUM) $\Leftrightarrow$ RIGID



4 bodies : Regular tetrahedron is a C.C. BUT homographic motions with $e \neq 1$ possible only in $\mathbb{R}^4$ or $\mathbb{R}^6$

and of the form : $$x(t) = \varsigma(t)\, x(0),$$

where $\varsigma(t) \in \mathbb{C}$ satisfies Kepler equation

$$\ddot{\varsigma} = -\,\mathrm{cst}\, \frac{\varsigma}{|\varsigma|^3}$$

and $\mathbb{R}^{2d}$ is identified to $\mathbb{C}^d$ by the choice of a complex structure J.

SURPRISE

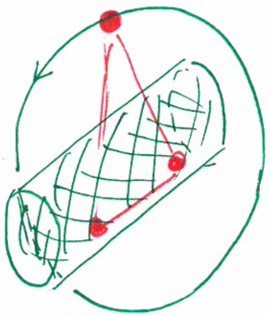
(ALBOUY-CHENCINER INVENTIONES 1998)

In $\mathbb{R}^{2d\geq 4}$, $\exists$ more configurations admitting (quasi-periodic) relative equilibrium motions ($e=0$), the BALANCED CONFIGURATIONS

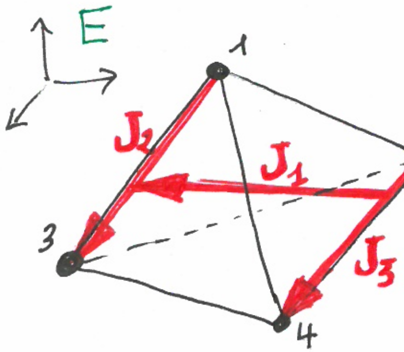
$$\boxed{x(t) = e^{\Omega t}x(0)\,,\quad \Omega \text{ antisymmetric non degenerate}}$$

(FOR THE CASE OF CENTRAL CONFIGURATIONS, $\Omega = \omega J$, $J^2 = -Id$)

Ex: Isosceles triangles with 3 equal masses



SHAPES



$$X = \begin{array}{c} \quad J_1 \;\; J_2 \;\; J_3 \\ \left(\begin{array}{ccc} | & | & | \\ | & | & | \end{array} \right) \end{array}$$

translation invariant

SIDE OF THE BODIES

$$B = X^{tr} X$$

INTRINSIC INERTIA MATRIX

$O(E)$-invariant

SIDE OF AMBIENT SPACE

$$S = X X^{tr}$$

INERTIA MATRIX

$O(\mathcal{D}^*)$-invariant

democracy group

FORCES

$$\ddot{\vec{r}}_i = \sum_{j \neq i} m_j \frac{\vec{r}_j - \vec{r}_i}{\|\vec{r}_j - \vec{r}_i\|^3}$$

$$\Updownarrow$$

$$\left(\ddot{\vec{r}}_1 \cdots \ddot{\vec{r}}_4 \right) = \left(\vec{r}_1 \cdots \vec{r}_4 \right) \begin{pmatrix} -\sum_{l \neq 1} \frac{m_l}{r_{1l}^3} & \frac{m_1}{r_{12}^3} & \frac{m_1}{r_{13}^3} & \frac{m_1}{r_{14}^3} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{m_4}{r_{41}^3} & \frac{m_4}{r_{42}^3} & \frac{m_4}{r_{43}^3} & -\sum_{l \neq 4} \frac{m_l}{r_{4l}^3} \end{pmatrix}$$

$$\Updownarrow$$

$$\ddot{X} = 2XA$$

WINTNER-CONLEY MATRIX

$$U(X) = \hat{U}(B)$$

$$d\hat{U}(B)\Delta B = \operatorname{trace} A \Delta B$$

$$\boxed{X(t) = e^{\Omega t} X(0)}$$

RELATIVE EQUILIBRIA

$$\ddot{X} = \Omega^2 X = 2XA$$

SIDE OF THE BODIES $\Leftarrow$ $\Rightarrow$ SIDE OF AMBIENT SPACE

$$X^{tr}\Omega^2 X = 2BA \qquad\qquad \Omega^2 S = 2XAX^{tr}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

DEFINITION OF BALANCED CONFIGURATIONS $\rightarrow$ $[A, B] = 0$ $\qquad\qquad$ $[\Omega^2, S] = 0$

$$\Updownarrow$$

B critical point of $\hat{U}|$ isospectral manifold of B

$\Updownarrow$ 4 bodies

$$\begin{cases} I = \dfrac{1}{M} \displaystyle\sum_{i_1 < i_2} m_{i_1} m_{i_2} \|\vec{r}_{i_1} - \vec{r}_{i_2}\|^2 = \text{cste} \\ \dfrac{1}{M} \displaystyle\sum_{i_1 < i_2 < i_3} m_{i_1} m_{i_2} m_{i_3}\, \mathrm{aire}^2 (\triangle_{i_1 i_2 i_3}) = \text{cste} \\ \dfrac{1}{M} m_1 m_2 m_3 m_4\, \mathrm{vol}^2 (1234) = \text{cste} \end{cases}$$

$\mathbb{R}^6$ or $\mathbb{R}^4$?

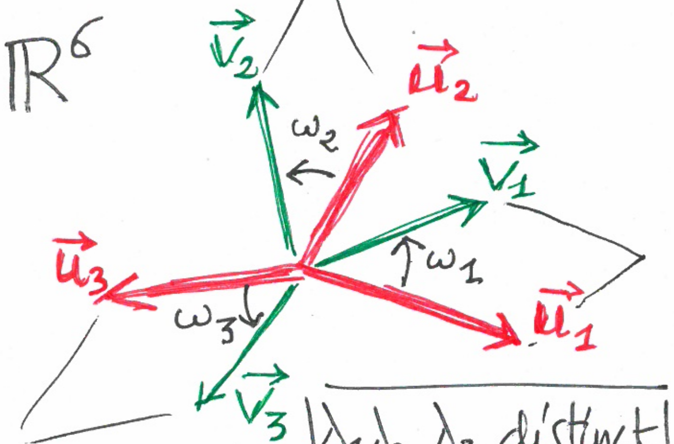
$\boxed{\text{QUESTION}}$ MINIMAL DIMENSION FOR WHICH EXISTS A RELATIVE EQUILIBRIUM MOTION OF X BALANCED CONFIGURATION CLOSE TO X_0 = REGULAR TETRAHEDRON ?

$\Omega^2 X = 2XA \Longrightarrow \exists$ bases $\ni$ $\Omega^2 = \mathrm{diag}(-\omega_1^2,-\omega_2^2,-\omega_3^2)$, $2A = \mathrm{diag}(-\lambda_1,-\lambda_2,-\lambda_3)$

$S = \mathrm{diag}(\sigma_1,\sigma_2,\sigma_3)$, $B = \mathrm{diag}(b_1,b_2,b_3)$

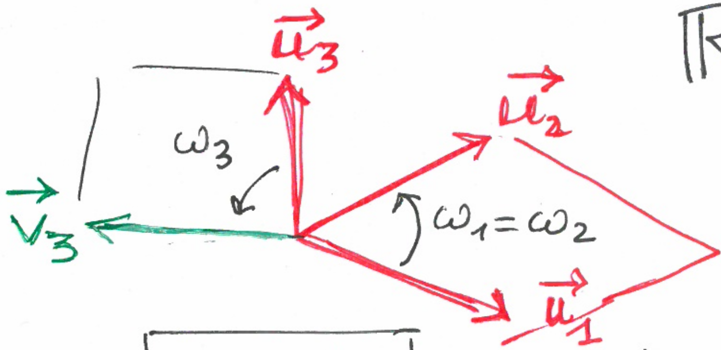
with $\boxed{\omega_i^2 = \lambda_i \ , i = 1,2,3}$

$\mathbb{R}^6$



$\boxed{\lambda_1, \lambda_2, \lambda_3 \text{ distinct}}$ (6-BALANCED)

$\mathbb{R}^4$

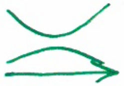
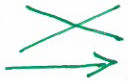


$\boxed{\lambda_1 = \lambda_2}$ (4-BALANCED)

C.C. $\Updownarrow$ $\lambda_1 = \lambda_2 = \lambda_3$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ common eigenbasis of S and Ω^2

AVOIDED CROSSINGS : $\left\{ \begin{array}{l} \text{SYMMETRIC MATRICES} \\ \text{WITH A DOUBLE EIGENVALUE} \end{array} \right\}$ CODIM 2

(IN GENERIC 1-PARAMETER FAMILIES  AND NOT )

$\Downarrow$

4-BODIES BALANCED CONFIGURATIONS : DIM 3

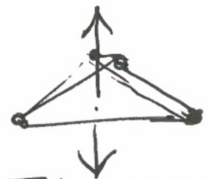
$\downarrow$

AFTER SCALING : DIM 2 $\Longrightarrow$

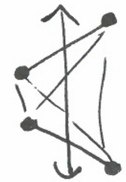
COULD EXPECT THAT NEAR REGULAR  ALL 4-BALANCED CONF. ARE REGULAR 

NEVERTHELESS, IF EQUAL MASSES

$\exists$ NON TRIVIAL FAMILIES :



$\mathbb{Z}/3\mathbb{Z}$-SYM



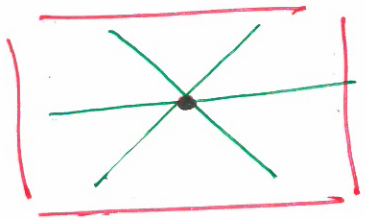
$\mathbb{Z}/4\mathbb{Z}$-SYM

FIRST GUESS: DUE TO SYMMETRY ?

AFTER FIRST TRIES WITH (UNTRACTABLE) RESULTANT I GAVE JACQUES LINEARIZED EQUATIONS ASKING HIM TO SHOW THAT FOR GENERIC MASSES $\exists$ ONLY TRIVIAL SOLUTIONS

NO!

BUT HE FOUND 3 NICE FAMILIES OF NON-TRIVIAL SOLUTIONS



$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

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> simplify(expand(R));
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$$48\, b e c a d f - 4 d^4 c^2 - 4 f^4 b^2 - 4 a^4 e^2 - a^4 d^2 + 2 a^3 d^3 - a^2 d^4 - a^4 f^2 + 2 a^3 f^3$$

$$- a^2 f^4 - d^4 f^2 + 2 d^3 f^3 - d^2 f^4 - b^4 a^2 - b^4 d^2 + 8 b^4 f^2 - c^4 a^2 + 8 c^4 d^2 - c^4 f^2$$

$$+ 8 e^4 a^2 - e^4 d^2 - e^4 f^2 - 12 b^4 c^2 - 12 b^4 e^2 - 12 b^2 c^4 - 12 b^2 e^4 - 12 c^4 e^2$$

$$- 12 c^2 e^4 - 12 b e c a^2 d - 12 b e c a^2 f - 12 b e c a d^2 - 12 b e c a f^2$$

$$- 12 b e c d^2 f - 12 b e c d f^2 - 4 b^6 - 4 c^6 - 4 e^6 + 8 b e c a^3 + 8 b e c d^3$$

$$+ 8 b e c f^3 + 10 d c^2 a^2 f - 20 d^2 c^2 a f + 10 d c^2 a f^2 + 10 f b^2 a^2 d$$

$$+ 10 f b^2 a d^2 - 20 f^2 b^2 a d - 20 a^2 e^2 d f + 10 a e^2 d^2 f + 10 a e^2 d f^2$$

$$+ 2 b^2 c^2 a d + 2 b^2 c^2 a f + 38 b^2 c^2 d f + 2 b^2 e^2 a d + 38 b^2 e^2 a f + 2 b^2 e^2 d f$$

$$+ 38 c^2 e^2 a d + 2 c^2 e^2 a f + 2 c^2 e^2 d f + 36 b^3 e c a + 36 b^3 e c d - 72 b^3 e c f$$

$$+ 36 c^3 b e a - 72 c^3 b e d + 36 c^3 b e f - 72 e^3 b c a + 36 e^3 b c d + 36 e^3 b c f$$

$$+ 2 a^4 d f - 2 a^3 d^2 f - 2 a^3 d f^2 - 2 a^2 d^3 f + 6 a^2 d^2 f^2 - 2 a^2 d f^3 + 2 a d^4 f$$

$$- 2 a d^3 f^2 - 2 a d^2 f^3 + 2 a d f^4 - 2 d c^2 a^3 - 2 d^2 c^2 a^2 + 8 d^3 c^2 a + 8 d^3 c^2 f$$

$$- 2 d^2 c^2 f^2 - 2 d c^2 f^3 - 2 f b^2 a^3 - 2 f^2 b^2 a^2 + 8 f^3 b^2 a - 2 f b^2 d^3 - 2 f^2 b^2 d^2$$

$$+ 8 f^3 b^2 d + 8 a^3 e^2 d + 8 a^3 e^2 f - 2 a^2 e^2 d^2 - 2 a^2 e^2 f^2 - 2 a e^2 d^3 - 2 a e^2 f^3$$

$$- 2 b^2 c^2 a^2 - 20 b^2 c^2 d^2 - 20 b^2 c^2 f^2 - 20 b^2 e^2 a^2 - 2 b^2 e^2 d^2 - 20 b^2 e^2 f^2$$

$$- 20 c^2 e^2 a^2 - 20 c^2 e^2 d^2 - 2 c^2 e^2 f^2 + 10 b^4 a d - 8 b^4 a f - 8 b^4 d f - 8 c^4 a d$$

$$+ 10 c^4 a f - 8 c^4 d f - 8 e^4 a d - 8 e^4 a f + 10 e^4 d f + 2 a^3 d b^2 - 8 a^2 d^2 b^2$$

$$+ 2 a d^3 b^2 + 2 a^3 f c^2 - 8 a^2 f^2 c^2 + 2 a f^3 c^2 + 2 d^3 f e^2 - 8 d^2 f^2 e^2 + 2 d f^3 e^2$$

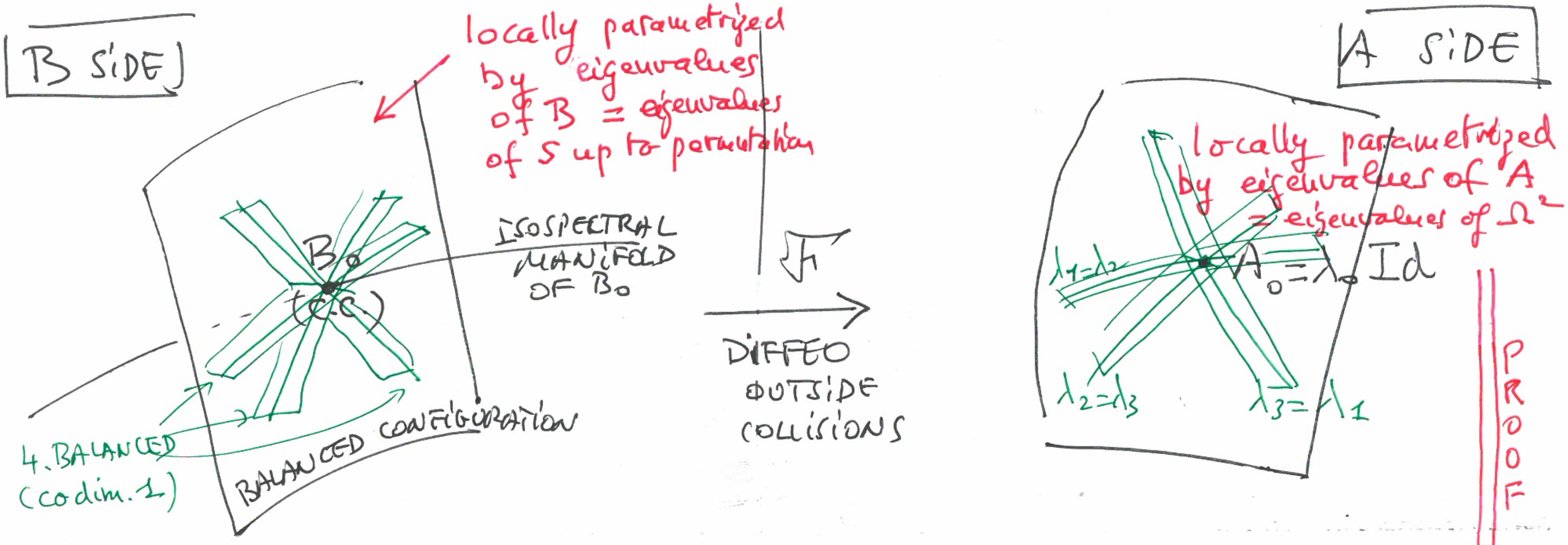
$$+ 84 b^2 c^2 e^2$$

AND HE WAS RIGHT !

GENERIC $B_0 \iff$ (NO 3 MASSES $=$)

- IN THE <u>UNIQUE</u> ORTHONORMAL EIGENBASIS OF B_0, BOTH B_0 AND A_0 ARE DIAGONAL
- IF B BALANCED CLOSE TO B_0, $\exists\, 1!\ R = R(B)$ SUCH THAT

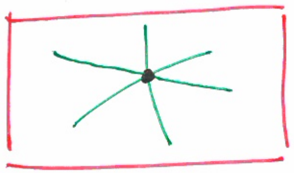
$$RBR^{-1} = \mathrm{diag}(b_1, b_2, b_3)\ ,\quad RAR^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$



IF $\mathcal{A}(B) = R(B)A(B)R(B)^{-1}$, $\quad d\mathcal{A}(B_0)\Delta B = \Delta A + \cancel{[\Delta R, A_0]}$

$= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ $\qquad \Delta A = d\mathcal{F}(B_0)\Delta B$ (ISO), $\quad \Delta R = dR(B_0)\Delta B$, $\quad A_0 = \lambda_0 Id$

HENCE, AFTER SCALING

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