To Jacob, some questions for his 60th anniversary

Perverse solutions of the planar n-body problem

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Let $X(t) = \big(\vec{r}_1(t), \vec{r}_2(t), \ldots, \vec{r}_n(t)\big)$ be a solution of the n-body problem with newtonian potential and masses $m_1, m_2, \ldots, m_n$. We ask the following questions:

Question 1. *Does there exist another system of masses, $(m'_1, m'_2, \ldots, m'_n)$, for which $X(t)$ is still a solution ?*

Question 2. *Same as question 1 but insisting that the sum $M = \sum_{i=1}^n m_i$ of the masses and the center of mass $\vec{r}_G = (1/M)\sum_{i=1}^n m_i \vec{r}_i$ do not change.*

Definition. *If the answer to the first (resp. second) question is yes, we shall say $X(t)$ is a* perverse *(resp.* really perverse*) solution and the allowed systems of masses will be called* admissible.

Remark. If the inverse problem raised by Question 1 may seem very natural, Question 2 needs some motivation. The possible existence of *choreographies* whose masses are not all equal is at the origin of the notion of perverse solution. Recall that a planar choreography is a periodic solution $\mathcal{C}(t) = \big(q(t+T/n), \ldots, q(t+(n-1)T/n), q(t+T) = q(t)\big)$ of the n-body problem such that all n bodies follow the same closed plane curve $q(t)$ with equal time spacing ([S1],[S2],[CGMS]). It is noticed in [C] that if a choreography exists whose masses are not all equal, it is a really perverse choreography: by replacing each mass by the mean mass M/n we obtain new admissible masses, while keeping the center of mass and total mass unchanged.

In the sequel, we shall consider only the planar problem. We shall identify the plane of motion with the complex plane $\mathbb{C}$, hence the positions $\vec{r}_G, \vec{r}_i,\ i = 1, \ldots, n$, with complex numbers $z_G, z_i,\ i = 1, \ldots, n$, and $X(t)$ with an element of $\mathbb{C}^n$. We shall use the following notations (we always assume that $z_i \neq z_j$):

$$\begin{cases} z_{ij} = z_i - z_j, \quad a_{ij} = \dfrac{z_{ij}}{|z_{ij}|^3} \text{ if } i \neq j,\ a_{ii} = 0, \quad m = (m_1, m_2, \ldots, m_n), \\ \mathcal{A}_0 = \big(z_{ij}\big)_{1 \le i,j \le n}, \quad \mathcal{A} = \big(a_{ij}\big)_{1 \le i,j \le n}. \end{cases}$$

We shall identify $\mathcal{A}_0$ and $\mathcal{A}$ with linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. This allows them to act on the vector m. The definition of the center of mass may be rewritten

$$\sum_{j=1}^n m_j z_{ij} = M(z_i - z_G), \quad M = \sum_{j=1}^n m_j, \quad \text{that is} \quad \mathcal{A}_0(t)m = M\big(X(t) - z_G(t)(1, \ldots, 1)\big),$$

and the equations of motion in a galilean frame are

$$\forall t, \forall i,\ \ddot{z}_i(t) = -\sum_{j\neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^3}, \quad \text{that is} \quad \mathcal{A}(t)m = -\ddot{X}(t).$$

Hence, if another set $m'_1, m'_2, \ldots, m'_n$ of masses admits the same solution $X(t)$, the difference

$$\mu = m - m' = (\mu_1, \mu_2, \ldots, \mu_n) \in I\!R^n$$

is a real non-zero vector in the kernel of any of the complex matrices $\mathcal{A}(t)$. If, moreover, M and $z_G(t)$ are the same for the two sets of masses, μ is also in the kernel of any of the matrices $\mathcal{A}_0(t)$. It will be important to remember that $\mathcal{A}_0$ and $\mathcal{A}$ are antisymmetric (${}^t\mathcal{A}_0 = -\mathcal{A}_0$, ${}^t\mathcal{A} = -\mathcal{A}$). This will cause the parity of n to play a role. We start with the obvious

Proposition 1. *If $n = 2$, no solution is perverse. In other words, any planar solution of the 2-body problem determines the masses.*

Proof. If $n = 2$, the matrix $\mathcal{A}(t)$ is of maximal rank whenever it is defined, that is provided $z_{12}(t) \neq 0$.

As soon as $n \geq 3$, perverse solutions do exist, as shown by the following “trivial” examples (thanks to Reinhart Schäfke for proposing immediately the example of an equilateral triangle rotating around a fourth body):

Example 1. $X(t) = \left(re^{i\omega t}, re^{i\omega t + \frac{2i\pi}{n-1}}, \ldots, re^{i\omega t + (n-2)\frac{2i\pi}{n-1}}, 0\right)$ is a relative equilibrium solution with n masses $(m_1, m_1, \ldots, m_1, m_0)$ if and only if the following “Kepler-like” condition is satisfied:

$$r^3\omega^2 = \frac{U_n}{I_n} = m_0 + \frac{m_1}{n-1} \sum_{1\leq j<k\leq n-1} \frac{1}{|z_{jk}|}.$$

In the above formula,

$$U_n = m_1 m_0 (n-1) + m_1^2 \sum_{1\leq j<k\leq n-1} \frac{1}{|z_{jk}|} \quad \text{and} \quad I_n = m_1(n-1)$$

stand respectively for the potential and the moment of inertia with respect to the center of mass, of the configuration normalized by $|z_{in}| = 1$ if $1 \leq i \leq n-1$. This leaves a one parameter family of admissible sets of masses. Moreover, for the regular $(n-1)$-gon inscribed in the unit circle, we have

$$\sum_{1\leq j<k\leq n-1} \frac{1}{|z_{jk}|} = \frac{n-1}{2}\Big(\frac{1}{2\sin\frac{\pi}{n-1}} + \frac{1}{2\sin\frac{2\pi}{n-1}} + \cdots + \frac{1}{2\sin\frac{(n-2)\pi}{n-1}}\Big) = (n-1)^2(\delta_{n-1} + 1),$$

where we have set

$$\delta_n = -1 + \frac{1}{4n} \sum_{l=1}^{n-1} \frac{1}{\sin\frac{\pi l}{n}}.$$

Hence,

$$r^3\omega^2 = m_0 + (n-1)m_1(\delta_{n-1}+1) = M + (n-1)m_1\delta_{n-1}.$$

Provided δ_{n-1} is different from 0, the right hand side of the above formula is a linear form in the masses which is linearly independent of the total mass $M = m_0 + m_1(n-1)$.
But δ_{n-1} is strictly negative if $n-1 \leq 472$ and strictly positive if $n-1 \geq 473$ (see [MS]; the first occurence of the magic number 472 seems to be in [M]). It follows that M may be chosen as a natural parameter of the set of admissible masses. In particular, these examples are *perverse* but not *really perverse*.

Remark. For non-newtonian potentials of the form $1/r^{2\beta}$, $\beta \neq 1/2$, the analogue of δ_n becomes

$$\delta_n = -1 + \frac{1}{2^{2\beta+1}n}\sum_{l=1}^{n-1}\frac{1}{(\sin\frac{\pi l}{n})^{2\beta}},$$

and may become zero for some value of β (see [CS]).

Example 2. Similar to Example 1 are the relative equilibrium solutions whose configuration is made of one central mass m_0 and k regular homothetic n-gons, the masses in the j-th polygon being all equal to m_j, for $j = 1, \ldots, k$. In this case, the equations insuring relative equilibrium motion may be put in the form (see [BE] or [CS]):

$$\rho_j^3\omega^2 = m_0 + \sum_{s=1}^{k} m_s H_n(\rho_s/\rho_j), \quad j = 1, \ldots, k,$$

where ρ_j is the radius of the $j-$th polygon and

$$H_n(x) = \sum_{l=1}^{n^*(x)}\frac{1 - x\cos\frac{2\pi l}{n}}{(1 + x^2 - 2x\cos\frac{2\pi l}{n})^{\frac{3}{2}}}, \quad n^*(x) = n \text{ if } x \neq 1,\ n^*(x) = n-1 \text{ if } x = 1.$$

In the "generic" case, such solutions will be perverse and not really perverse. But, as soon as $k \geq 3$, one gets really perverse solutions for special choices of the radii ρ_j and the integer n (see the last section).

When $n = 3$, the situation is still easy to deal with, thanks to Albouy and Moeckel [AM].

Proposition 2. *The perverse solutions of the planar 3-body problem are exactly the collinear homographic solutions. The center of mass is the same for all admissible sets of masses, but not the total mass, which is a natural parameter for such sets. In particular, really perverse solutions do not exist.*

Proof. If $n = 3$, the matrix $\mathcal{A}(t)$ is of rank 2 as soon as the configuration is not a triple collision. The existence of a fixed non-zero real vector μ in the kernel of $\mathcal{A}(t)$ implies immediately that the three bodies stay collinear, with a fixed configuration up to similarity. This implies that the motion is homographic. Moreover, the center of mass is dynamically defined as the unique common focus of the similar conics described by the bodies in a galilean frame where the center of mass corresponding to one admissible choice of masses is fixed.

Conversely, each collinear homographic solution of the 3-body problem is perverse: this is a direct consequence of Theorem 2 and Proposition 4 of [AM] which, together, say that the set of masses for which a given configuration of three bodies is central is of dimension 2 and may be parametrized by the "multiplier" λ (which is determined by the equation $\ddot{X} = -\lambda X$ as soon as the homographic solution X is given) and the total mass M. To finish the proof, it remains to recall that the center of mass of such a 3-body configuration does not depend on the choice of masses for which it is central (see [AM] where this observation is attributed to C. Marchal).

The case $n = 4$. The determinant of the antisymmetric 4×4 matrix $\mathcal{A}$ is equal to the square of the Pfaffian (if we extend the notation K_4 of [AM] to the complex domain, $P = K_4/2$)

$$P(z_1, z_2, z_3, z_4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Hence, if a solution of the 4-body problem admits two different sets of masses, its configuration must satisfy $P\big(z_1(t), z_2(t), z_3(t), z_4(t)\big) = 0$ at each instant t.

As in [AM], but in the complex setting, let us use the following notations :

$$A = z_{12}z_{34},\ B = z_{13}z_{24},\ C = z_{14}z_{23}.$$

The above condition becomes

$$P = \frac{A}{|A|^3} - \frac{B}{|B|^3} + \frac{C}{|C|^3} \equiv 0.$$

On the other hand, as $\mathcal{A}_0$ represents the bivector $(1,1,1,1) \wedge (z_1, z_2, z_3, z_4)$, it is of rank 2, that is

$$A - B + C \equiv 0.$$

Together, the two identities above imply that A, B, C cannot be $\mathbb{R}$-dependent, i.e. that the three vectors in $\mathbb{R}^2$ represented by the complex numbers A, B, C can never be collinear; indeed, if m_1, m_2, m_3, m_4 lie in this order on a line, A, B, C are real and positive; then $B = A + C$ and $B^{-2} = A^{-2} + C^{-2}$, which is impossible. But then $A - B$ and $A/|A|^3 - B/|B|^3$, being respectively equal to $-C$ and $-C/|C|^3$, must be collinear and this can happen only if $|A| = |B|$, which implies immediately that $|A| = |B| = |C|$ (this remark has already been used in [V] and [AM]). We have proved the

Lemma 1. *For any perverse solution of the planar 4-body problem, the configuration is such that at any time*

$$|z_{12}||z_{34}| = |z_{13}||z_{24}| = |z_{14}||z_{23}|. \qquad (*)$$

Configurations which satisfy $(*)$ do exist – for example, an equilateral triangle with the fourth mass at the center, a rhombus with small angle $\pi/6$, an isosceles triangle with two angles equal to $\pi/6$ and fourth mass at the middle point of the base – but, as we have just seen, they cannot be collinear.

Definition. *A 4-body configuration is called* stricly convex *(resp.* strictly non-convex*) if none of the bodies (resp. if one of the bodies) belongs to the interior of the convex hull of the three others.*

A planar 4-body configuration is either strictly convex, or stricly non-convex, or partially collinear (i.e. such that at least three bodies are collinear).
If, for a given t, the configuration $\big(z_1(t), z_2(t), z_3(t), z_4(t)\big)$ is strictly convex (resp. strictly non-convex) and the real vector $(\mu_1, \mu_2, \mu_3, \mu_4)$ belongs to the kernel of $\mathcal{A}(t)$, each μ_i is different from zero. This is because if, for example, $\mu_1 = 0, \mu_3 \neq 0, \mu_4 \neq 0$, the bodies 2,3,4 are such that $\mu_3 a_{23}(t) + \mu_4 a_{24}(t) = 0$ and hence collinear. And if only one of the μ_j is different from zero, say μ_4, then all a_{i4} must be zero, which means total collision. Moreover, strict convexity is equivalent to three μ_i being of the same sign and strict non-convexity to only two μ_i being of the same sign. For example, 1 lies in the interior of the triangle defined by 2,3,4 if and only if μ_2, μ_3 and μ_4 are of the same sign.

As the μ_i are independent of t, the nature (strictly convex, strictly non-convex, or partially collinear) of the configuration of a perverse solution does not change along the motion. The possibility of collinearities is exluded by the following lemma.

Lemma 2. *In a perverse solution of the planar 4-body problem, three of the bodies can never become collinear. In other words, either the configuration stays strictly convex for all t, or it stays strictly non-convex for all t.*

Proof. Let us suppose now that, for example, 2,3,4 are collinear at some instant t. Then $\mu_1 = 0$, otherwise one would deduce from the equation $a_{21}\mu_1 + a_{23}\mu_3 + a_{24}\mu_4 = 0$ that all four bodies are collinear at this instant and we have already excluded this possibility. This implies that (μ_2, μ_3, μ_4) belong, for any t to the kernel of the antisymmetric matrix $(a_{ij}(t))_{2\leq i,j\leq 4}$, which means that it is proportional to $(a_{34}(t), a_{42}(t), a_{23}(t))$. As in the proof for the case $n = 3$, one concludes that the configuration of the three last bodies remains similar to a given collinear configuration. This in turn implies that the whole configuration remains self-similar: indeed, the relations

$$\frac{|z_{13}|}{|z_{12}|} = \frac{|z_{43}|}{|z_{42}|}, \quad \frac{|z_{13}|}{|z_{14}|} = \frac{|z_{23}|}{|z_{24}|},$$

say that the the fourth body lies at the intersection of two circles centered on the line which contains the three first.
Finally, the solution should be homographic, but this is impossible because it follows immediately from Dziobek's equations in terms of triangle areas [D] that a configuration of four bodies with three bodies collinear is never a central configuration (thanks to Alain Albouy for reminding me of this fact). This proves the lemma.

There exists at least one perverse – but not really perverse – solution with non-convex configuration: it is our "trivial" example of three equal masses in an equilateral triangle uniformly rotating around the fourth, located at their center of mass. This is the sole homographic perverse solution because in [MB], McMillan and Bartky prove that this is the only configuration which is central for more than one set of (non-homothetic) masses. No other example, in particular no convex example, is known.

Question. *Is the MacMillan and Bartky example the only perverse solution of the planar 4-body problem? In other words, do non-homographic perverse solutions of the planar 4-body problem exist?*

As a consolation for this disappointing situation, we now prove the

Proposition 3. *If $n \leq 4$, the planar n-body problem does not possess any really perverse solution.*

Proof. If $(m_1, \ldots, m_n)$ and $(m'_1, \ldots, m'_n)$ are admissible masses for a really perverse solution $X(t)$, their differences $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ belong, at any time, to the kernel of both matrices $\mathcal{A}_0(t)$ and $\mathcal{A}(t)$, that is

$$\mu_2 z_{12} + \mu_3 z_{13} + \mu_4 z_{14} = 0, \quad \mu_2 \frac{z_{12}}{|z_{12}|^3} + \mu_3 \frac{z_{13}}{|z_{13}|^3} + \mu_4 \frac{z_{14}}{|z_{14}|^3} = 0, \quad \text{etc...}$$

As none of the real numbers $\mu_1, \mu_2, \mu_3, \mu_4$ is equal to zero (because three bodies are never collinear) this implies, in the same way as above, that

$$\begin{cases} |z_{12}| = |z_{13}| = |z_{14}|, \\ |z_{21}| = |z_{23}| = |z_{24}|, \\ |z_{31}| = |z_{32}| = |z_{34}|, \\ |z_{41}| = |z_{42}| = |z_{43}|. \end{cases}$$

Hence, the configuration should be a regular tetrahedron. As it is planar, this is impossible.

What about 5 bodies ? The homographic perverse solutions include on the one hand all the collinear ones (same reasoning as in the case of three bodies, using [AM]), on the other hand the "trivial" example of four equal masses on a square uniformly rotating around the fifth one located at the center of mass. None of these is really perverse.
Only in the case of choreographies – whose definition was recalled at the beginning of the paper – are we able to say more.

Proposition 4 (see [C]). *For $n \leq 5$, the planar n body problem does not possess any perverse choreography.*

This is done by interverting the roles of the z_{ij} (resp. the a_{ij}) and the masses, that is replacing the equations $\mathcal{A}_0 m = 0$ (resp. $\mathcal{A} m = 0$) by equations which involve the circulant $n \times n$ matrix defined by the n masses. One then uses the spectral structure of such matrices.

More bodies: really perverse solutions of the planar n-body problem do exist. It is shown in [CS] that relative equilibria of a central mass and at least three homothetic regular n-gons, with equal masses on each of them, may be really perverse if n is well chosen. The simplest such example seems to be 3 regular 456-gons, that is 1369 bodies.

Finally, we ask the

Question. *Do non-homographic perverse solutions of the planar n-body problem exist?*

This is probably a difficult question, as are all the questions where one is asked to understand the structure of the solutions of the n-body problem whose configuration remains all the time in a given subset of the configuration space. A famous example of such a question is the Saari conjecture which states that a solution with constant moment of inertia with respect to the center of mass should be rigid (and hence a relative equilibrium by [AC]). The only available method seems to be taking enough time derivatives of the constraints and hoping for some new exploitable constraints to emerge.

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